Berlin Mathematics Research Center

MATH+

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Challenges and Solutions

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Berlin Mathematics Research Center

MATH+

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Contents

Illustration: Julia Nurit Schönnagel

1 Game of Chests

Authors: Fabien Nießen and Silas Rathke (FU Berlin)

Challenge

Every year, the *Frosty Bank of Borrealis*, the venerable financial capital of the North Pole, pays the elves their salaries. But this winter, Santa Claus has issued a new decree: No one can simply receive their wages — every elf must earn them in a game. For the young workshop elf Johanna Snow, the bank has come up with a particularly clever challenge. At the beginning, there are three chests in front of her, completely empty. On the table next to them are 2025 coins with integer values from 1 to 2025, each value occurring exactly once.
The rules of the game are the following:

- Each turn, the bank chooses one of the three chests.
- After that, Johanna must take one of the coins left on the table and place it in the chosen chest.
- This is repeated until all coins have been distributed among the chests.

Only then, at the very end, Johanna is allowed to choose one of the three chests and receives its contents as her salary.
Since the attempt to switch to self-driving sleds using AI failed miserably, leaving a big hole in the North Pole's finances, the bank will try to keep Johanna's payout as low as possible. What is the highest amount that Johanna can still guarantee for herself, assuming she plays

with an optimal strategy? What are the last two digits of this amount?

Note: A strategy is called optimal if there is no other strategy that guarantees Johanna a higher minimum amount.

Possible Answers:

1. 00
2. 12
3. 25
4. 38
5. 47
6. 50
7. 63
8. 75
9. 86
10. 99

Solution

The correct answer is: 3.

The exact amount that Johanna can secure for herself with certainty is 685 125. Therefore, the amount ends in 25.

To prove this, we show how the bank can prevent Johanna's salary from exceeding 685 125 and then describe Johanna's strategy to ensure that she receives this amount.

The bank's strategy is as follows: on each turn, it always chooses one chest with the lowest total value at that moment. Our claim: This ensures that throughout the entire game, *the difference between the most valuable and the least valuable chest never exceeds* 2025. At the beginning, this statement is obviously true. Now let us assume that this claim holds before an arbitrary turn. The reasons for this assumption will be discussed further below.
Then, if the total values of the three chests before the turn are denoted by $A \leq B \leq C$, the inequality $C - A \leq 2025$ holds (the difference between the most valuable and the least valuable chest is at most 2025, as assumed). Following its strategy, the bank now chooses the least valuable chest with the value of A. Johanna places a coin with a value of x in it, where $1 \leq x \leq 2025$.
We now consider all possibilities for the new difference between the most valuable and the least valuable chest after this turn:

- **Case 1:** $A + x \leq B$

  Then the value of the least valuable chest is $A + x$ and that's why the new difference is

  $$C - (A + x) = (C - A) - x \leq 2025.$$

- **Case 2:** $B \leq A + x \leq C$

  Now the value of the least valuable chest is B and that's why the new difference is

  $$C - B \leq C - A \leq 2025.$$

- **Case 3:** $A + x \geq C$

  In this case, the value of the least valuable chest is B and the value of the most valuable chest is $A + x$. Therefore, we get

  $$(A + x) - B = (A - B) + x \leq 2025,$$

  as $A - B \leq 0$ and $x \leq 2025$.

In all cases, the difference remains at most 2025.
We can now use this argument to prove the claim. We already know that the claim holds at the beginning of the game. The argument then allows us to conclude that the claim also holds after the first move. Since the claim is true for move 1, the argument can be applied again, implying that the claim also holds for move 2. This process can be continued successively up to the final move of the game, thereby proving the claim. This type of proof is formally

known as *mathematical induction*.

Therefore, the most valuable chest can contain a maximum value of

$$\frac{1+2+\cdots+2024}{3}+2025=685\,125$$

in the end.

Johanna's strategy to actually obtain this amount is as follows. At the beginning, she mentally partitions the coins with values from 1 to 2024 into three groups with equal total value and assigns each group to one of the three chests. For example, she may first assign the coins of values 8 and 4 to one group, the coins of values $2, 3$ and 7 to another group, and the coins of values $1, 5$ and 6 to the remaining group. She then distributes each subsequent block of nine larger coins $m_1, m_2, \ldots, m_9$ listed in increasing order of value, according to the following scheme: m_1, m_5 and m_9 go to one group, m_2, m_6 and m_7 go to another group, and m_3, m_4 and m_8 go to the remaining group.
Whenever the bank selects a chest, Johanna places a coin from the corresponding group into that chest.
As soon as the bank selects a chest that already contains all the coins of the corresponding group, Johanna places the coin with value 2025 into that chest. From this point on, her goal is achieved: this chest contains at least one third of the sum of the numbers from 1 to 2024 plus 2025. What happens next is irrelevant, as Johanna can choose this chest at the end and thus obtain an amount of at least 685 125.

Illustration: Friederike Hofmann

2 Ready, SET, Bake!

Author: Lukas Protz
Project: MATH+

Challenge

Annika is a diligent elf who loves to make her fellow elves happy. This holiday season, she decides to bake a variety of delicious cookies for all her friends - including Santa! To create a wide range of cookie types, she has gathered:

- 2 different cookie cutters: a Santa and an elf shape
- ingredients for 2 types of dough: vanilla and chocolate
- 3 flavors of icing: vanilla, chocolate and lime
- 3 types of sprinkles: red hearts, silver pearls and ordinary sprinkles

Given these options, Annika can create exactly 36 unique types of cookies, where each cookie type is defined by a unique combination of cutter, dough, icing, and sprinkles. Two cookies are considered the same type if they use the same cutter, dough, icing, and sprinkles.
To ensure that each type tastes delicious, Annika bakes one of each. Once they are finished, she places all 36 cookies into a bowl.
Annika loves the game SET, and the cookies remind her of it. Now she wonders how many cookies she needs to pick from the bowl to guarantee a SET among the chosen cookies.

A SET of cookies is a collection of three cookies, such that for each category (cutter, dough, ...) either all cookies share the same feature or each of the cookies has a different feature. If we represent a cookie via its features (cutter, dough, icing, sprinkles), then for example the cookies

(Santa, vanilla, vanilla, hearts),
(Santa, vanilla, chocolate, pearls),
(Santa, vanilla, lime, ordinary)

form a SET. However, the following three cookies do not form a SET:

(elf, vanilla, vanilla, hearts),
(Santa, vanilla, chocolate, pearls),
(Santa, vanilla, lime, ordinary),

This is, because the first and the second cookie have a different feature in the cutter category, whereas the second and third cookie have the same feature in the cutter category.

Of course, Annika could just pick three of the cookies deliberately to obtain a SET, so to make everything more interesting she picks the cookies at random and without looking. What is the minimum number of cookies Annika needs to select to guarantee that there is at least one SET among the chosen cookies?

Possible Answers:

1. 14 or more
2. 13
3. 12
4. 11
5. 10
6. 9
7. 8
8. 7
9. 6
10. 5 or less

Solution

The correct answer is: 1.

We will show, that Annika needs to choose a minimum of 17 cookies to ensure a SET among them.
To do this, we first look at the simpler case, that there is only one cookie cutter and one dough but still 3 flavors of icing and 3 types of sprinkles. In other words we now want to find a SET in the categories flavors and sprinkles. There are 9 different types of cookies that Annika could bake with this restriction. Now, to visualize the selection process one can set up a table as depicted in Figure 1.

	hearts	pearls	ordinary
vanilla			
chocolate			
lime			

Figure 1: Each space in the table corresponds to a different type of cookie.

If Annika selects a cookie, we make a cross at the corresponding square in the table. A SET in the table is then represented by

- a complete row or column of crosses,
- a complete diagonal of crosses or
- a cross in the upper left corner, a cross in the last row and central column, a cross in the central row and last column as well as rotations of the table with this configuration (see Figure 2). Such a configuration will be called *off-diagonal*.

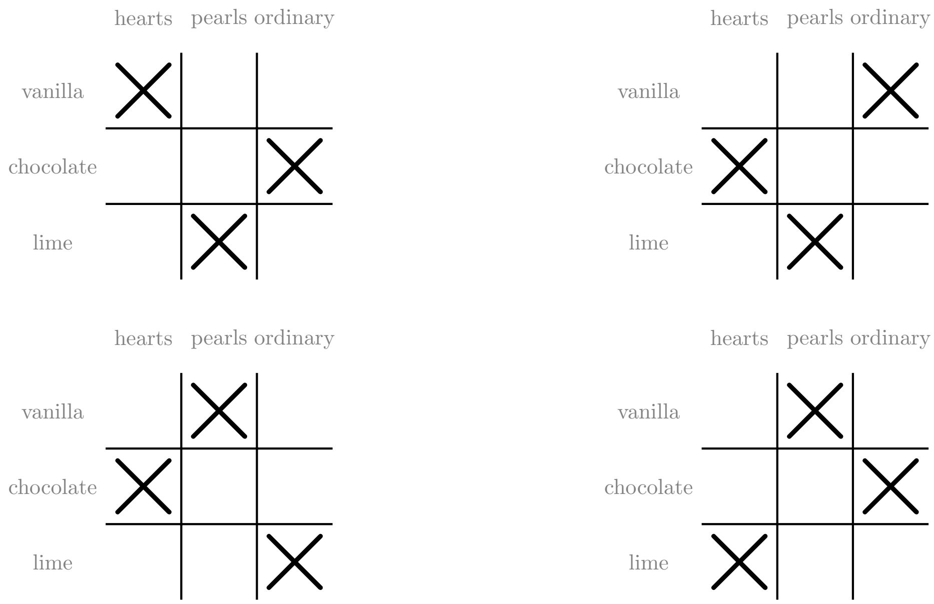


Figure 2: The off-diagonals

At first, it is easy to see that Annika can choose four different types of cookies without achieving a SET, as Figure 3 shows:

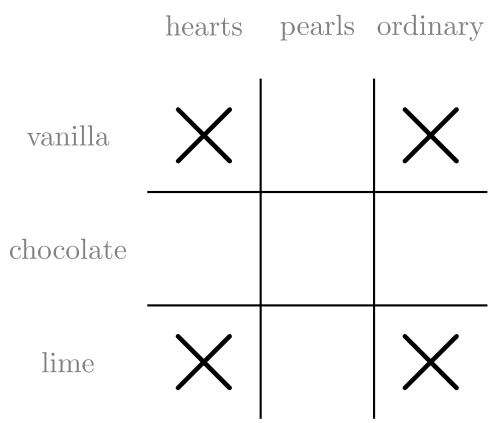


Figure 3: Four types of cookies that do not contain a SET.

However, it is not possible to choose five cookies without getting a SET. We show this directly, by trying to place five crosses in the table without creating a SET. To this end, assume that Annika first chooses the cookie corresponding to the central square in the table. (This can be achieved by reassigning the features to the table in a suitable order.) Can we then place four more crosses without creating a SET? There can be at most two crosses in the corners, as otherwise there will be a complete diagonal in the table. Thus, at least two cookies are either in the central row or central column. However, they cannot both be in the central row or both be in the central column. Therefore we can assume (again by reassigning features if necessary) that one cross is in the top row and central column and one cross is in the central row and left column (see Figure 4):

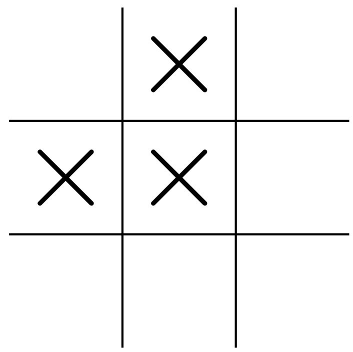
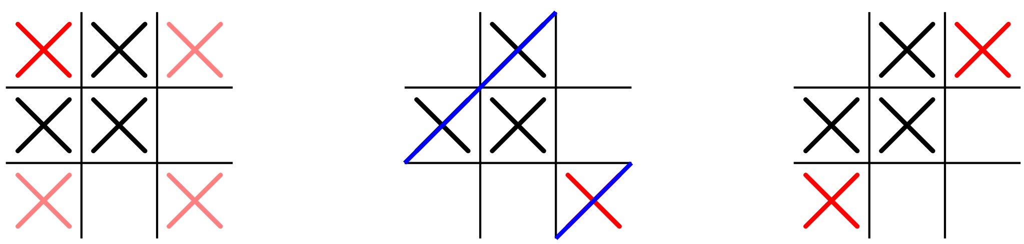


Figure 4: Three crosses in the central row or central column.

To avoid creating a SET the two other crosses must be located in the corners. Neither of them can go in the top left corner as the other one would inevitably create a full row, column or diagonal (see Figure 5(i)). It is also not possible to place a cross at the lower right corner, as this would create an off-diagonal (see Figure 5(ii)). Finally, it is not possible to place the crosses in the remaining two squares, as this again creates a diagonal (see Figure 5(iii)).



(i) (ii) (iii)

Figure 5: The three examples show, that it is impossible to place two more crosses without creating a SET.

To then answer the original question, we have to reintroduce the categories cookie cutters and dough. Because each of these categories has only two features, a SET can only be achieved if all cookies have the same feature of each of these categories.

In total, there a four types of cookies with different combinations of cookie cutters and dough. Therefore, if Annika chooses at least 17 cookies, by the pigeon hole principle, there must be at least 5 cookies with the same dough that also where cut out by the same cookie cutter. But then there must be three cookies among them, that form a SET, because we just proved, that there has to be a SET among the other two categories. Furthermore, less than 17 cookies cannot be sufficient, as there may be no five cookies with the same dough that where cut out by the same cookie cutter. And for four or even less cookies with this property, we can find a combination of icings and sprinkles as in Figure 3 such that these cookies do not contain a SET. Hence, 17 is the answer to the challenge.

Illustration: Ivana Martić

3 A Colorful Christmas

Author: Hajo Broersma
Project: 4TU.AMI

Challenge

Mondrian the painter-elf has designed a rectangular Christmas card as shown in Figure 6. The 8×12 area is divided into the following regions: 72 unit (1×1) squares, three 2×2 squares, and six 1×2 rectangles as shown.

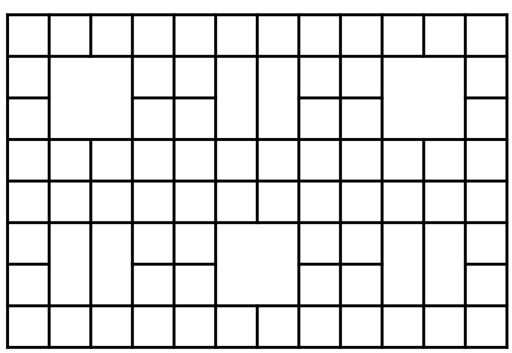


Figure 6: The uncolored Christmas card.

Mondrian asks three pixies to color all the different regions of the card with red, yellow and blue, such that each 1×1 unit square, each 2×2 square and each 1×2 rectangle receives one of these three colors. The completed coloring is called *proper* if regions of the same color share no line segment (though they may share a point); see Figure 7 for examples.

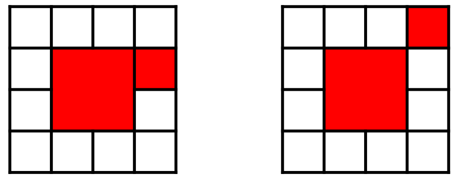


Figure 7: The (partial) coloring on the left *is not* allowed, but the (partial) coloring on the right *is* allowed.

There are many different proper colorings possible. Mondrian especially likes the color red, as it reminds him of father Christmas' outfit. The first thing he wants to know is whether there exists a proper coloring in which exactly half of the 72 unit squares are colored red. Secondly, he is interested to know what the largest area is that can be colored red in a proper coloring. In such a coloring each unit square has an area of 1, each 2×2 square has an area of 4 and each 1×2 square has an area of 2. Now there are two questions the pixies have to answer:

- Question 1: Is there a proper coloring in which exactly 36 unit squares are colored red?
- Question 2: What is the largest area that can be colored red in a proper coloring?

Possible Answers:

1. The correct answer to question 1 is NO; the largest red area in a proper coloring is 39.
2. The correct answer to question 1 is NO; the largest red area in a proper coloring is 40.
3. The correct answer to question 1 is NO; the largest red area in a proper coloring is 41.
4. The correct answer to question 1 is NO; the largest red area in a proper coloring is 42.
5. The correct answer to question 1 is NO; the largest red area in a proper coloring is 43.
6. The correct answer to question 1 is YES; the largest red area in a proper coloring is 39.
7. The correct answer to question 1 is YES; the largest red area in a proper coloring is 40.
8. The correct answer to question 1 is YES; the largest red area in a proper coloring is 41.
9. The correct answer to question 1 is YES; the largest red area in a proper coloring is 42.
10. The correct answer to question 1 is YES; the largest red area in a proper coloring is 43.

Solution

The correct answer is: 8.

The easy part of the challenge is to answer Question 1. It is possible to find a proper coloring in which 36 unit squares are colored red. Simply start in the upper-left corner with the red color for this unit square. Next, alternate between black (for the moment) and red for the unit squares that have a line segment in common with the already colored unit squares. This results in half of the unit squares being colored red and half of them being colored black.
Now, assign yellow to the 2×2 squares and for every pair of $1{\times}2$ rectangles that share a line segment, assign blue to one rectangle and yellow to the other. This partial coloring can be completed to a proper coloring by assigning forced colors: yellow to black unit squares sharing a line segment with both red and blue regions, and blue to those sharing a line segment with both red and yellow regions. Finally, the remaining black unit squares are colored blue. See Figure 8 for an example of a proper coloring with 36 red unit squares.

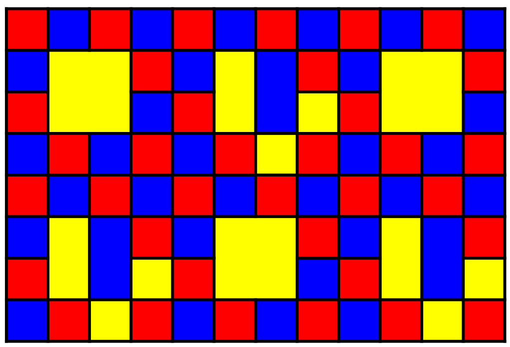


Figure 8: An example of a proper colouring with 36 red unit squares.

It remains to show that the largest possible red area in a proper coloring is 41. For this, we divide the Christmas card into six square regions, each of which has an area of 4×4.

Let us first consider a region consisting of a 2×2 square and the neighboring 12 unit squares, as depicted in Figure 7. If we do not assign red to the 2×2 square, then we can use red on at most 6 of the neighboring unit squares, so we obtain a red area of at most 6. However, if we assign red to the 2×2 square, then we cannot use red on the neighboring unit squares, except at the four corners, so we obtain a red area of at most 8.

Next, let us look at a region consisting of two 1×2 rectangles and the neighboring 12 unit squares. If we do not assign red to one of the 1×2 rectangles, then we can use red on at most 6 of the neighboring unit squares, so we obtain a red area of at most 6. However, if we assign red to one of the 1×2 rectangles, then we cannot use red on the neighboring 1×2 rectangle and unit squares, except at the two corners, where the unit squares only share a point with the red rectangle, and we can also assign red to half the other six unit squares, obtaining a red area of at most 7.

Using the above analysis, we can determine an upper bound on the maximal contribution of the left 8×4 area to the total red-colored area. Suppose that this contribution is 15 (or more). Then we are forced to use a coloring according to the above patterns for the upper and lower 4×4 areas, yielding contributions of 8 and 7, respectively, which sum up to 15. However, it is easy to check that the two patterns cannot be combined into a proper coloring

of the 8×4 area, since we cannot avoid neighboring red unit squares.
We therefore conclude, that the maximal contribution of the left 8×4 area to the total red area is at most 14. See Figure 9 for an example of a partial coloring of the left 8×4 area where the red area is exactly 14. By the same reasoning, the maximal contribution of each of the middle and right 8×4 areas to the red-colored area is also at most 14. It follows that the largest possible red area in a proper coloring is bounded from above by $3 \cdot 14 = 42$.

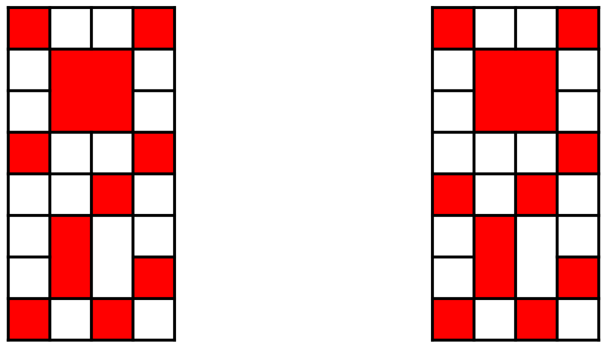


Figure 9: Two examples of partial colorings of the left 8×4 area where the red colored area is exactly 14 (with the top and bottom halves contributing $8 + 6$ to the red area in the left figure, and $7 + 7$ in the right figure).

Finally, it remains to be proven that a red area of 42 is impossible, whereas a red area of 41 can be attained.

Assume there exists a proper coloring with a red area of 42. Then each of the left, middle, and right 8×4 areas must contribute exactly 14 to the red area. In each 8×4 area, which consists of two 4×4 subregions, the contributions must be either $6 + 8$ or $7 + 7$, so that their sum is 14. There are many cases to analyze, but a convenient approach to obtaining a contradiction is to assume that the middle 8×4 area contributes $7 + 7$. Using symmetry and the previously determined forced coloring of the upper middle 4×4 region, it follows that either the upper-left or upper-right 4×4 region cannot contribute more than 6 to the red area. However, a contribution of at least 7 for both regions would be necessary to reach a total of 42. We conclude that the middle 8×4 area must contribute $6 + 8$ to the red area in order to reach 42.
Similarly, considering the possible colorings of the upper middle 4×4 region that contribute 6 to the red area, either the left or right upper 4×4 region cannot contribute more than 7. By symmetry, let us assume that the right upper 4×4 region contributes 7. Note that it cannot contribute 8, since one of the left corners cannot be colored red.
In this case, both of the lower corners of this region have to be red. The same holds for both right corners of the lower middle 4×4 region, since it contributes 8 by assumption. It follows that the lower-right 4×4 region can contribute at most 6 to the red area, yet it must contribute at least 7. This contradiction shows that a red area of 42 cannot occur.

Using the above reasoning and starting with a contribution of $6 + 8$ for the middle 8×4 area, it can be shown that a red area of 41 is possible. For example, this can be achieved with a contribution of $8 + 6$ for the left 8×4 area and a contribution of $7 + 6$ for the right 8×4 area. Figure 10 shows a partial coloring in which the total red area is 41. This partial coloring can easily be extended to a proper coloring by assigning yellow and blue to the remaining uncolored squares and rectangles, as illustrated in Figure 11.

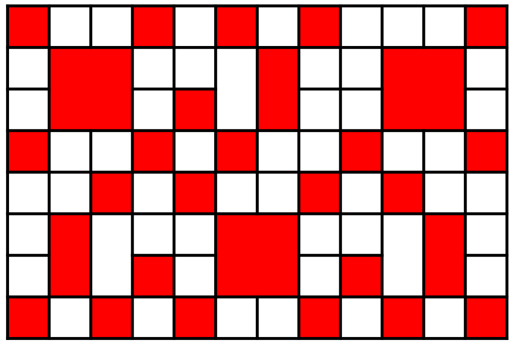


Figure 10: A partial colouring with a total red area of 41. This colouring can be extended to the proper colouring shown in Figure 11.

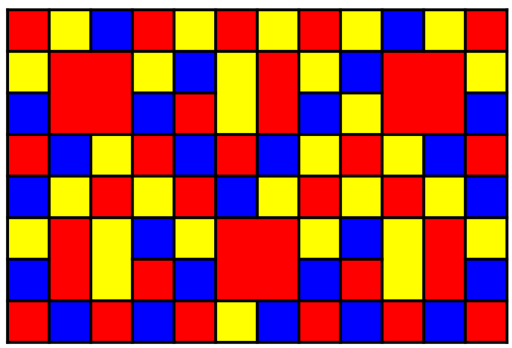


Figure 11: An example of a proper coloring with the largest possible red area of 41.

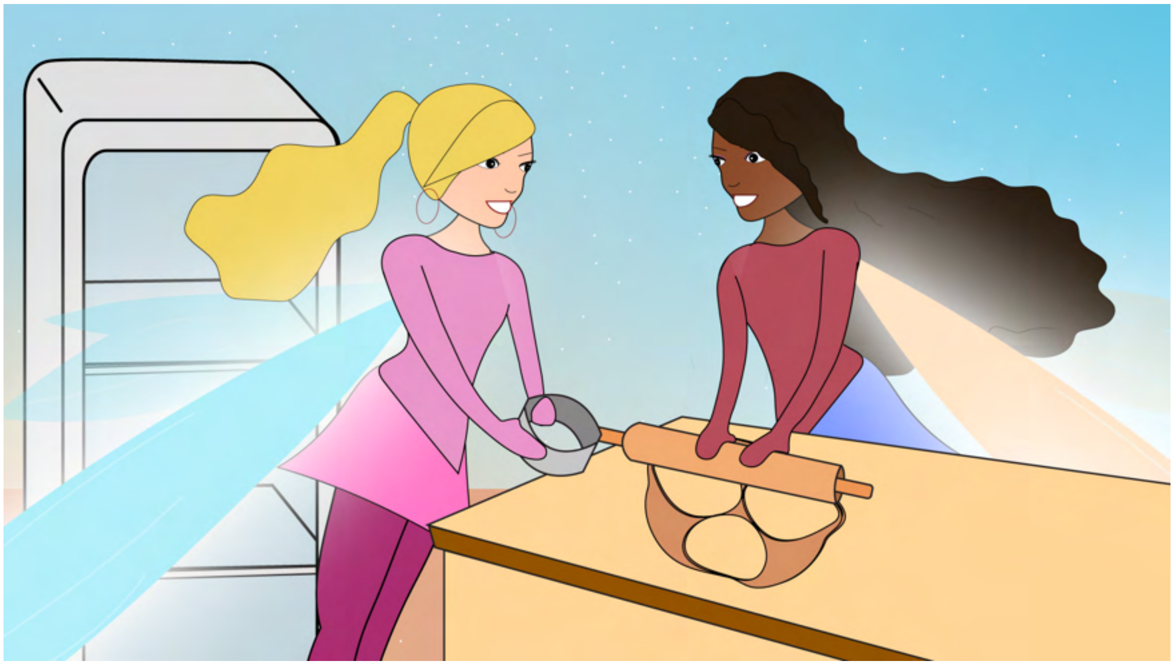


Illustration: Ivana Martić

4 Cutting Christmas Cookies, Again!

Authors: Pim van 't Hof (University of Twente)
Project: 4TU.AMI

Challenge

Pixies Pi and Pie are at it again: just like last year, they are baking perfectly round Christmas cookies. For this reason, they prepared a large circular piece of dough yesterday and put it in the fridge overnight to chill. When they opened the fridge this morning, they discovered that a large circular piece of dough had been cut out. (They suspect pixie Pastry to have pinched that circular piece of dough to use it as a base for her annual Christmas pie.)

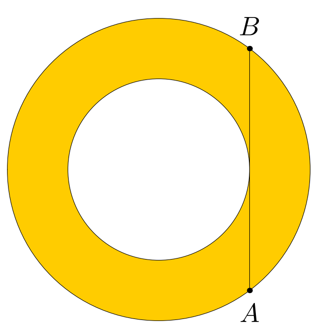


Figure 12: The piece of dough the pixies Pi and Pie found in the fridge.

Figure 12 shows the piece of dough that pixies Pi and Pie found in the fridge this morning.

The boundary of the original piece of dough and the boundary of the large circular hole form concentric circles, i.e. they share the same center but have different radii. The line segment AB is a chord of the large circle that is tangent to the boundary of the circular hole. The length of AB is 36 cm.

Because the pixies Pi and Pie still want perfectly round Christmas cookies but only have two relatively large circular cookie cutters, they decide to knead the remaining piece of dough into a ball and roll it out into a circular piece of dough with exactly the same area as the piece of dough they found in the fridge this morning. From this circular piece of dough, they cut out three cookies: a large one and two smaller ones. Figure 13 shows the piece of dough that remains after the cookies have been put in the oven. The boundary of the large hole goes through the center of the circular piece of dough and touches its boundary. The boundaries of the two (congruent) smaller holes touch each other, the boundary of the large hole, and the boundary of the circular piece of dough.

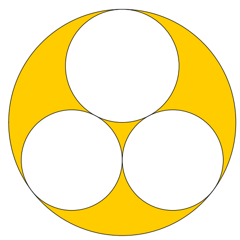


Figure 13: The remaining piece of dough is shown in yellow. The white circles represent the holes.

What is the area of the remaining piece of dough after the three circular cookies have been cut out?

Possible Answers:

1. 112π cm^2
2. 115π cm^2
3. 118π cm^2
4. 120π cm^2
5. 125π cm^2
6. 128π cm^2
7. 130π cm^2
8. 132π cm^2
9. 136π cm^2
10. There is not enough information in the problem statement to compute the area of the remaining piece of dough.

Solution

The correct answer is: 2.

We first need to determine the area of the circular piece of dough from which pixies Pi and Pie cut out the three cookies, which is the same as the area of the piece of dough they found in the fridge this morning (depicted in Figure 12). Let r and R be the radii of the inner and outer circle in Figure 12, respectively. The area X of the piece of dough is then given by

$$X = \pi R^2 - \pi r^2 \,. \tag{1}$$

Now let O be the center of the two concentric circles, and let M be the point where the chord AB touches the inner circle; see Figure 14.

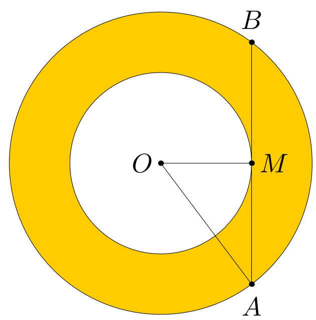


Figure 14: The area of the (yellow) piece of dough is 324π cm^2.

Since the two circles are concentric and line segment AB is tangent to the inner circle, we know that $AM = BM = 18$ and that OM is perpendicular to AB. By the Pythagorean Theorem, we have that

$$R^2 = (OA)^2 = (OM)^2 + (AM)^2 = r^2 + 18^2 = r^2 + 324 \,. \tag{2}$$

Combining Equations (1) and (2) gives us the area X of the piece of dough:

$$X = 324\pi \,.$$

Now we need to determine how much of this area remains after the three cookies have been cut out. From the area X, we can compute that the radius of the circular piece of dough from which the cookies are cut out is 18. Since the boundary of the largest hole goes through the center of this circular piece of dough and touches its boundary, the large hole has radius 9. It remains to show that the two smaller holes have radius 8, since that implies that the area of the remaining piece of dough is

$$324\pi - 81\pi - 2 \cdot 64\pi = 115\pi \,.$$

Showing that the small holes have radius 8 can be done in several ways. We give a proof that only relies on the Pythagorean Theorem and the fact that if any two circles touch, then the centers of the two circles and the point of contact are collinear, i.e., lie on a single line.
Let us consider the right half of the remaining piece of dough; see Figure 15. Let C, D and E be the centers of the circular piece of dough, the large hole, and the small hole on the bottom

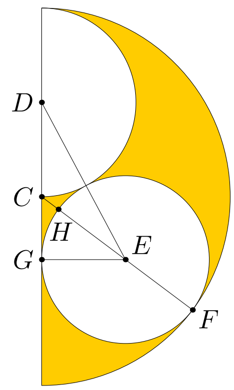


Figure 15: The right half of the remaining piece of dough.

right, respectively. Let F and G be the points where the boundary of the small hole touches the boundary of the piece of dough and the diameter of the piece of dough that goes through C and D, respectively. Finally, let H be the point where CE intersects the boundary of the small hole.

Since C and E are the centers of two circles that touch each other in point F, we know that C, E and F lie on a single line. Since CF is a radius of the circular piece of dough, the length of CF is 18. Since EF and EH are radii of the small hole, we have that $CH = 18 - 2x$, where x is the radius of the small hole. Now consider triangle CEG. By the Pythagorean Theorem, we have that

$$(CG)^2 = (CE)^2 - (EG)^2 = ((18 - 2x) + x)^2 - x^2 = 324 - 36x\,. \tag{3}$$

If we apply the Pythagorean Theorem to triangle DEG, we find that

$$(DG)^2 = (DE)^2 - (EG)^2 = (9 + x)^2 - x^2 = 81 + 18x\,. \tag{4}$$

Combining Equations (3) and (4) with the fact that $DG = 9 + CG$ yields

$$\sqrt{81 + 18x} = 9 + \sqrt{324 - 36x}\,.$$

If we solve this equation for x, we find that $x = 8$. This completes the proof.

Illustration: Julia Nurit Schönnagel

5 Sporty Lunch Break

Author: Thomas Nowotka (MATH+)
Project: MATH+

Challenge

Even at the North Pole, it is now recognized that physical activity is extremely beneficial for well-being and concentration. Therefore, it is mandatory for all employees of Santa's workshop to participate in a shared movement game during the lunch break.
Today, the head elf Alvie is responsible for organizing the break. He lines up in a circle with all the elves who are on shift today. At the beginning of the game, Alvie holds a magical ball made of stardust in his hand. This ball is now thrown back and forth over several rounds according to the following fixed rules:

1. The k-th round begins with Alvie passing the ball to the k-th colleague to his left.
2. In the k-th round, each elf who receives the ball throws it to the k-th elf to their left.
3. A round ends as soon as the ball has returned to Alvie.

Figure 16 shows, as an example, how the second and third rounds proceed when 8 elves are at work.

Alvie wants to pass the ball to each elf exactly once, therefore as many rounds are played as there are elves present excluding Alvie. Since the elves have a strong sense of justice, they cheer at the end of a round if every elf was able to throw the ball during that round. Figure

16 illustrates how the third round proceeds when 8 elves are at work. One can see that each elf gets to throw the ball once, so at the end of the third round they cheer.

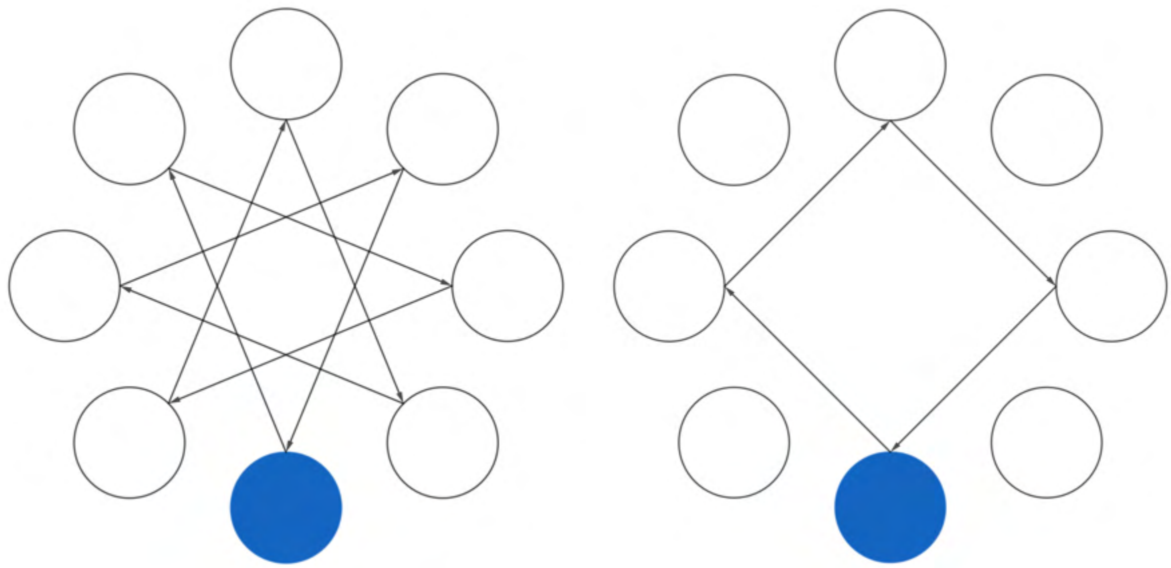


Figure 16: The third and the second round when 8 elves (circles) are at work. Throws are depicted as arrows. Alvie is shown in blue.

At the end of the day, when Alvie has to write a report for Santa, he realizes that he does not exactly remember how many elves, including himself, worked in the workshop today.
After much thought, he can narrow it down to ten possibilities (see answer choices). He also remembers that he heard 80 cheers during the lunch break today.
Can you tell Alvie how many elves (including him) were at work today?

Remark: It can be shown that, in every round of the game, the ball returns to Alvie after a finite number of throws. You may assume this for this problem.

Hint: Every positive integer greater than one can be written as a unique product of prime numbers (unique up to the order of the factors). This product is called prime factorization and may be helpful in solving this problem.

(*The possible answers are on the next page.*)

Possible Answers:

1. 148
2. 151
3. 152
4. 157
5. 164
6. 166
7. 173
8. 179
9. 181
10. 184

Solution

The correct answer is: 5.

Let n be the number of elves who were at work today.
Then $n-1$ rounds were played during today's lunch break.

We now consider that in the k-th round cheering occurs exactly when k and n are coprime.

To this end, we first assume that k and n are not coprime and show that in this case no cheering occurs after round k.
By the definition of coprimality, there exists a common divisor $d > 1$. Now we color, starting from Alvie and proceeding clockwise, every d-th elf red until we return to Alvie. Since $d \mid n$, there are $\frac{n}{d}$ red elves.[1] Since $d > 1$, these are not all elves, i.e., there exist elves that were not colored (see Fig. 17).
If the ball is thrown by Alvie in round k according to Rule 2, then because $d \mid k$ it always lands only with red-colored elves. This means that in this round not every elf is allowed to throw the ball and therefore no cheering occurs.

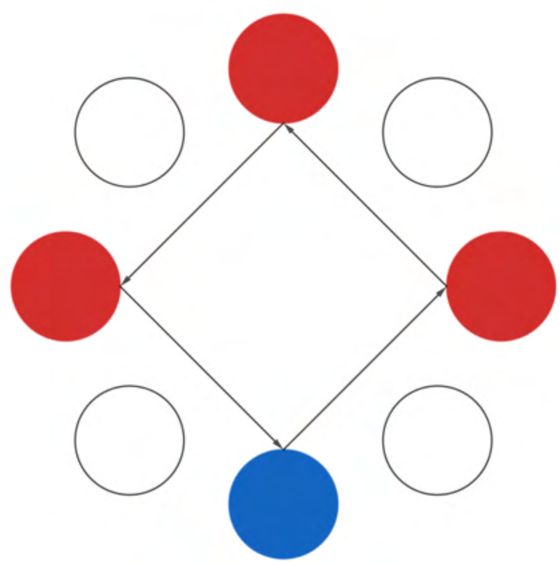


Figure 17: Round $k = 6$, when $n = 8$ elves (circles) are at work. Throws are depicted as arrows. Starting from Alvie (blue), every 2nd elf is colored red counterclockwise.

Now we consider the case that k and n are coprime and show that in this case cheering occurs after round k. For this purpose, we introduce the following notation: For a positive integer l, we say that the ball has passed l elves if an elf passes the ball to the l-th elf to his left.

For every positive integer m it holds: in the k-th round, the ball is again with Alvie after m throws following Rule 2 if and only if the ball has passed the complete circle of n elves once

[1] $d \mid n$ ("d divides n") means that d is a divisor of n.

or several times, i.e., if $n \mid k \cdot m$ holds. Due to the coprimality of n and k, this is exactly the case when $n \mid m$ holds: indeed, if one writes $\frac{km}{n}$ as a fraction, common factors of km and n can be canceled. Since k and n are coprime, cancellation is only possible between m and n. The statement $n \mid k \cdot m$ means that the fraction can be reduced such that the denominator equals 1. Since only m can be canceled, it follows that this is exactly possible when $n \mid m$ holds. Consequently, the ball is for the first time again with Alvie after n throws, i.e., the round consists of exactly n throws.
Now we show by contradiction that no two throws can be made by the same elf. If that were the case, there would exist a sequence of throws that leads from one elf back to himself without Alvie receiving the ball. How would the game continue? The situations in which the elf receives the ball for the first and the second time lead to exactly the same sequence of subsequent throws. Therefore, this elf would always receive the ball again and the ball would never return to Alvie, which is a contradiction to the remark in the problem statement. Since no two of the n throws are carried out by the same elf and n elves participate, each present elf throws the ball once and cheering occurs at the end of the round.

Thus, today cheering occurred exactly as often as there are positive integers smaller than n that are coprime to n. This number is denoted by $\varphi(n)$.
We therefore want to determine the number n from the possible answers for which $\varphi(n) = 80$ holds.

If p is a prime number, then every positive integer smaller than p is coprime to p, i.e., we obtain

$$\begin{aligned}
&\varphi(151) = 150 \neq 80,\\
&\varphi(157) = 156 \neq 80,\\
&\varphi(173) = 172 \neq 80,\\
&\varphi(179) = 178 \neq 80, \text{ and}\\
&\varphi(181) = 180 \neq 80.
\end{aligned}$$

Thus, the remaining possibilities are $148 = 2^2 \cdot 37,\ 152 = 2^3 \cdot 19,\ 164 = 2^2 \cdot 41,\ 166 = 2 \cdot 83$ and $184 = 2^3 \cdot 23$, whose prime factorizations are all of the form $2^q \cdot p$ with a positive integer q and a prime number $p \neq 2$.
Therefore, we now determine $\varphi(2^q p)$: a positive integer m smaller than $2^q p$ is not coprime to $2^q p$ exactly when m and $2^q p$ share a common divisor. This is exactly the case when $2 \mid m$ or $p \mid m$ holds. The justification for this is similar to above with the reduction of fractions.
There are $\frac{2^q p}{2} - 1 = 2^{q-1}p - 1$ positive integers smaller than $2^q p$ that are divisible by 2, and $\frac{2^q p}{p} - 1 = 2^q - 1$ positive integers smaller than $2^q p$ that are divisible by p. There are $\frac{2^q p}{2p} - 1 = 2^{q-1} - 1$ positive integers smaller than $2^q p$ that are divisible by both p and 2.
Adding the first two quantities and subtracting the third to avoid double counting, we obtain the number of positive integers smaller than $2^q p$ that are divisible by 2 or p, and hence not coprime to $2^q p$. This number equals

$$(2^{q-1}p - 1) + (2^q - 1) - (2^{q-1} - 1) = 2^{q-1}p - 1 + 2^{q-1}.$$

Subtracting this quantity from the number of positive integers smaller than $2^q p$, we obtain

$\varphi(2^q p)$. Thus,

$$\varphi(2^q p) = (2^q p - 1) - (2^{q-1} p - 1 - 2^{q-1}) = 2^{q-1} p - 2^{q-1} = 2^{q-1}(p-1).$$

It follows that

$$\begin{aligned}
\varphi(148) &= \varphi(2^2 \cdot 37) = 2^1 \cdot 36 = 72 \neq 80, \\
\varphi(152) &= \varphi(2^3 \cdot 19) = 2^2 \cdot 18 = 72 \neq 80, \\
\varphi(166) &= \varphi(2^1 \cdot 83) = 2^0 \cdot 82 = 82 \neq 80, \\
\varphi(184) &= \varphi(2^3 \cdot 23) = 2^2 \cdot 22 = 88 \neq 80, \text{ and} \\
\varphi(164) &= \varphi(2^2 \cdot 41) = 2^1 \cdot 40 = 80.
\end{aligned}$$

Thus, today 164 elves were at work.

Illustration: Friederike Hofmann

6 Random Saint Nicholas House Drawing

Author: Silas Rathke (FU Berlin)

Challenge

In the bar "At the Dripping Icicle", Saint Nicholas is indulging in his favorite pastime: he talks incessantly about his house. What an architectural masterpiece it is! In its elegant simplicity, it combines the defining elements of Neo-Romanesque style with Scheldt Gothic! And despite this intellectual brilliance, every child still knows exactly what it looks like! Full of pride, he finally presents a drawing of his beloved house to his bored audience (see Fig. 18).

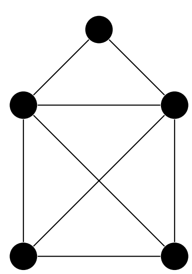


Figure 18: The House of Saint Nicholas consisting of five circles and eight connecting lines.

"And the best is yet to come!" he informs his unwilling audience. "Once you have drawn the five black circles, you can draw all eight lines between the circles without lifting your pen and without drawing any line twice!"

At this point, the mathematician elf Uhur has had enough and replies with one of his favorite sentences: “Prove it!”

Saint Nicholas freezes. After all, during his long-winded speech he has drunk one mulled wine or another, and now he simply cannot remember how to draw his house according to the rule mentioned above. He has no choice but to draw at random and hope for the best.
He pulls a thick red felt-tip pen out of his sack and decides to trace over the thin black lines in the drawing of the House of Saint Nicholas with the felt-tip pen.
To do this, he chooses one of the five circles uniformly at random[2] as a starting point and places his pen there. From then on, he always draws the next line in the following way: he looks at where his pen currently is, checks which lines in the black drawing have this circle as an endpoint and have not yet been traced. From these, he chooses one uniformly at random and independently of his previous decisions, and traces it with the felt-tip pen. He continues in this manner until his pen reaches a point where all lines have already been traced in red.

Let p be the probability that he has then actually traced the complete Saint Nicholas House. What is the 100th digit after the decimal point in the decimal expansion of p?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

[2] In this case, uniformly at random means that all circles are chosen with the same probability.

Solution

The correct answer is: 9.

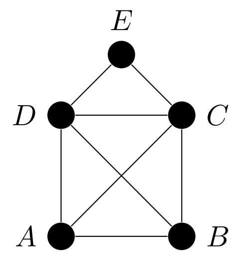


Figure 19: The Saint Nicholas House with circles A to E.

If a line has a circle as an endpoint, we say that the line is *incident* to that circle. The *degree* of a circle is the number of edges incident to it. In the Saint Nicholas House, circle A for example has degree 3.

To solve this problem, we can ask Uhur's mathematician friend: Euler. According to his theorem, one can draw a figure such as the Saint Nicholas House without lifting the pencil if and only if the figure is connected and at most two circles have an odd degree. If exactly two circles have odd degree, then these are precisely the circles where the pencil must start and end.
In the Saint Nicholas House, exactly A and B have odd degree. If he starts at A, then according to Euler's theorem his pencil must end at B. Why is that? He cannot end at A again, because whenever the pencil is at A, an even number of incident lines have already been drawn. Thus, there is always at least one line left through which the pencil can leave A again. Whenever the pencil is at another circle K different from A, it has already drawn an odd number of incident lines to K there. This means that in the end the pencil can only be at circle B, because it is the only circle with an odd number of incident lines.

If Saint Nicholas does not choose A or B at the beginning, he will not be able to draw the complete house in any case. This happens with probability $\frac{3}{5}$.
Now we must compute the probability that he does draw the complete house if he starts at A or B. Since the house is symmetric, it suffices to consider only the case where he starts at A.

Let p_A be the probability that Saint Nicholas draws the entire house under the condition that he started at A. We will first compute the complementary probability $\overline{p_A}$, i.e., the probability that Nikolaus <u>does not</u> draw the complete house although he started at A. We already know that his drawing must end at B. This implies that also in his drawing, A and B must have odd degree, while C, D and E must have even degree. Furthermore, B must have degree 3, for otherwise one could continue drawing. Let us now consider the lines of the Saint Nicholas House that are not incident to B:
To construct an example where the entire house is not drawn, we must select some of these lines to omit from the drawing. We select them in such a way that at each circle an even number of incident lines is chosen. It is easy to see that this is possible in exactly three ways (cf. Figure 21).

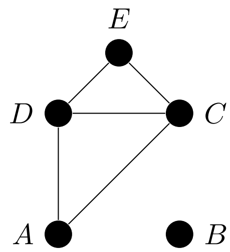


Figure 20: Lines of the Saint Nicholas House that are not incident to B.

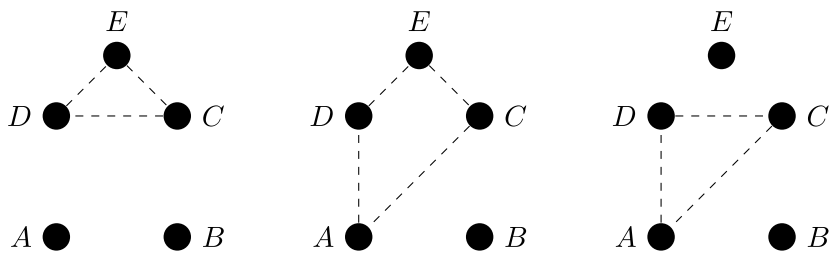


Figure 21: Clever selection of lines from Figure 20.

These three cases correspond to three possible final states in which the complete house is not drawn (cf. Figure 22).

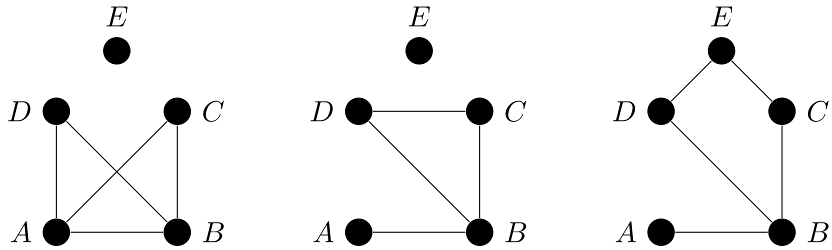


Figure 22: If Saint Nicholas starts at A, these are the only final states in which the entire house is not drawn.

Now we must compute the probability that these figures are drawn when starting at A. We do this in the following table:

Order of Circles	**Probability**	=
A, B, C, A, D, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$
A, B, D, A, C, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$
A, C, B, A, D, B	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$
A, C, B, D, A, B	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1$	$\frac{1}{54}$
A, D, B, A, C, B	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$
A, D, B, C, A, B	$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1$	$\frac{1}{54}$
A, B, C, D, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{54}$
A, B, D, C, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{54}$
A, B, C, E, D, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$
A, B, D, E, C, B	$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 \cdot \frac{1}{3}$	$\frac{1}{54}$

Thus, we see that $\overline{p_A} = 10 \cdot \frac{1}{54} = \frac{5}{27}$. Hence,

$$p_A = 1 - \overline{p_A} = \frac{22}{27}.$$

He has the same probability of success when starting at B. Since only circles A and B out of the 5 circles can lead to a successful drawing, the overall probability of success is

$$p = \frac{2}{5} \cdot \frac{22}{27} = \frac{44}{135} = 0.3\overline{259}.$$

Thus, the 100th digit after the decimal point is 9.

Hopefully, after this complicated calculation, Saint Nicholas will think twice next time about whom he bores with his ramblings...

Alternative solution:

In the solution above, we already observed that one can draw the Saint Nicholas House completely without lifting the pencil only if one starts at A or B in Figure 19. There are 44 possible ways to draw the Saint Nicholas House when starting at A. These are shown in Figure 23.

We first want to compute the probability for each of these 44 (possible) drawings when starting at A. For this we consider, for each circle, the lines that still must be drawn leaving that circle:

- Starting at A, one chooses one of the three incident lines with probability $\frac{1}{3}$. When the drawing returns to A via one of the remaining two lines, only one line will remain, which will be chosen with probability 1.
- Circle B is first reached via one incident line. Then one of the remaining two incident lines is chosen with probability $\frac{1}{2}$.
- Circle C is first reached via one incident line. Then one of the remaining three incident lines is chosen with probability $\frac{1}{3}$. Later, C is reached again, and the last remaining incident line is chosen with probability 1.
- For circle D, the situation is the same as for C: one line chosen with probability $\frac{1}{3}$, the other with probability 1.
- Circle E is reached via one incident line. Then only one remaining incident line is left, which is chosen with probability 1.

Each of the 44 drawings from Figure 23 is obtained with the following probability:

$$\underbrace{\frac{1}{3} \cdot 1}_{\text{from } A} \cdot \underbrace{\frac{1}{2}}_{\text{from } B} \cdot \underbrace{\frac{1}{3} \cdot 1}_{\text{from } C} \cdot \underbrace{\frac{1}{3} \cdot 1}_{\text{from } D} \cdot \underbrace{1}_{\text{from } E} = \frac{1}{54}.$$

Since the house is symmetric, there are also 44 possibilities possible drawings when starting at B.
We start our drawing with probability $\frac{2}{5}$ at either A or B. Thus the desired probability is:

$$p = \frac{2}{5} \cdot 88 \cdot \frac{1}{54} = 0.3\overline{259}.$$

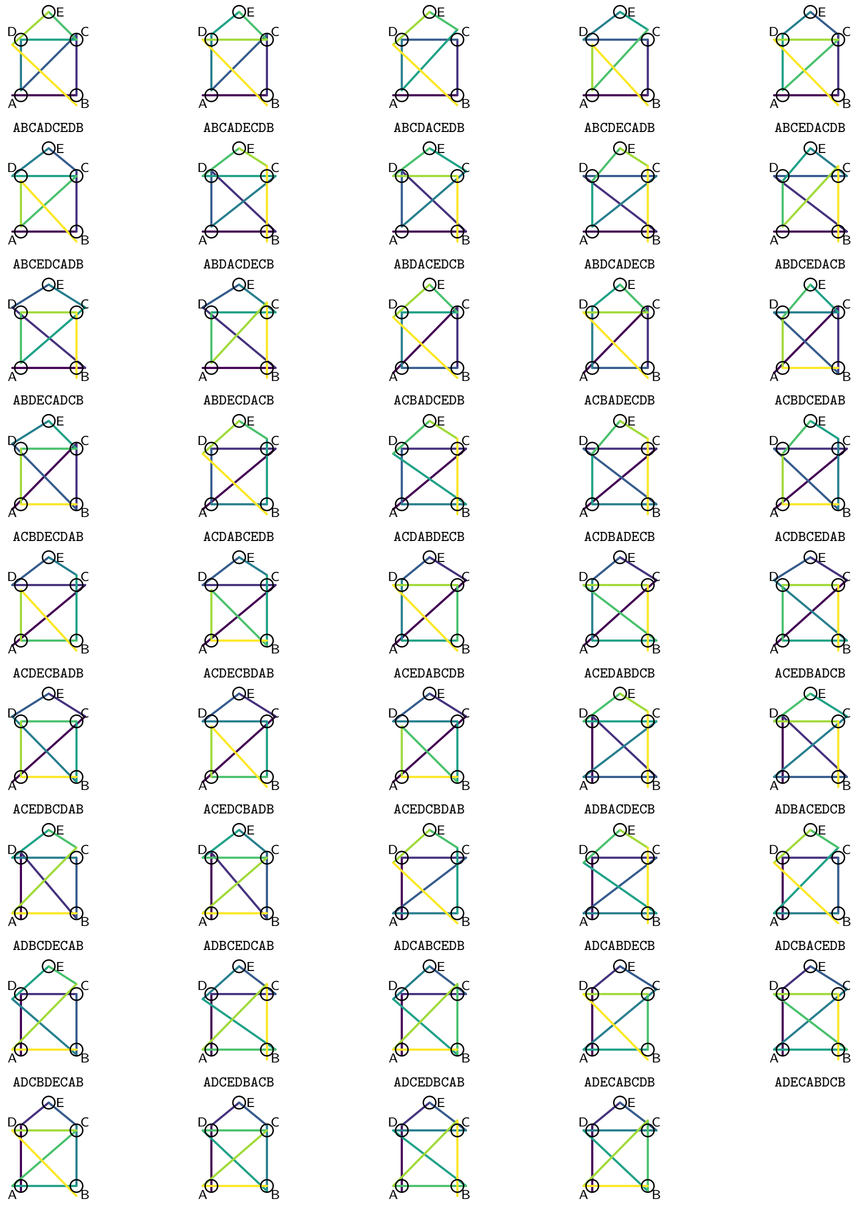


Figure 23: The 44 possibilities of drawing the Saint Nicholas House when starting at the lower left corner.

Illustration: Ivana Martić

7 The Wrapping Room Riddle

Author: Nikola Sadovek

Challenge

In Santa Claus's wrapping room, the elves are rushing to finish wrapping the last presents in time for Christmas Eve. But there is a problem – **the wrapping-paper machine is broken!**

Instead of cutting all possible shapes of wrapping paper, the machine can only cut the paper into **five special shapes** (see Figure 24). Each shape consists of so-called identical unit elf squares (all small squares have exactly the same size), and the machine can **neither rotate nor flip** them – every piece must be used exactly as it is shown in Figure 24.

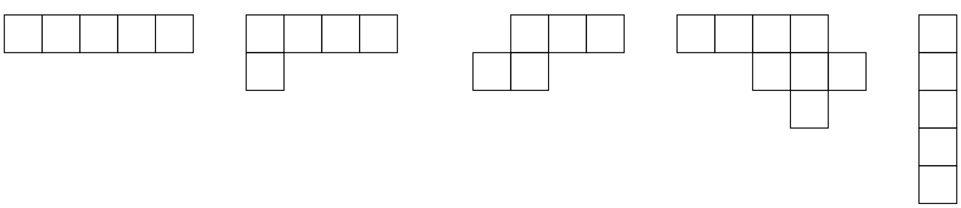


Figure 24: The five special shapes.

Fortunately, each of these pieces can still be used for wrapping certain presents:

- **Long strip** – used to wrap toy trains.

- **L-shape** – fits around boxes with corners.
- **Small step shape** – covers stacked books.
- **Water pistol** – for the water pistols in the gift boxes.
- **Straight column** – for candy canes.

The elves now need to cut some **rectangular pieces of size** $m \times n$ with the machine, where $m, n \geq 1$. Here, m is the number of rows and n is the number of columns of the rectangular piece.
They want to determine **for which pair** (m, n), listed in the possible answers below, the entire rectangle can be filled **using only these five shapes**, **with no leftover space**.
Can you help them figure it out before Santa leaves for his midnight journey?

Possible Answers:

1. $m = 2021,\ n = 2024$
2. $m = 2021,\ n = 2029$
3. $m = 2022,\ n = 2026$
4. $m = 2022,\ n = 2028$
5. $m = 2023,\ n = 2024$
6. $m = 2023,\ n = 2027$
7. $m = 2024,\ n = 2026$
8. $m = 2024,\ n = 2027$
9. $m = 2024,\ n = 2029$
10. None of the other answers are true.

Solution

The correct answer is: 10.

We will show that the solutions are precisely those pairs (m, n) for which 5 divides mn.

We start by showing that if 5 divides mn, then we can fill the rectangle of dimension $m \times n$ with only long strips or only straight columns.
Now, if 5 divides nm it must divide either n or m. To see this, consider the following to cases:

1. 5 divides m: In this case, the statement is obviously true.

2. 5 does not divide m: In this case m and 5 do not have any common (positive) divisors other than 1. However, the fraction $\frac{mn}{5}$ can be reduced to a whole number. Since m and 5 cannot be reduced, n and 5 must be reducible. Therefore, n divides 5 and the statement is true.

If case 1 is correct, we can fill each column with $m/5$ straight columns, thereby filling the entire rectangle. Similarly, if case 2 is correct, we can fill each row with $n/5$ long strips, thereby filling the entire rectangle.

Now, we show that if an $m \times n$ rectangle can be filled using only the five given shapes, then 5 divides mn. Divide the rectangle into mn equally sized squares. Then, assign values to the squares according to figure 25. If we introduce coordinates to the squares with the lower left square corresponding to $(1, 1)$ and the upper right square corresponding to (m, n), then the value of the square with arbitrary coordinates (i, j) is $2^{i+j-2} \bmod 31$. Here, the operation $\bmod 31$ refers to the remainder of the division of 2^{i+j-2} by 31.

1	2	4	8	16	1	2	4
16	1	2	4	8	16	1	2
8	16	1	2	4	8	16	1
4	8	16	1	2	4	8	16
2	4	8	16	1	2	4	8
1	2	4	8	16	1	2	4

Figure 25: Assigning values to the squares of the rectangle. The numbers $1, 2, 4, 8, 16$ repeat periodically.

If we now place any of the 4 shapes covering 5 squares on the rectangle, it will always cover 5 squares with different values. This can be easily verified by hand, since by the periodicity of the values there are only very few cases to consider. Thus, the sum of these values is always 31. For the shape covering 8 squares the situation is a little different. Here, the values of the covered squares always sum to a multiple of 31, as can also be easily verified by considering all possible cases. This shows that the sum of the values of the covered squares by any of the given shapes is always divisible by 31. Therefore, a filling of an $m \times n$ rectangle can only exist, if the sum of all values is divisible by 31. We will use this fact to prove our claim.

Because the sum of the numbers $1, 2, 4, 8, 16$ is 31 and therefore divisible by 31, we can deduce that the sum of the values of 5 successive columns is divisible by 31, as the numbers $1, 2, 4, 8, 16$ appear exactly once in each row. Similarly, the sum of the values of 5 successive rows is also divisible by 31. In this way, we can reduce the size of the rectangle by first removing each run of 5 successive columns and afterwards removing each run of 5 successive rows of the rectangle without changing the remainder of the sum of all leftover values when divided by 31 (cf. figure 26). The reduced rectangle has dimensions $k \times l$, with $k, l < 5$. If we can show that either k or l are equal to 0, then, because we removed a multiple of 5 rows and a multiple of 5 columns, the number of rows or columns must be divisible by 5, proving the claim.

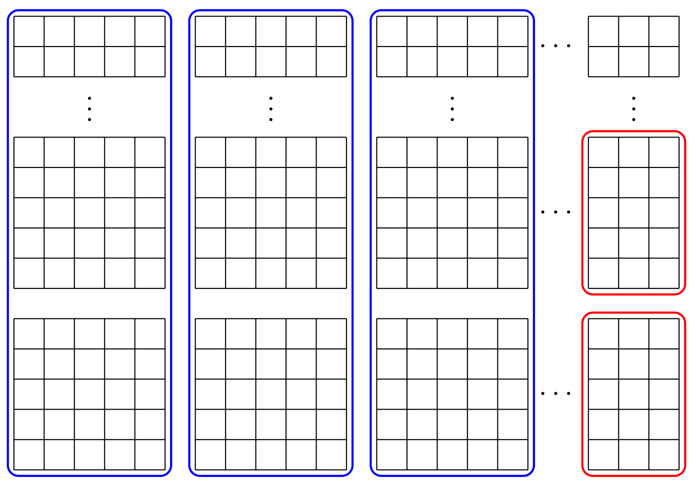


Figure 26: Reducing the dimensions of the rectangle. First all runs of 5 successive columns (blue) are removed. Then all runs of 5 successive rows (red) are removed.

Checking the cases:

1. $k = 1$ **or** $l = 1$: This case corresponds to having one row or one column with at most 4 squares. The values in these squares must all be different and at least one value is missing. Therefore, the sum of these values is less than 31. The only option for the rectangle to be divisible by 31 is, if the values sum to 0, i.e. $k = 0$ or $l = 0$.

2. $k = l = 2$: By construction, there are essentially 3 possibilities, all of which do not lead to a sum divisible by 31, ruling out this case. Figure 27 illustrates this.

$2x$	$4x$
x	$2x$

16	1
8	16

1	2
16	1

Figure 27: The sum of the values in each case is not divisible by 31. Especially for the first case, $9x$ is not divisible by 31, as x is not divisible by 31.

3. $k = 2, l = 3$ **(or** $k = 3, l = 2$**)**: This case can be ruled out by extending the 2×3 by a 2×2 rectangle to a 2×5 rectangle. By our earlier arguments, 31 divides the sum of the values of the 2×5 rectangle. Thus, if the sum of values of the 2×3 rectangle was divisible by 31 so is the sum of values of the 2×2 rectangle, which is a contradiction to the previous case. Hence, this case can also be ruled out. (Similarly we can rule out $k = 3, l = 2$.)

4. **All other cases**: The argument above can be applied more generally: By extending a $k \times l$ rectangle with a $(5 - k) \times l$ or a $k \times (5 - l)$ rectangle we get a rectangle with either 5 rows or 5 columns, whose values sum to a number divisible by 31. Thus, the values of the $k \times l$ sum to number divisible by 31 if and only if the sum of the values of the $(5 - k) \times l$ and the sum of the values of the $k \times (5 - l)$ rectangle is divisible by 31. In this way we can reduce all other cases to the previously considered cases.

Checking all the cases, we deduce that either $k = 0$ or $l = 0$, finishing the proof.

Illustration: Zyanya Santuario

8 Save The Presents!

Author: Lukas Protz
Project: MATH+

Challenge

When the security elf realized what had happened, he was terrified. The Grinch had managed to sneak into the vault room, where all the Christmas presents are stored. Not only that, he had cracked the code to open the vault, changed it, and reprogrammed it in such a way that everything inside would be destroyed if incorrect codes were entered too many times.
The security elf begged the Grinch to give him the correct code, but the Grinch viciously replied: "Gather 100 elves and take them to the vault, then we can play a little game and maybe you can save your presents." The security elf immediately followed the Grinch's instructions and returned to the vault with 100 elves.

"The rules are the following," the Grinch says. "The code for the vault consists of 100 non-repeating characters. In total, there are 199 characters to choose from." The Grinch points towards a panel with 199 buttons, each labeled with exactly one character. "There is also a screen that displays the previously entered code. It highlights characters that are in the correct position in green and characters that appear in the code but are not correctly positioned in blue.
Each of you only gets one chance to open the vault. Moreover, during the game you are not allowed to communicate with the other elves in any way. To make things more difficult, I will make sure that only the elf whose turn it is, can see the screen. During your turn, the screen

only shows the characters being entered — with no indication whether they are correct or not. Only after you finish entering all 100 characters and step away will the vault evaluate your attempt. The screen will then highlight the characters for the next elf to see. As soon as the correct code is entered, the vault opens.
If no one of you enters the correct code, all presents will be automatically destroyed. You get 10 elf-minutes (the standard time unit at the North Pole) to come up with a strategy."

After quite some time of thinking, one elf comes up with an idea: "Let us make a list of all characters. The first elf will try the first 100 characters on the list. For each subsequent try we do the following: If a character is highlighted green, it remains in the same position. If a character is highlighted blue, it will be placed in the next free position to the right in the code. A free position is a position, where no green character was placed yet. A blue highlighted character at the end of the code might not have a free position to the right. In this case, the character is placed in the first free spot in the code, starting from the beginning.
The remaining positions will be filled with the next unused characters on the list. Each of these characters is inserted into the first still free position in the code, scanning from left to right." Running out of time, the elves agree on this strategy and start playing the game.

Hopefully, the elves can save their presents! Let m be the minimal number of elves that would be needed to guarantee that the elves crack the code with their strategy. What is the units digit of m?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Solution

The correct answer is: 10.

We will answer the question more generally: Let n be the total number of characters and k the length of the code. The process of finding the code remains the same. Furthermore, let $m_{n,k}$ denote the minimal number of tries needed to guarantee that the vault opens. We will show, that if $k \geq n - k + 1$, then $m_{n,k} = k$. This is sufficient for our problem, since $n = 199$ and $k = 100$ satisfy $k = 100 \geq 100 = n - k + 1$. In fact $m_{199,100} = 100$.

Before we begin the proof, we introduce the following notion: By a *shift* we mean the repositioning of a correctly chosen, but wrongly positioned character to the position directly to its right or, if there is no such position, to the first position of the code. Note that the shift of a character is not necessarily allowed by the rules, but this notation will still be useful.
To begin with, here are some easier observations:

Observation 1: After at most $n - k + 1$ attempts, all the correct characters are determined. After try one, k characters have been checked. In each subsequent try, at least one more character is checked if not all correct characters have been determined yet. Thus, after $n - k + 1$ tries at least $1 \cdot k + (n - k) \cdot 1 = n$ characters have been checked.

Observation 2: $m_{n,k} \leq n$. By observation 1, after at most $n - k + 1$ tries all characters have been found. Then, after at most $k - 1$ more tries, every character is correctly placed. This is because each incorrectly placed character is repositioned to a new position where it has never been placed before. Since there are only k positions, each character has to be in the correct position after at most $k - 1$ shifts. Hence, $m_{n,k} \leq n - k + 1 + k - 1 = n$.
Note that this upper bound cannot be improved in this generality, as $m_{n,n} = n$. This can be seen as follows: After the first try, all correct characters are determined. There is one scenario in which all characters are exactly one position to the right of their correct position, except for the character in the first position. Its correct position is the last position in the code. It then takes $n - 1$ shifts to place every character correctly.

Now we can start the proof of the claim:

Consider some scenario of the game. There are non-negative integers $a, l_1, l_2, \ldots, l_a$ such that in try i exactly l_i correct characters are found and in try $a+1$ all remaining correct characters are found. Since we assume $k \geq n - k + 1$, it follows from observation 1 that after k tries, all correct characters have been identified. Furthermore, after k tries, at least the l_1 characters found in the first try have been placed correctly. The characters found in the second try need at most one more shift, the characters found in the third try at most two more shifts, and generally the characters found in the i-th try need at most $i - 1$ shifts.

The key observation now is that in the next try, each incorrectly positioned character must be shifted a certain number of times until it reaches a position currently occupied by another incorrectly positioned character. That character, in turn, is shifted to the position of yet another incorrectly positioned character, and so on. Continuing this process, the remaining incorrectly placed characters form a cycle. This means that the sum of all of the individual

shifts is k.
We will show that after the k-th try, the sum of individual shifts to correctly position each incorrectly positioned character is strictly less than k. Subsequently, we then conclude that after try k, all characters are correctly placed.

Because of the previous consideration, the total number S of shifts still needed after the k-th try is bounded by

$$S \leq l_1 \cdot 0 + l_2 \cdot 1 + l_3 \cdot 2 + \ldots + l_a \cdot (a-1) + (k - l_1 - l_2 - \ldots - l_a) \cdot a.$$

where the last term corresponds to the remaining correct characters that need to be found, multiplied by their maximum number of shifts needed after the k-th try.

Now we want want to reduce the general case to more specific ones. Let $a, l_1, \ldots, l_a$ be as before. Instead of finding all the remaining correct characters in try $a+1$, we now want to look at the scenario, where we find all the remaining correct characters except the last one. The last one is found in a later try, say try b. This only affects the last term of the upper bound on S, which is replaced by

$$(k - l_1 - l_2 - \ldots - l_a - 1) \cdot a + 1 \cdot (b-1).$$

This term is now larger than before, since $b-1$ is greater than a. This raises the question, how large b can become:
k characters are getting checked in the first attempt, $k - l_1$ in the second. After continuing in this way, we see that $k - l_1 - l_2 - \cdots - l_{i-1}$ characters are getting checked in try i for $i \leq a$. After try $a+1$, a total of

$$k + (k - l_1) + (k - l_1 - l_2) + \ldots + (k - l_1 - l_2 - \ldots - l_a) = (a+1) \cdot k - a \cdot l_1 - \ldots - 1 \cdot l_a$$

characters were checked. In each subsequent try exactly one more character is checked until the last correct character is found. Since there are n characters in total, one needs a maximum of

$$n - (a+1) \cdot k + a \cdot l_1 + \ldots + 1 \cdot l_a$$

further tries to determine the last correct character Putting things together, we can conclude that

$$b \leq a + 1 + n - (a+1) \cdot k + a \cdot l_1 + \ldots + 1 \cdot l_a.$$

Finally, we can compute

$$\begin{aligned} S \leq \quad & l_1 \cdot 0 + l_2 \cdot 1 + l_3 \cdot 2 + \ldots + l_a \cdot (a-1) + (k - l_1 - l_2 - \ldots - l_a - 1) \cdot a \\ & + 1 \cdot (a + 1 + n - (a+1) \cdot k + a \cdot l_1 + \ldots + 1 \cdot l_a - 1) \\ = \quad & (k-1) \cdot a + a + 1 + n - (a+1) \cdot k - 1 \\ = \quad & n - k, \end{aligned}$$

since the l_i-terms cancel in the first equality. This means that

$$S \leq n - k < n - k + 1 \leq k,$$

or in short $S < k$. This completes the proof.

Illustration: Zyanya Santuario

9 Elves on the Elevator

Authors: Stella Kapodistria (TUe) and Marko Boon (TUe)
Project: 4TU.AMI

Challenge

It's Christmas Eve at the North Pole and Santa Claus is preparing for his grand journey. His sleigh, polished to a twinkling shine, waits atop the workshop tower, on the 10th floor. Down on the ground floor, the 0th floor, Santa stands in front of the elevator, rubbing his weary knees after a long night of toy inspections. As the doors open with a cheerful ding, he takes a first step towards them with a sigh of relief - at last, a lift bringing him to his sleigh instead of endless stairs and narrow chimneys!

But before he can step inside, a giggling swarm of eight mischievous elves darts past him. In a flurry of tiny hands and jingling laughter, they jab at the control panel, lighting up a merry constellation of buttons - every floor, even the 10th, fair game to their mischief. Each elf presses exactly one of the buttons $1, 2, \ldots, 10$ (uniformly at random) with the light-hearted whimsy typical of Christmas elves.

Santa groans and covers his eyes with a gloved hand, thinking, "I should have taken the stairs…". Before he can manage to look at the lit-up panel to see which buttons the elves have pushed, surrounded by his merry band of button-happy troublemakers, he asks himself, "What are the chances that, before reaching my sleigh on the 10th floor, the elevator will stop at exactly five different other floors along the way?

Remark: The elevator starts on the 0th floor and, going from bottom to top, makes exactly one stop at every floor whose button has been pushed at least once. As Santa enters the elevator, he presses the button for the 10th floor, where his sleigh is waiting.

Possible Answers:

1. $\frac{6067920}{8^8} \approx 0.3617 \approx 36\%$.
2. $\frac{452307240}{8^{10}} \approx 0.4212 \approx 42\%$.
3. $\frac{17781120}{9^8} \approx 0.4131 \approx 41\%$.
4. $\frac{40007520}{10^8} \approx 0.4001 \approx 40\%$.
5. $\frac{44007520}{10^8} \approx 0.4401 \approx 44\%$.
6. $\frac{52007520}{10^8} \approx 0.5201 \approx 52\%$.
7. $\frac{52007520}{11^8} \approx 0.2426 \approx 24\%$.
8. $\frac{62007520}{11^8} \approx 0.2893 \approx 29\%$.
9. $\frac{64007520}{11^8} \approx 0.2986 \approx 30\%$.
10. $\frac{80015040}{11^8} \approx 0.3733 \approx 37\%$.

Solution

The correct answer is: 4.

First, we derive a formula for a more general setting: Let $M + 1$ denote the top floor, m the number of stops, and E the number of elves.
Afterwards, for the purpose of our specific problem, we plug in $M = 9$, $m = 5$ and $E = 8$.

The probability in question can be computed as the fraction of the number of all the ways that the elves choose exactly m of the floors $1, 2, ..., M$ and possibly the top floor divided by the number of the different ways the elves can choose any floor.

The denominator can be computed as $(M + 1)^E$, as each of the E elves presses one of $M + 1$ buttons.

For the numerator, note that out of all the $(M + 1)^E$ ways, only a handful are fitting the requirement of the E elves activating the buttons of exactly m of the floors $1, 2 \ldots, M$ and possibly the top floor.
Let us first select the m floors we want the elevator to stop at out of the possible M floors. This can occur in $\binom{M}{m} = \frac{M!}{m!(M-m)!}$ different ways.
Now we are interested in computing in how many ways the elves can choose out of the selected m floors plus the top-floor ensuring that all the selected floors appear at least once:

For this, we first compute the number of ways the elves can choose from the m floors plus the top-floor, which turns out to be $(m + 1)^E$.
Here, we included all the cases where at least one of the selected floors is not pressed, so we have to substract the number of these cases which we denote by C.
We compute C by the so called *inclusion-exclusion principle*: For each floor j of the m selected floors, we count the number of cases where the floor j is not visited which results in $\binom{m}{1}(m + 1 - 1)^E$.
Now, for each pair of floors j, k of the m selected floors, all cases where both j and k were not visited were counted twice. That's why we count the number of cases where this happens for each pair of floors, add these numbers up, which results in $\binom{m}{2}(m + 1 - 2)^E$, and then subtract it from the previous number.
Now we take a look at each triple of floors j, k, l of the m selected floors. For every case where none of the three floors j, k, l was visited, we added three in the first step and subtracted three in the second step. Therefore, we must count the number of cases where this happens for each triple of floors, add these numbers up, which results in $\binom{m}{3}(m + 1 - 3)^E$, and add it to the number computed previously. If we continue doing this for every set of i floors for $i = 4, \ldots, m$, we get

$$C = \sum_{i=1}^{m} (-1)^{i-1} \binom{m}{i} (m + 1 - i)^E.$$

Consequently, the number of ways the elves can choose from the selected m floors plus the top-floor, where all the selected floors appear at least once, is

$$(m + 1)^E - C = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m + 1 - i)^E.$$

Finally, the probability under question is equal to

$$\frac{\binom{M}{m}}{(M+1)^E}\sum_{i=0}^{m}(-1)^i\binom{m}{i}(m+1-i)^E.$$

If we insert $M = 9, m = 5$ and $E = 8$, we get

$$\frac{\binom{9}{5}}{10^8}\sum_{i=0}^{5}(-1)^i\binom{5}{i}(6-i)^8 = \frac{40007520}{10^8} = \frac{250047}{625000} \approx 0.4000752 \approx 40\%.$$

Alternatively one can compute the probability with a recursion by adding one elf:
Denote by $\mathbb{P}(M, E, m)$ the probability under question, namely the probability that the elevator stops at exactly m distinct floors from $\{1, \dots, M\}$ before reaching floor $M + 1$.

Consider adding the E-th elf after $E - 1$ elves have already pressed. There are exactly two different cases:
If the first $E - 1$ elves already produced exactly m distinct stops among $\{1, \dots, M\}$, the new elf must press the button of one of the already pressed floors or the top floor. This means $m + 1$ of the $M + 1$ equally likely choices are favorable. Therefore, the conditional probability that the new elf does not increase the number of stops at floors among $\{1, \dots, M\}$ is $(m + 1)/(M + 1)$.
If the first $E - 1$ elves produced exactly $m - 1$ distinct stops, the E-th elf increases the count to m if and only if they choose one of the $M - (m - 1) = M - m + 1$ previously unchosen floors among $\{1, \dots, M\}$. Here, the conditional probability is $(M - m + 1)/(M + 1)$.

Thus, for every M, $1 \leq m \leq M$ and $E \geq 1$ we get

$$\mathbb{P}(M, E, m) = \frac{m+1}{M+1}\,\mathbb{P}(M, E-1, m) + \frac{M-m+1}{M+1}\,\mathbb{P}(M, E-1, m-1).$$

Since

$$\mathbb{P}(M, E, 0) = 1/(M+1)^E$$

and

$$\mathbb{P}(M, 0, m) = \mathbf{1}_{\{m=0\}}$$ [3]

are known, we can directly compute $\mathbb{P}(M, E, m)$ for any M, E, m with $0 \leq m \leq M$ using the given formula. In particular, we get $\mathbb{P}(9, 8, 5) \approx 0.4$.

Alternatively one can compute the probability by progressing floor-by-floor:
Let $\mathbb{P}(M, f, E, m)$ be the probability of obtaining exactly m stops among floors $\{f+1, \dots, M\}$ starting at floor f with E elves inside the elevator. Each elf will leave at floor $(f + 1)$ with probability $1/(M + 1 - f)$. Let i be the number of elves exiting at floor $(f + 1)$. Then, for all

[3] $\mathbf{1}_{\{m=0\}}$ is an indicator function taking value one if $m = 0$ and value zero otherwise

M, f, E, m with $f < M$:

$$\mathbb{P}(M, f, E, m) = \Big(1 - \frac{1}{M+1-f}\Big)^E \mathbb{P}(M, f+1, E, m) + \sum_{i=1}^{E} \binom{E}{i} \Big(1 - \frac{1}{M+1-f}\Big)^{E-i} \Big(\frac{1}{M+1-f}\Big)^i \mathbb{P}(M, f+1, E-i, m-1).$$

Since we have

$$\mathbb{P}(M, M, E, m) = \mathbf{1}_{\{m=0\}}$$

for all M, E and m, we can directly compute $\mathbb{P}(M, f, E, m)$ for all M, f, E, m with $f \leq M$. In particular, we get $\mathbb{P}(9, 0, 8, 5) \approx 0.4$.

A different approach is based on first-principles multinomial count and equivalence:
Alternatively, we choose a m-element subset S of $\{1, 2, ..., M\}$ and distribute the E elves into counts:

$$m_1, \dots, m_m \geq 1, \quad m_{m+1} \geq 0, \quad m_1 + \cdots + m_{m+1} = E,$$

where the first m counts correspond to floors in S and m_{m+1} corresponds to floor $M+1$. For each such tuple $(m_1, ..., m_m, m_{m+1})$ the number of cases where the choices of the E (distinct) elves result in these counts is:

$$\frac{E!}{m_1! \cdots m_{m+1}!}.$$

Since there are $\binom{M}{m}$ possibilities to choose S, the total number of favorable cases concerning our problem is:

$$\binom{M}{m} \sum_{\substack{m_1+\cdots+m_{m+1}=E \\ m_1,\dots,m_m \geq 1,\, m_{m+1} \geq 0}} \frac{E!}{m_1! \cdots m_{m+1}!}.$$

This multinomial expression counts *exactly the same cases* as our first approach using the inclusion-exclusion principle. This equivalence follows because:

- our first approach counts cases by subtracting those cases in which some of the chosen floors are not visited from all cases.
- our current approach directly counts the cases where each chosen floor is visited.

Both views enumerate the identical combinatorial set, therefore the sums coincide.

Illustration: Ivana Martić

10 Christmas Light Show

Author: Lukas Protz
Project: MATH+

Challenge

The eager elf Eifi bought a lot of drones to help with all the work at the North Pole. Unfortunately, the elves at the IT department are not as eager as Eifi and tell him that they currently have no capacity to program the drones. Therefore, Eifi wants to program the drones himself. But not to help with the work - no, he wants to make a light show with the drones to entertain his fellow elves and help them relax and forget about the Christmas stress for a while.

Eifi has a few flight routes for the drones in mind. He introduces a coordinate system centered exactly at the North Pole. For simplicity, he assumes the area around the North Pole to be a perfect flat plane. The units of the coordinate system are the standard distance units at the North Pole, an elf-mile. The flight routes he has in mind depend on two positive integers a, b and can be parametrized by time in the following way:

$$\gamma(t) = (\sin(at), \sin(bt)), \quad 0 \leq t < 2\pi.$$

Note: For each value of t, the formula describes the two coordinates of the flight route at time t. Of course, the time t is measured in the standard time units at the North Pole, the elf-minute. Further, $\sin(t)$ is a function w.r.t. (with respect to) arc length and not w.r.t. degrees, e.g. $\sin(\pi) = 0$ but $\sin(180) \approx -0.8011 \neq 0$.

Eager as Eifi is, he tries out a few combinations of a, b and lets the drones fly one after the other , but . . . oh no! For one of the combinations, at one point of the route, the drones start reversing their direction and travel back exactly the way they came from, thereby crashing into other drones. Eifi immediately stops his experiment. He wonders: For what combinations of a, b can this happen? Can you help Eifi figure out all pairs of a, b, where the drones start to travel back the path they came from at some point, so that he can continue his experiments to find the most beautiful flight route without causing accidents again?

Remark: For any real number α, $\sin(\alpha)$ is defined as the vertical coordinate of the point on the unit circle obtained by measuring an angle of α radians counterclockwise from the intersection of the unit circle and the horizontal axis on the right side of the vertical axis. The unit circle is defined as the circle with radius 1, which is centered at the origin.
Figure 28 illustrates this definition and in Figure 29 the graph of the sine-function is shown for α between $-\pi$ and 3π.

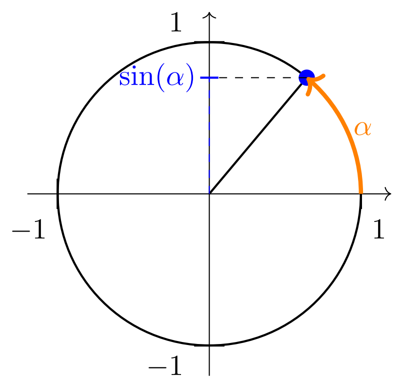


Figure 28: Illustration of the definition of $\sin(\alpha)$.

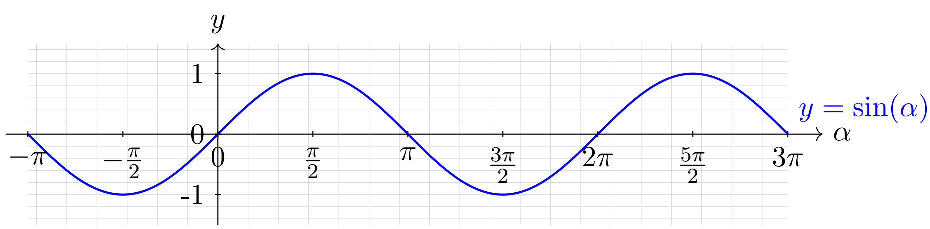


Figure 29: The graph of the sine-function.

(*The possible answers are on the next page.*)

Possible Answers:

1. No combinations of a and b achieve this. Eifi just programmed the drones incorrectly.
2. There is exactly one combination of a and b and Eifi was just unlucky to have tested the drones with this combination.
3. There are only finitely many combinations of a and b that achieve this.
4. This can only happen if $a = b$.
5. This can only happen if a and b are coprime, i.e. have no common positive divisors other than the number 1.
6. This can only happen if a and b are not coprime, i.e. a and b do have a common divisor greater than 1.
7. This exactly happens if a and b are both powers of 2 or if $a = b$.
8. This happens exactly when the highest power of two dividing a is also the highest power of two dividing b.
9. This happens for all combinations of a and b if both numbers are bigger than 2025^{2025}.
10. None of the other answers is true.

Solution

The correct answer is: 8.

We show, that if and only if the highest power of 2 that divides a is also the highest power of 2 that divides b, then the drones travel back the path they came from at some point. In other words

$$a = 2^n c \quad b = 2^n d,$$

with n being a non-negative integer and c, d being odd positive integers.

For the drones to travel back the path they came from, there has to be a time $t_0 \in (0, 2\pi)$ such that

$$\gamma(t_0 + t) = \gamma(t_0 - t) \quad \text{for all } t \text{ with } t_0 + t, t_0 - t \in [0, 2\pi).$$

This implies the following system of equations:

$$\begin{array}{lrcl} \text{I} & \sin(a(t_0 + t)) &=& \sin(a(t_0 - t)) \\ \text{II} & \sin(b(t_0 + t)) &=& \sin(b(t_0 - t)). \end{array}$$

If two values $\sin(x), \sin(y)$ of the sine-function are equal, the two arguments x, y must be related in the following way:

$$\left\{ \begin{array}{lcrlrl} x &=& & y + & 2k\pi & \text{for an integer } k, \text{ or} \\ x &=& - & y + & (2k+1)\pi & \text{for an integer } k. \end{array} \right\}$$

These facts follow from the 2π-periodicity of the sine function, the identity $\sin(x) = \sin(\pi - x)$, and the observation that the equation $\sin(x) = v$ for some v has at most two solutions for $x \in [0, 2\pi)$.

Applying the relation between the arguments to equation I and II, we get

$$\begin{array}{l} \text{I} \left\{ \begin{array}{lcrlrl} a(t_0 + t) &=& & a(t_0 - t) + & 2k\pi & \text{or} \\ a(t_0 + t) &=& - & a(t_0 - t) + & (2k+1)\pi & \end{array} \right\} \text{ and} \\ \text{II} \left\{ \begin{array}{lcrlrl} b(t_0 + t) &=& & b(t_0 - t) + & 2l\pi & \text{or} \\ b(t_0 + t) &=& - & b(t_0 - t) + & (2l+1)\pi & \end{array} \right\}. \end{array}$$

Note, that l is also supposed to be an integer.
We can rule out the first equation of I above, as it can be rearranged into:

$$t = \frac{k}{a}\pi.$$

These are only discrete values for t, which means that the drones would only cross the path at discrete points and thus could not travel back the path they came from. By the same token, we can rule out the first equation of II.

Rearranging the second equation of I and the second equation of II instead yields:

$$t_0 = \frac{2k+1}{2a}\pi \quad \text{and} \quad t_0 = \frac{2l+1}{2b}\pi.$$

Equating the two expressions for t_0 and rearranging the new equation results in

$$\frac{a}{b} = \frac{2k+1}{2l+1}.$$

If we fully reduce $\frac{2k+1}{2l+1}$, the numerator and denominator clearly remain odd and we obtain $\frac{2k'+1}{2l'+1}$ with integers k', l' and $\gcd(2k'+1.2l'+1) = 1$. From

$$\frac{a}{b} = \frac{2k'+1}{2l'+1}$$

it follows that

$$a = \lambda(2k'+1) \quad \text{and} \quad b = \lambda(2l'+1)$$

for some integer λ.
Because $2k'+1$ and $2l'+1$ are odd, the highest power of 2 dividing a must also divide λ and vice versa. Of course, the same holds for b and λ and hence, the highest power of 2 dividing a is also the highest power of 2 dividing b.

Now, all that is left to check is that, if $a = 2^n c$ and $b = 2^n d$ with c, d being odd positive integers, then, after a certain amount of time t_0, the drones travel back the path they came from. To show this, we need the fact that $\sin\left(m\frac{\pi}{2} + x\right) = \sin\left(m\frac{\pi}{2} - x\right)$ for any odd integer m. This can be easily seen by noting that $\sin\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} - x\right)$ and $\sin\left(-\frac{\pi}{2} + x\right) = \sin\left(-\frac{\pi}{2} - x\right)$ and using the 2π-periodicity of the sine-function.
Considering again the above arguments, especially the formula for t_0, one might try setting $t_0 = \frac{\pi}{2^{n+1}}$. Indeed, we get

$$\sin\left(a\left(\frac{\pi}{2^{n+1}} + t\right)\right) = \sin\left(c\frac{\pi}{2} + at\right) = \sin\left(c\frac{\pi}{2} - at\right) = \sin\left(a\left(\frac{\pi}{2^{n+1}} - t\right)\right),$$

and analogously

$$\sin\left(b\left(\frac{\pi}{2^{n+1}} + t\right)\right) = \sin\left(d\frac{\pi}{2} + bt\right) = \sin\left(d\frac{\pi}{2} - bt\right) = \sin\left(b\left(\frac{\pi}{2^{n+1}} - t\right)\right).$$

This completes the proof.

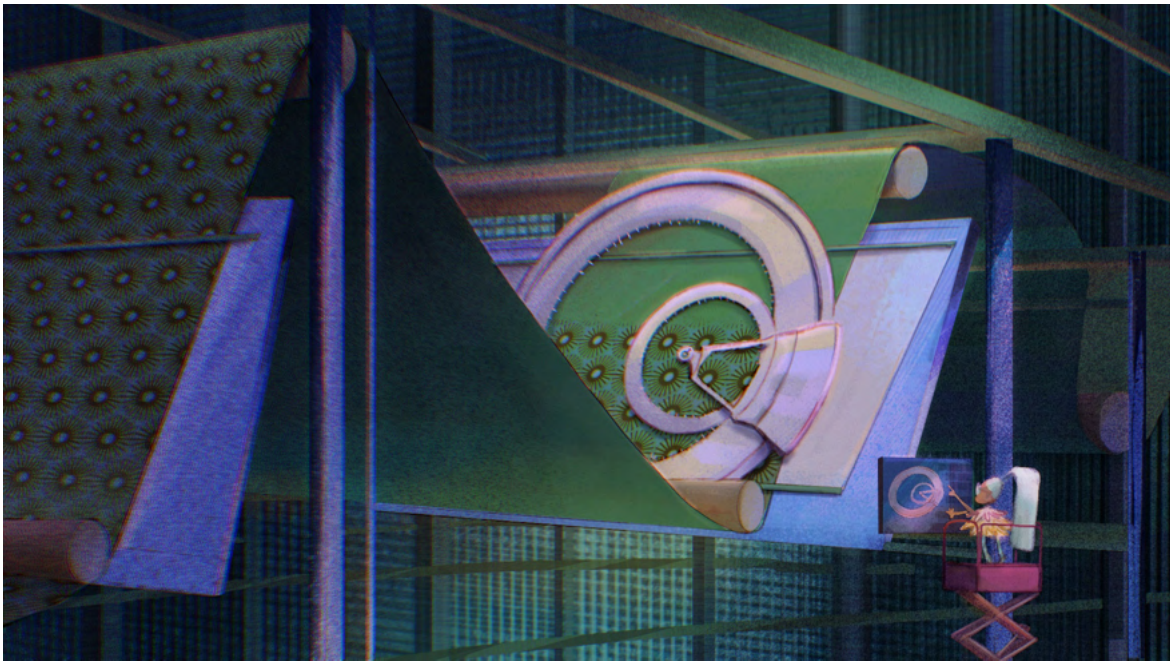


Illustration: Zyanya Santuario

11 Beautiful Gift-Wrapping Using a Spirograph

Author: Eva Deinum (Wageningen University & Research)
Project: 4TU.AMI

Challenge

The elves use a Spirograph to decorate gift-wrapping papers for the Christmas presents. This Spirograph consists of a movable inner cogwheel with n teeth and a fixed outer ring with N teeth, where $N > n$. The inner cogwheel as well as the outer ring are circular. The teeth of the inner cogwheel and the outer ring fit perfectly into each other, so that the inner cogwheel is not able to slip between the teeth of the outer ring but can only be rolled along the outer ring. The inner cogwheel has an off-centered hole for a pen. In Figure 1, the Spirograph setup is shown on the left, and the hole for the pen is marked with a blue dot.

By rolling the inner cogwheel along the outer ring while pressing down the tip of the pen through the hole, the elves can draw closed curves like the purple curve shown in the left part of Figure 30. The closed curves are obtained by rolling the inner cogwheel until it eventually returns to its starting position (with also the hole for the pen returning to its staring position). The so created curves can have various kinds of symmetry and can also intersect with themselves, e.g. the purple curve shown in the left part of Figure 30 intersects with itself and has a so called 7-fold (rotational) symmetry. Here, an n-fold rotational symmetry is a rotation around the center of the outer ring by $\frac{360^\circ}{n}$, that maps the closed curve to itself. When the rotation corresponding to the symmetry is applied n times, it is just a 360° rotation.

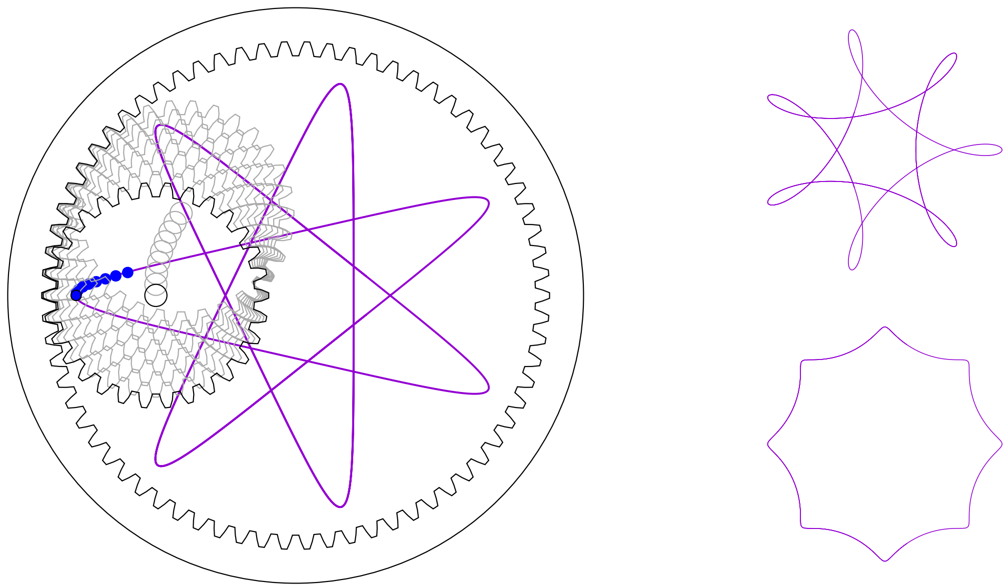


Figure 30: Spirograph setup (left), with in purple the curve it produces. Example of curves for other n, N values (right).

By choosing cogwheels and rings with different values of n and N, the elves can create different curves. In the right part of Figure 30, we show two more examples of curves that can be created using a Spirograph with different values of n and N; the top one has 7-fold symmetry and intersects with itself, the bottom one has 8-fold symmetry and doesn't intersect with itself.

Today, a young elf is in charge of decorating the gift-wrapping paper. Unfortunately, he doesn't fully understand the Spirograph yet and there are no other elves around to help him. He wants to make a curve that has the following two properties:

1. the curve has 5-fold symmetry;
2. the curve intersects with itself.

He can choose between three outer rings $N = 100$, $N = 115$, and $N = 144$, respectively, as well as six inner cogwheels with $n = 20$, $n = 23$, $n = 36$, $n = 40$, $n = 46$ and $n = 69$, respectively. How many different combinations of outer ring and inner cogwheel can the young elf choose to draw a curve that has the two desired properties as mentioned above?

(*The possible answers are on the next page.*)

Possible Answers:

1. 1 combination
2. 2 combinations
3. 3 combinations
4. 4 combinations
5. 5 combinations
6. 6 combinations
7. 7 combinations
8. 8 combinations
9. 9 combinations
10. Without knowing where the hole in the inner cogwheel is located, one cannot determine the number of combinations uniquely.

Solution

The correct answer is: 8.

Each closed curve that can be drawn by the Spirograph can (also) be drawn by starting at a position of the inner cogwheel, where the hole for the pen is closest to the outer ring. Starting from such a position, we now want to determine the number T of teeth that the inner cogwheel was rolled along before closing up. The curve closes up exactly when the current position of the inner cogwheel matches its starting position for the first time. On the one hand, this implies that T is a multiple of N, i.e. there is a positive integer β such that $T = \beta \cdot N$. On the other hand, since after each n teeth the hole for the pen of the inner cogwheel is again closest to the outer ring, this implies T is also a multiple of n, i.e. there is a positive integer α, such that $T = \alpha \cdot n$. Altogether, using these two conditions on T we get the following equation:

$$\alpha \cdot n = \beta \cdot N.$$

Moreover, the curve closes up the first time that the current position of the inner cogwheel matches its starting position. This means, that we are looking for the smallest value of α such that the above equation has an integer solution for β. Rearranging the equation for β we obtain

$$\beta = \frac{\alpha n}{N}.$$

Reducing the fraction by dividing both N and n by their greatest common divisor $\text{gdc}(n, N)$ we arrive at

$$\beta = \frac{\alpha \cdot n/\gcd(n, N)}{N/\gcd(n, N)}.$$

Because $N/\gcd(n, N)$ and $n/\gcd(n, N)$ do not have any positive common divisors other than 1, for β to be an integer, α must be divisible by $N/\gcd(n, N)$. Because we are looking for the smallest positive value for α, we obtain

$$\alpha = \frac{N}{\gcd(n, N)}.$$

Using this result, we can calculate β as

$$\beta = \frac{n}{\gcd(n, N)}.$$

Now, for the curve to have self intersections, it must not close up after only N teeth. This means that $\beta > 1$. Further, for the curve to have 5-fold symmetry, α must be divisible by 5. Calculating α and β for all combinations of n and N leads to the values depicted in figure 31. In total, 8 combinations of n and N lead to values of α and β that meet the above conditions, thus confirming that the answer 8 is correct.

	$N = 100$	$N = 115$	$N = 144$
$n = 20$	$\alpha = 5,\ \beta = 1$	$\alpha = 23,\ \beta = 4$	$\alpha = 36, \beta = 5$
$n = 23$	$\alpha = 100,\ \beta = 23$	$\alpha = 5,\ \beta = 1$	$\alpha = 144, \beta = 23$
$n = 36$	$\alpha = 25,\ \beta = 9$	$\alpha = 115,\ \beta = 36$	$\alpha = 4, \beta = 1$
$n = 40$	$\alpha = 5,\ \beta = 2$	$\alpha = 23,\ \beta = 8$	$\alpha = 18, \beta = 5$
$n = 46$	$\alpha = 50,\ \beta = 23$	$\alpha = 5,\ \beta = 2$	$\alpha = 72, \beta = 23$
$n = 69$	$\alpha = 100,\ \beta = 69$	$\alpha = 5,\ \beta = 3$	$\alpha = 48, \beta = 23$

Figure 31: The table shows the values of α and β in dependence of the values for n and N. The entries with α being divisible by 5 and $\beta > 1$ are highlighted in blue.

Note: The above considerations also show, that the position of the hole relative to the inner cogwheel is irrelevant for the symmetry and self intersections of the curve, as long as the hole is located in the interior of the inner cogwheel.

Illustration: Friederike Hofmann

12 Baking by Chance

Author: Tobias Paul (formerly HU Berlin)

Challenge

In the Christmas bakery, as every year, preparations for the Christmas cookies begin on December 1, 2025. To ensure that enough cookies are ready on Christmas Eve, the elves decide to bake two sheets at a time.

Therefore, before each baking cycle, two baking sheets are prepared, one on the left side of the oven for the upper rack and one on the right side of the oven for the lower rack. Unfortunately, there are only two different cookie cutters: stars and Christmas trees. For the first baking cycle, the sheets are randomly[4] filled with cookies in these two shapes. Each baking sheet holds exactly 1000 cookies, regardless of the shape. Both baking sheets are then placed in the oven and the cookies are baked for 15 minutes at 175 degree Celsius.

Simultaneously, new sheets are prepared for the next baking cycle, one on the left and one on the right of the oven, each holding 1000 cookies again. The shape of each cookie on the left baking sheet is determined by copying the shape of a randomly selected cookie from the upper sheet in the oven. Similarly, for the right baking sheet, the shape of each new cookie is determined by the shape of a randomly selected cookie from the lower sheet in the oven.

In accordance with elf labor regulations, exactly 20 baking cycles are performed each day. The operation begins on December 1, 2025.

[4] Throughout this problem, "randomly" always means uniformly at random.

What is the probability (rounded to one decimal place) that two randomly chosen cookies finished in the final baking process on December 24, 2025, have the same shape if they are

(a) from the same baking sheet, and

(b) from different baking sheets?

Possible answers:

1. (a) $0,5$ (b) $0,5$
2. (a) $0,6$ (b) $0,5$
3. (a) $0,7$ (b) $0,5$
4. (a) $0,5$ (b) $0,6$
5. (a) $0,6$ (b) $0,6$
6. (a) $0,7$ (b) $0,6$
7. (a) $0,5$ (b) $0,7$
8. (a) $0,6$ (b) $0,7$
9. (a) $0,7$ (b) $0,7$
10. None of the above solutions.

Project Reference:

Project EF4-7 dealt with various population models. The process described in this exercise is similar to a Wright-Fisher model, which was one of the first population models in mathematics.

Solution

The correct answer is: 3.

Solution to (a): First, we consider the opposite case, namely that the two selected cookies have different shapes. The two selected cookies have different shapes if and only if the "template" cookies on the previous baking sheet already had different shapes. For this to happen, it is necessary that the same cookie on the previous tray was not responsible for determining the shape of both cookies. This occurs with probability $1 - \frac{1}{1000}$, since the probability that a selected cookie is chosen again is $\frac{1}{1000}$.
This argument can now be continued iteratively, so that the probability that, throughout the baking history, the shape of two cookies was never determined by the same cookie on the previous sheet is

$$(1 - 1/1000)^{479} \approx 0,62$$

after the 479th copying process. Here we refer to the 479th copying process because the elves carried out 20 baking processes on 24 days, and the first copying was done during the preparation for the second baking process.
In the case where there was never a common "template", the two selected cookies have different shapes if and only if their (different) "templates" on the very first sheet had different shapes. The probability of this is 0.5, since the shapes on the first sheet were assigned randomly, each with probability 0.5.

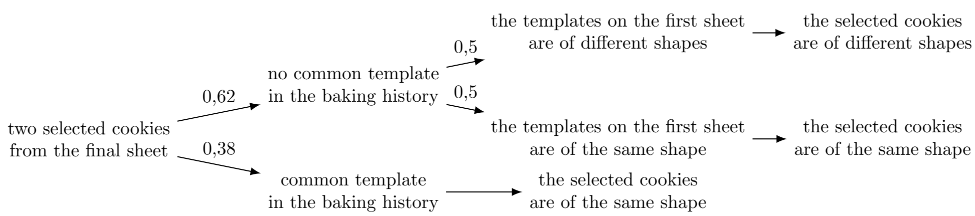


Figure 32: A tree diagram to illustrate the situation.

The situation considered so far is illustrated in Figure 32 in the form of a tree diagram. Using the path rules, it follows that the probability that the two selected cookies have different shapes is approximately

$$0.62 \cdot 0.5 = 0.31.$$

Therefore, the probability that the two selected cookies have the same shape is approximately $1 - 0.31$ which gives 0.7 after rounding to one decimal place.

Solution to (b): Using a similar argument, one can conclude that the two selected cookies from the last baking process have the same shape if and only if their "templates" on the first left sheet and the first right sheet had the same shape.
Since the shapes were randomly assigned at the beginning with probability 0.5, the two "templates" and therefore also the two selected cookies have the same shape with probability 0.5.

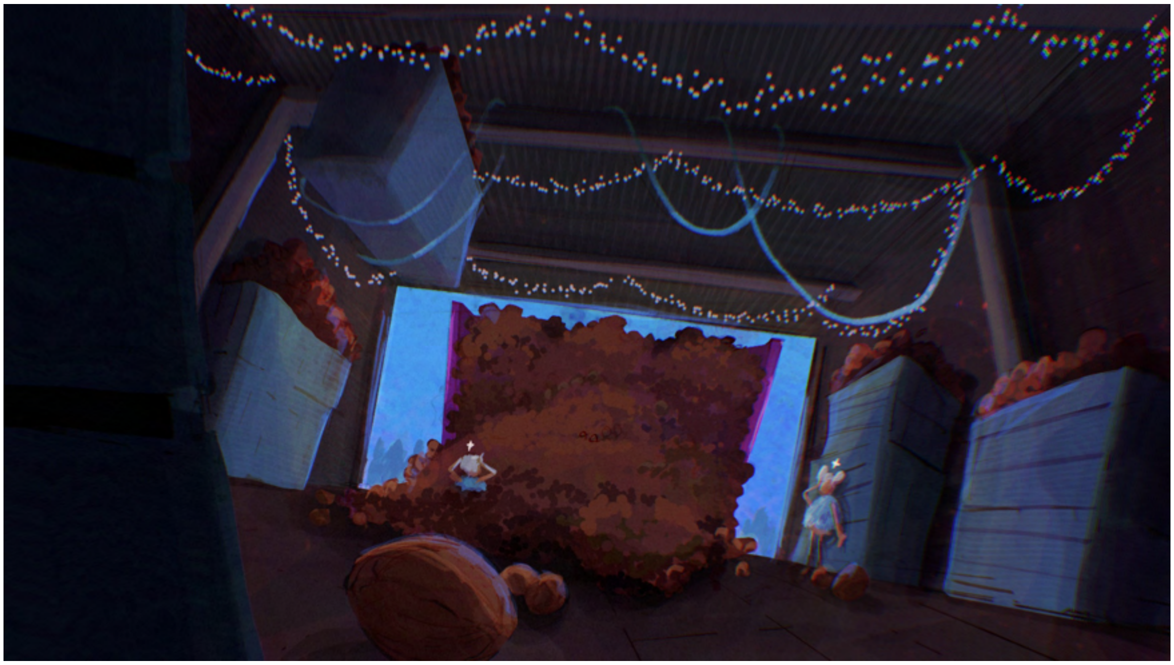


Illustration: Zyanya Santuario

13 The Great Walnut Delivery

Author: Mehmet Akif Yıldız

Challenge

The legendary Nutcracker Workshop is preparing for the grand Christmas Parade! To be ready for any walnut emergency, they have 2025 walnuts — randomly shoved into three supply crates, none of them empty. By randomly we mean the following: if one lists the numbers a, b and c of walnuts contained in the three crates in increasing order, say $a \leq b \leq c$, then every triple (a, b, c) of positive integers with $a + b + c = 2025$ is equally likely.

The Nutcracker Headquarters has just informed them that a huge walnut delivery is already on the way. The delivery elves have one very simple rule: they will only unload into completely empty crates — and this shipment is so massive that they will need two of them!
If fewer than two crates are empty when they arrive, they will simply dump the entire shipment onto the floor, causing a rolling walnut avalanche across the Nutcracker Workshop.

Luckily, there are still a few hours left. So two clever workshop coordinators, Aria and Bram, decide to make a game out of clearing the crates:

The Game

Aria and Bram alternate turns. On a player's turn, a legal move consists of the following:

- Choose two non-empty crates. Suppose they contain a and b walnuts with $1 \leq a \leq b$.
- Choose a positive integer c with $1 \leq c \leq a$.
- Transfer c walnuts from the crate containing less walnuts to the other crate. After the move, those two crates contain $a - c$ and $b + c$ walnuts.

A player loses if they cannot make a legal move anymore — that is, if there is only one non-empty crate left. Aria starts the game.
Assume Aria and Bram play optimally. Let p be the probability that Bram wins the game. Which of the options below is closest to p?

Possible Answers:

1. 0.1%
2. 0.2%
3. 0.3%
4. 0.5%
5. 1%
6. 2%
7. 5%
8. 10%
9. 25%
10. 50%

Solution

The correct answer is: 2.

Sketch Solution: First, we fix some terminology. A distribution of nuts in the three crates is called a *winning position* for one player if there is a strategy for this player to guarantee their win. On the other hand, a *losing position* for one player is a distribution of the nuts such that their turn will definitely lead to a winning position for the other player. Because both Aria and Bram play optimally, being in a losing position results in losing the game.

The first observation to be made is that the game is deterministic after the walnuts are divided into three piles, i.e. each position is either a winning or losing position: To prove this, we assume the contrary and construct a contradiction.
Suppose that there is a position that is neither a winning nor a losing position. Any game that Aria and Bram play from this position must end in a losing position (namely if only one of the crates is non-empty). Let p be the first position in the game of Aria and Bram that is a winning or losing position. If p is a losing position, then the position before p was a winning position, contradicting that p is the first position that is losing or winning. Similarly, if p was a winning position, then the position before p was losing, as otherwise neither Bram nor Aria would have picked p as the next position. This also contradicts that p is the first position that is winning or losing. Therefore, the assumption that there is a position that is neither winning nor losing was false, proving the claim.

Now suppose that after being randomly divided, the piles contain $1 \leq a \leq b \leq c$ walnuts with $a + b + c = 2025$. We claim that Aria wins the game if and only if $a < b$.
We use the triple (x, y, z), with $x \leq y \leq z$, to represent the current configuration of the piles at any moment during the game and the triple (x', y', z') with $x' \leq y' \leq z'$ to represent the configuration resulting from (x, y, z) after a player's turn.
Let us assume that a player receives the triple (x, y, z) with $x < y \leq z$. They can then transform this configuration into $(x, x, y + z - x)$ by moving $y - x$ nuts from the crate containing y nuts into the crate containing z nuts. Thus, $x' = y' = x$ and $z' = z + y - x$. The other player then has to choose a positive integer $k \leq x'$ and can only perform the transformation $(x' - k, x' + k, z')$ or $(x' - k, x', z' + k)$. All these transformations lead to a configuration (x'', y'', z'') with $x'' < y''$. Hence, a player starting with a configuration where exactly one of the crates contains the smallest number of nuts can always achieve that the next configuration they receive has exactly one crate containing the smallest number of nuts again. Since the game ends if two crates are empty, this player can never lose. Therefore, configurations (x, y, z) with $x < y$ are winning positions. On the other hand, configurations (x, y, z) with $x = y$ are losing positions, as they always lead to a winning position. This proves the claim.

As a result, Aria wins the game if and only if the partition of 2025 walnuts into three non-empty parts has a unique smallest part. In other words, the only initial partitions in which Bram wins are

$$(1, 1, 2023), (2, 2, 2021), \ldots, (674, 674, 677), (675, 675, 675).$$

To count the total number of partitions of 2025 into three non-empty parts, we focus on the size k of the smallest part. Suppose that the smallest part is k. Then we must distribute the remaining

$$2025 - k$$

walnuts in two parts y and z, with $k \leq y \leq z$.
Since $y \geq k$, let

$$y = k + u$$
$$z = k + v$$

where $u \leq v$ and $u, v \geq 0$.
Substituting gives

$$k + (k + u) + (k + v) = 2025 \implies u + v = 2025 - 3k.$$

Since $u \leq v$, we have $0 \leq u \leq \left\lfloor \frac{2025-3k}{2} \right\rfloor$.[5] Thus, for a fixed k, the number of possible pairs (u, v) with $u \leq v$ is exactly

$$\left\lfloor \frac{2025 - 3k}{2} \right\rfloor + 1.$$

Each such pair produces a unique triple $(k, k + u, k + v)$.
Substituting $j = 675 - k$ we can transform the above expression into

$$\left\lfloor \frac{2025 - 3k}{2} \right\rfloor + 1 = \left\lfloor \frac{3 \cdot (675 - k)}{2} \right\rfloor + 1 = \left\lfloor \frac{3j}{2} \right\rfloor + 1.$$

Since $0 \leq k \leq 675$, we also have $0 \leq j \leq 675$. Evaluating the above expression for small values of j, we obtain the following sequence:

$$1, 2, 4, 5, 7, 8, \ldots$$

Indeed, this is just the natural numbers without multiples of 3 as can be seen in the following way:
The numbers in the sequence can be obtained by adding 1 and 2 alternately to the previous number in the sequence: Looking at the similar expression $\frac{3j}{2} + 1$, we see that, increasing j by 2, the expression increases by 3. In fact, increasing j by 2 has the same effect on both expressions. However, because of the floor in the first expression, increasing j by 1 increases the new expression by $\frac{3}{2}$, but the original expression only by 1. This means that every third positive integer is skipped in the sequence. Since the original sequence starts with 1,2,4, this means that all multiples of 3 are skipped.
Summing all the values in the sequence yields the total number of distributions of the nuts in the three crates. Using the observation above, this reduces to summing all positive integers from 1 to $\left\lfloor \frac{3 \cdot 675}{2} \right\rfloor + 1 = 1013$ and subtracting all multiples of 3 less than 1013. This can be done using the Gauss summation formula, which states that for a positive integer n we have

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Thus, the total number of distributions is given by

$$\frac{1013 \cdot 1014}{2} - 3 \cdot \frac{337 \cdot 338}{2} = 342732.$$

So, the desired probability is $\frac{675}{342732} = 0.0019694688... \approx 0.2\%$.

[5] For any real number x the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$.

Illustration: Julia Nurit Schönnagel

14 Ice Basketball Substitution

Author: Silas Rathke

Challenge

Álex is the coach of an ice basketball team from the North Pole. His team consists of 99 players. For everyone who is not familiar with the local rules of ice basketball in the Christmas Village, the most important ones are listed here:

- During the game, each team has exactly three players on the court.
- During each timeout, each team is allowed to substitute at most one player. Players who have previously participated in the game may be substituted back in.
- Each of the team's 99 players has to be on the court at least once during a match.

Álex is especially worried about the last rule. He has noticed that each three-player combination on his team either plays very well together or very poorly, and his goal is to ensure that every one of the 99 players appears on the court at least once during a single game without ever having a bad trio on the court.
While thinking about this, he notices the following: If every pair of players could play well with all of the other 97 players, it would be easy to achieve his goal: he could simply make any substitution he wanted and send one player onto the court at a time. However, if every pair of players would form a good trio with exactly one other player, he would have no chance: no matter which trio starts on the court, Alex could not make a substitution without having a bad three-player combination continue playing.

Find the smallest non-negative integer k with the following property: If every pair of players forms a good trio with at least k of the other 97 players, then it is always possible to choose a starting trio and then make a sequence of substitutions so that every one of the 99 players is on the court at least once, and at no time does a bad three-player combination play.
What is the units digit of k?

Remark: Ice basketball games in the Christmas Village are always extremely long and have many timeouts because of the slippery ice. No matter what sequence of substitutions Alex chooses, there will definitely be enough timeouts to carry them all out.

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Solution

The correct answer is: 3.

The correct value is $k = 33$. To prove this, we first show that even if every pair of players plays well with 32 other players, it is still possible that not all players can be substituted in if at no time a bad three-player combination shall play. We then prove that if each pair of players plays well with at least 33 other players, it is always possible.

Construction for 32

Assume the players are divided into three groups A, B and C, each consisting of 33 players. A team plays well together if and only if

- it contains 2 players from A and one player from B, or
- it contains 2 players from B and one player from C, or
- it contains 2 players from C and one player from A.

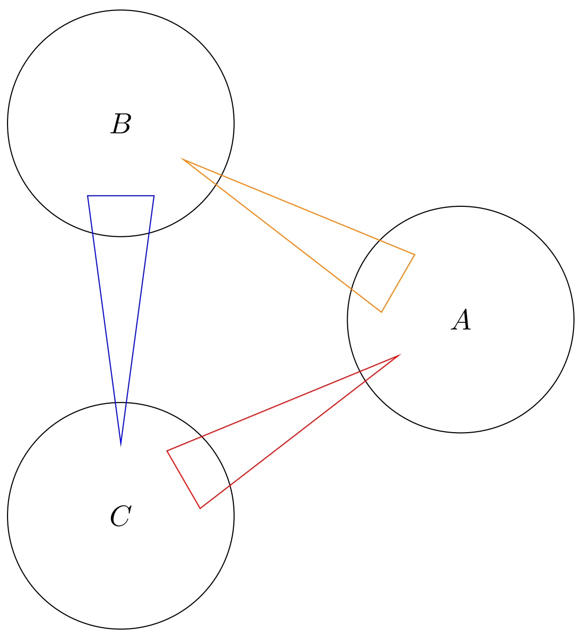


Figure 33: Illustration of the construction.

Figure 33 illustrates this concept.
Each pair of players plays well with 32 other players (or even 33, if both players of the pair come from the same group). However, it is impossible to choose a starting team from which all players can then be substituted: for example, if at the beginning a team is formed consisting of two players from A and one player from B, then the player from B can only be replaced by players from B, and the players from A can only be replaced by players from A. Consequently, a player from C can never be substituted in.

Proof for 33:

Suppose that each pair plays well with at least 33 other players. We represent the players as vertices in a complete graph [6] and we color the edges in such a way that two pairs P_1 and P_2 of vertices enclose edges of the same color if and only if it is possible, by substitutions, to switch from a team in which pair P_1 plays to a team in which P_2 plays.
In the example above, for instance, the edges between two vertices from A and all edges between a vertex from A and a vertex from B have the same color. However, an edge between a vertex from B and a vertex from C has a different color.
We need to show that there exists a color c such that every vertex is incident to at least one edge of color c.
First, we show that for every vertex, the outgoing edges are colored with at most two colors. Suppose there exists a vertex v such that the edges $\{u_1, v\}$, $\{u_2, v\}$ and $\{u_3, v\}$ to the vertices u_1, u_2 and u_3 are colored in three different colors. The player pair u_1v forms a good team of three with at least 33 other players. If it were to form a good team of three with u_2, then the edges $\{u_1, v\}$ and $\{u_2, v\}$ would have the same color. The same argument applies to u_3, Hence, the pair u_1v forms a good team of three with 33 players outside the set $\{u_1, u_2, u_3, v\}$.The same holds for u_2v and u_3v.
Since there are only 95 players outside $\{u_1, v\}$ and $\{u_2, v\}$ there must exist a player w who forms a good team with two of the three pairs u_1v, u_2v and u_3v. Suppose these are u_1v and u_2v. Then the edges $\{u_1, v\}$ and $\{u_2, v\}$ must be colored the same, since we can start with the team u_1vw and then replace u_1 by u_2. This is a contradiction. Therefore, the edges incident to any vertex are colored with at most two colors.

By the same argument, it is easy to show that there are no three edges $\{u, v\}, \{v, w\}$ $\{w, u\}$ that all have different colors.

Let v be an arbitrary vertex. If all its edges are colored the same, then all 99 vertices have at least one edge of this color, since the graph is complete. In this case, the claim is proven.
Hence, we may assume that v has outgoing edges in exactly two colors, c_1 and c_2, and in no other. Furthermore, assume that there exists a vertex w that has no outgoing edge colored in c_1. Otherwise, the claim would already be proven. Then the edge $\{v, w\}$ must be colored in c_2. Analogously, there must exist a vertex u that has no outgoing edge colored in c_2. In this case, the edge $\{v, u\}$ is colored in c_1 and the edge $\{u, w\}$ is colored neither in c_1 nor in c_2.
However, the edges $\{u, v\}$, $\{v, w\}$ and $\{w, u\}$ are then all colored differently, which contradicts the result shown above.

Since the assumption that no color exists such that every vertex has at least one outgoing edge of that color leads to a contradiction, there must exist a color such that every vertex has at least one outgoing edge of that color.

[6] Here, we refer to graphs from graph theory that consist of vertices and edges, where each edge connects two vertices. A graph is complete if all vertices are connected to each other by an edge.

Illustration: Julia Nurit Schönnagel

15 The Road from Bethlehem

Author: Matthew Maat (Universiteit Twente)
Project: Combining algorithms for parity games & linear programming

Challenge

After the visit of the shepherds and the wise men, Joseph is warned in a dream: he, his wife Mary and the baby Jesus must flee to Egypt to escape from the evil king Herod[7]. However, reaching Egypt from Bethlehem is not without peril. Because they do not know the way, they will have to trust in the directions of locals that they meet on their journey. Most of all, they hope to avoid Roman checkpoints where they will surely get caught.

[7]Book of Matthew, chapter 2, verse 13-15.

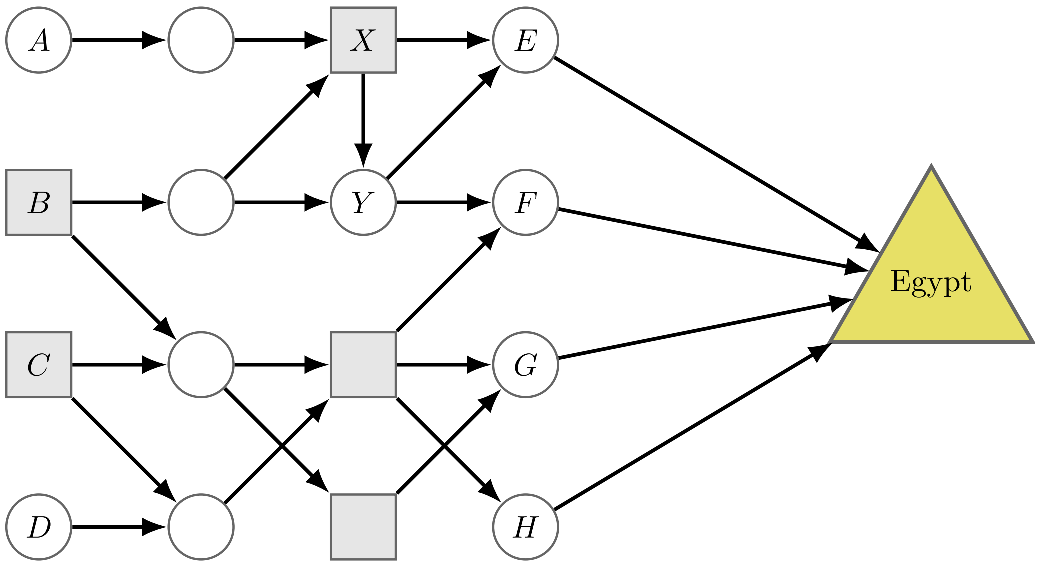


Figure 34: The road map. Circles and squares are cities. Arrows are one-way roads between cities, which can only be used in the direction given by the corresponding arrow.

The road network is shown in Figure 34. Mary and Joseph start in Bethlehem (B on the map). The 16 cities are marked by squares and circles. Each of the cities Emmaüs, Filadelfia, Gerar and Hebron (E, F, G and H) may or may not contain a Roman checkpoint (there are no checkpoints in other cities than these four). It is common knowledge to everyone in this region what the map looks like and where the checkpoints are. However, people from this region do not trust strangers easily, so they will not reveal their whole geographical knowledge to travelers. Instead, whenever travelers arrive, the people from the city will tell them which road to take from their city. All roads are one-way, they can only be used in one direction, not the other, and it is not possible to turn around in the middle of the way. Since travelers do not know the way, they will always follow any directions given to them.

In most cities (white circles), people are friendly, and they will send travelers in a direction that will let them avoid the checkpoints, if possible. However, some cities (grey squares) are completely inhabited by spies of Herod: they will point any travelers in a direction that leads them to a checkpoint, if possible. The friendly people and the spies are also aware of where the spies are and, considering the whole journey till the end, choose one of the best options to reach their goals.

Example: Suppose that Emmaüs has a Roman checkpoint and Filadelfia does not. Then if any travelers arrive in city X, then the spies will direct them to E. Or if the travelers arrive in city Y, then the friendly inhabitants will send them to F. Note: the spies will not send travelers from X to Y in this case, as they know the friendly inhabitants would send travelers to F.

In the end, Joseph and Mary arrive safely in Egypt with the child, without encountering any checkpoints. In Egypt, they also meet other travelers that started in Afek, Caesarea and Damascus (A, C, D on the map) and were also dependent on recommendations from locals. Strangely, all the other travelers had encountered Roman checkpoints on their journey to

Egypt. Given that Joseph and Mary did not encounter a Roman checkpoint and the other travelers did, which statement is true about the location(s) of the checkpoint(s)?

Note: Travelers are also given directions at their starting points.

Possible Answers:

1. There is only one possibility for the distribution of the checkpoints, in which there is precisely one checkpoint.
2. There must be exactly two checkpoints, which must be in E and in F.
3. There must be exactly two checkpoints, which must be in E and in G.
4. There must be exactly two checkpoints, which must be in E and in H.
5. There must be exactly two checkpoints, which must be in F and in G.
6. There must be exactly two checkpoints, which must be in F and in H.
7. There must be exactly two checkpoints, which must be in G and in H.
8. There is only one possibility, in which there are precisely three checkpoints.
9. There are precisely two possible distributions of checkpoints.
10. There are more than two possible distributions of checkpoints.

Project Reference:

This problem is related to reachability games.

Solution

The correct answer is: 4.

With the information given, we can deduce for each of the four cities whether or not there is a checkpoint.

- First of all, suppose that there is no checkpoint in Emmaüs. Then the friendly people would be able to guide any travelers from Afek to Emmaüs by recommending the roads marked on the left in Figure 35. But that would mean that travelers from Afek would not meet a checkpoint, which contradicts what we know, so that is not possible, and Emmaüs must have a checkpoint.
- Secondly, knowing that Emmaüs has a checkpoint, suppose that Filadelfia also has a checkpoint. Then the spies can recommend the roads marked on the right of Figure 35, which will make any travelers from Bethlehem end up in either Emmaüs or Filadelfia. But that cannot happen, since we know that Joseph and Mary did not meet a checkpoint. So there cannot be a checkpoint in Filadelfia.

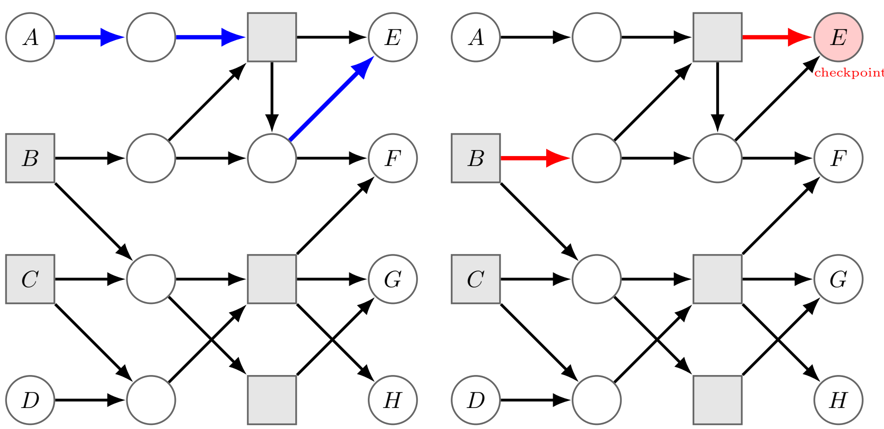


Figure 35: Left: the friendly inhabitants can guide people from Afek to Emmaüs by recommending the blue roads. Right: the spies can guide people from Bethlehem to Emmaüs or Filadelfia by recommending the red roads.

- Next, with the marked roads from the left of Figure 36, the spies can direct anyone from Bethlehem to Gerar. That means that Gerar cannot have a Roman checkpoint, since we know that Joseph and Mary arrived safely.
- Travelers from Damascus have no way of reaching Emmaüs. However, since we know that travelers from Damascus do encounter a checkpoint, there must be another checkpoint, which must then be in Hebron.

Combining all this, we see that there is only one possible distribution of checkpoints as shown on the right in Figure 36. Moreover, the friendly people can use the blue marked roads to guide people from Bethlehem to safe cities and the spies can guide travelers from other starting points to checkpoints with the red roads. So this is indeed a valid distribution of checkpoints, and it is the only one.

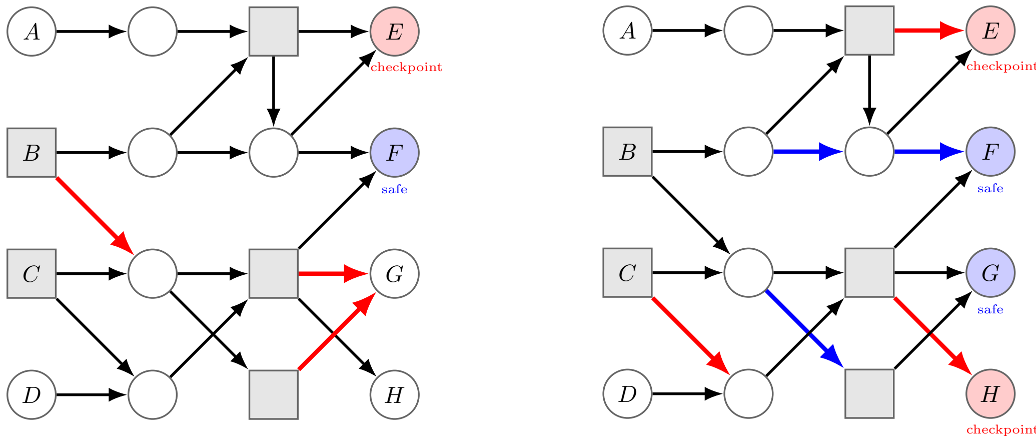


Figure 36: Left: The spies can direct travelers from Bethlehem to Gerar by recommending the red roads. Right: the distribution of checkpoints and the roads that friendly people and spies will direct travelers along.

Remark: in the context of reachability games, the set of cities from which the spies/friendly people can guarantee that you reach a certain set of cities S, is called the *attractor* of S. The related strategies of the friendly people and the spies considered in this proof are called *attractor strategies*. Attractors to a set of cities S are easy to compute, working from right to left: first you check which cities can be forced to go in one step to S, and then you add these cities to the attractor of S. Next, you check again which cities can be forced to go to the attractor of S and you add these, and so on. When no more cities are added, you have computed the attractor of S!

Illustration: Mar Curcó Iranzo

16 Santa's Delivery at the House Block

Author: Mar Curcó Iranzo

Challenge

Santa is flying over Verdedam, a city famous for its gardens, to deliver presents. Each block of houses in the city is represented by a rectangular grid, where every cell is either a chimney, where Santa needs to drop presents, or a small garden, where elves are waiting to load more presents onto his sleigh. Loading and delivery of presents are represented by positive and negative integers, respectively. For example, a cell with the number 4 stands for a small garden, where 4 presents are loaded onto Santa's sleigh. A cell with the number -4 stands for a chimney where Santa needs to drop 4 presents.
In Verdedam, the elves are highly organized. In every house block, they arrange the presents in the gardens in such a way that the following condition is satisfied: in every 4×3 or 3×4 rectangle of the corresponding grid, the sum of the entries is zero.

-9	-8	-11	24	-2	-10	-16
-16	16	29	-50	11	25	-7
22	-15	-3	21	-12	-22	38
-1	10	-14	1	6	8	-19
-26	-5	3	7	1	4	-33
30	1	-2	-4	-4	-6	39

5	1	4	-1	3
-2	3	1	-5	-6
-4	-2	-2	2	2

Figure 37: The corresponding grids of the first two house blocks. In every 4×3 or 3×4 rectangle of the grids, the sum of the entries is zero.

Santa arrives in Verdedam with an empty sleigh, flies from block to block and, within each block, always follows the same routine: he first visits all small gardens to load presents and only afterwards visits all chimneys, each exactly once. Considering the order in which Santa visits the blocks each year, his well-organized elves have of course distributed the presents such that at any chimney, Santa has enough presents in his sleigh to deliver the requested amount. The first block on his route, shown in Figure 37, is represented by a 6×7 grid whose entries sum to 1, so Santa leaves it with one present. The second block, also shown in Figure 37, is represented by a 3×5 grid whose entries sum to -1. Using the extra present from the previous block, Santa delivers all gifts and leaves this block with an empty sleigh.

As Santa continues his journey through Verdedam, strong winds force him to slightly change his route, and he reaches a different part of the city earlier than expected. There, he arrives at a third house block, represented by a 7×10 grid.

				20	25				

Figure 38: The grid corresponding to the third block of houses.

Santa knows that in the center of this block there are two small gardens where he loads 20 and 25 presents, as indicated in Figure 38. All 4×3 and 3×4 rectangles satisfy the usual condition.
Santa arrives at this third house block with an empty sleigh. Of course, the elves did not take this unforeseen event into consideration when distributing the gifts, so Santa may have or may have not enough presents to deliver the requested amounts in this block. How many presents will he fail to deliver, or how many presents will he have left over when he exits the block?

Hint: For the 7×10 grid in Figure 38, containing the fixed central entries 20 and 25, there exists at least one filling satisfying the condition that the sum of the entries in every 3×4 and every 4×3 rectangle is zero.

Possible Answers:

1. He will have failed to deliver 45 presents.
2. He will have failed to deliver 25 presents.
3. He will have failed to deliver 20 presents.
4. He will have failed to deliver 5 presents.
5. In the block, there are exactly as many presents as must be delivered.
6. He will have 5 presents left over in his sleigh.
7. He will have 20 presents left over in his sleigh.
8. He will have 25 presents left over in his sleigh.
9. He will have 45 presents left over in his sleigh.
10. It's not possible to determine with the information we have.

Solution

The correct answer is: 1.

The answer is 1., Santa will fail to deliver 45 presents.
To find the solution, notice that the question essentially asks for the sum of all the numbers in the grid, where in each cell we have a positive (integer) number if we pick up presents, and a negative (integer) number if we deliver presents. One way to get to the solution is to start by naming the numbers in the cells next to the ones with the known numbers as A and B, as shown in Figure 39 on the left.

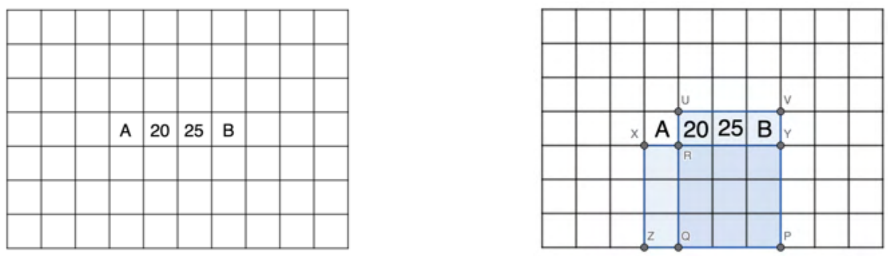


Figure 39: The 7×10 grid.

Now consider the 4×3 rectangle given by the points UVPQ and the 3×4 rectangle given by the points XYPZ shown in Figure 39 on the right. Since by the delivery rules the entries in both of them need to add up to zero and they overlap in the square RYPQ, we can conclude that the entries in the 1×3 rectangle UVYR and the entries in the 3×1 rectangle XRQZ add up to the same number. Hence the numbers in XRQZ have to sum up to $20 + 25 + B$.

Using similar constructions, we can see that in Figure 40 the numbers in each blue rectangle add up to $20 + 25 + B$ and the numbers in each orange rectangle add up to $20 + 25 + A$.

Now consider the 4×3 rectangle OLZN and the 3×4 rectangle MXQN shown in Figure 41 on the left. As before, these two rectangles overlap on the square MRZN. Hence, the numbers in the 1×3 rectangle OLRM and the numbers in the 3×1 rectangle RXQZ sum up to the same value again, which was $20 + 25 + B$.

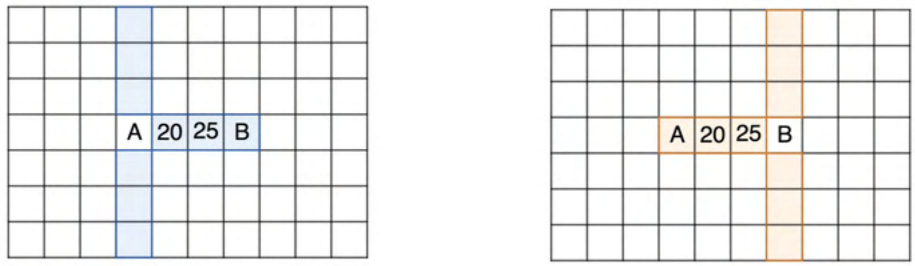


Figure 40: Rectangles with the same sum.

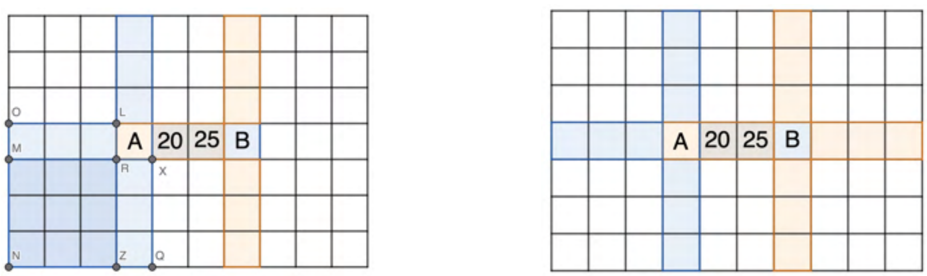


Figure 41: Another construction and more rectangles with the same sum.

So in Figure 41 on the right, the numbers in each blue rectangle add up to $20+25+B$. Using a similar construction, we get that the numbers in each orange rectangle add up to $20+25+A$.

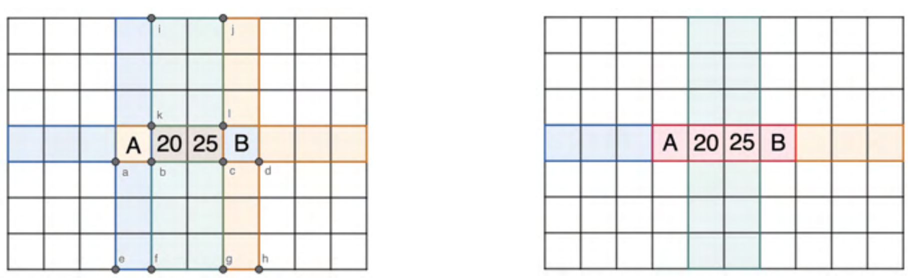


Figure 42: Another construction and the final computation.

Now consider the 3×4 rectangle adhe shown in Figure 42 on the left. Since the numbers in this rectangle must add up to zero and the numbers in the enclosed blue and the enclosed orange rectangle add up to

$$(20 + 25 + B) + (20 + 25 + A),$$

the numbers in the green rectangle bcgf must add up to $-(20 + 25 + B) - (20 + 25 + A)$. A similar argument shows that the numbers in the other green rectangle ijlk also add up to $-(20 + 25 + B) - (20 + 25 + A)$.

Finally, since the numbers in all the 3×4 white rectangles in Figure 42 on the right add up to zero, the sum of all the entries of the 7×10 grid, is the sum of the entries of the colored cells. From the information we collected we get the following sum

$$(A+20+25+B)+(20+25+B)+(20+25+A)+$$
$$(-(20+25+B)-(20+25+A))+(-(20+25+B)-(20+25+A))=-45.$$

So, Santa will be failing to deliver 45 presents.

Illustration: Ivana Martić

17 The Wreath of Two Christmas Magics

Author: Mehmet Akif Yıldız

Challenge

The elves are preparing a magnificent enchanted Christmas wreath to hang above the gate to the reindeer training grounds. Around the wreath, they place a circle of special numbered, magic ornaments.
Each ornament carries a number, which is freely chosen by the elves and may be a positive or negative real number — but never zero.

There are two kinds of magic ornaments:

- A blue ornament uses elf magic: It glows if its number is equal to the sum of the numbers of its two neighbors.
- A red ornament uses reindeer magic: It glows if its number is equal to the product of the numbers of its two neighbors.

To keep the magic flowing properly, blue and red ornaments must strictly alternate around the wreath — blue, red, blue, red ... all the way around. Since it is the year 2025, the elves would have liked to prepare a wreath with 2025 ornaments. However, such a wreath cannot satisfy the alternation of colors around it, so instead they consider wreaths of the two closest feasible sizes, one with $n_1 = 2024$ ornaments and one with $n_2 = 2026$ ornaments, and want to find an arrangement in which every ornament glows. We call such an arrangement a *fully glowing arrangement*.

For each wreath size, the elves ask themselves:

- Does there exist a fully glowing arrangement?
- If such an arrangement exists: What is the sum of all numbers on the wreath? Is this sum uniquely determined?

Possible Answers:

1. No fully glowing arrangement exists for either n_1 or n_2.
2. A fully glowing arrangement exists for exactly one of n_1 and n_2, but the sum is not uniquely determined.
3. A fully glowing arrangement exist for both n_1 and n_2; for exactly one of them the sum is uniquely determined and equals 0.
4. Fully glowing arrangements exist for both n_1 and n_2; for exactly one of them the sum is uniquely determined and equals 506.5.
5. Fully glowing arrangements exist for both n_1 and n_2; for exactly one of them the sum is uniquely determined and equals 759.
6. Fully glowing arrangements exist for both n_1 and n_2; for exactly one of them the sum is uniquely determined and equals 759.75.
7. Fully glowing arrangements exist for both n_1 and n_2. For both wreath sizes, the sum of the numbers is uniquely determined, and the two sums add up to 0.
8. Fully glowing arrangements exist for both n_1 and n_2. For both wreath sizes, the sum of the numbers is uniquely determined, and the two sums add up to 1265.5.
9. Fully glowing arrangements exist for both n_1 and n_2. For both wreath sizes, the sum of the numbers is uniquely determined, and the two sums add up to 1266.25.
10. Fully glowing arrangements exist for both n_1 and n_2. For both wreath sizes, the sum of the numbers is uniquely determined, and the two sums add up to 1518.75.

Solution

The correct answer is: 10.

First we show that fully glowing arrangements exist for both sizes. If the number on each blue ornament is 1/2 and the number on each red ornament is 1/4, the conditions under which every ornament glows are fulfilled, so by this we have a fully glowing arrangement for any even number, in particular for n_1 and n_2.

Now we consider an arbitrary fully glowing arrangement. We freely choose a blue ornament whose number we denote by $a \neq 0$. Let the following red ornament carry the number ab for some non-zero b. Now, the numbers on the following ornaments can be determined uniquely by the previous two using the conditions under which the ornaments glow, which results in Figure 43.

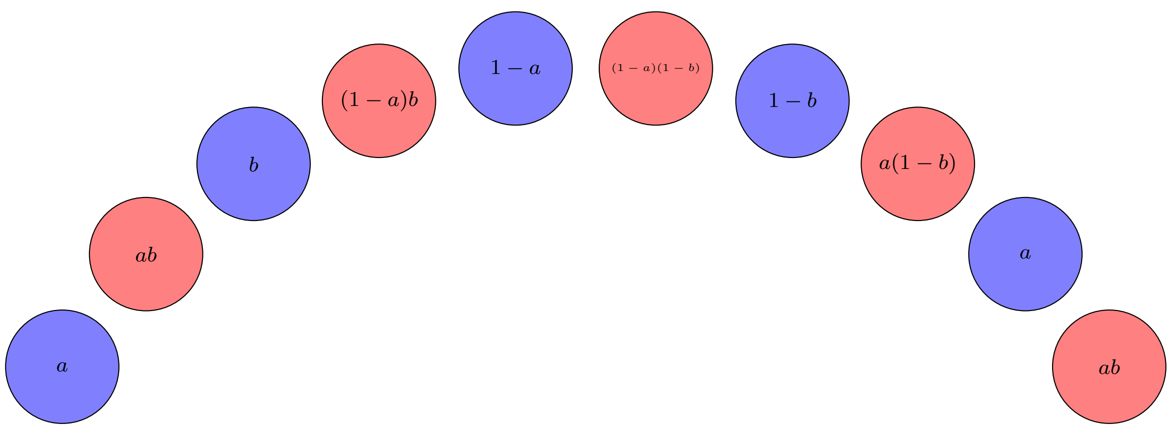


Figure 43: Ten consecutive ornaments in a wreath with $n \geq 10$ ornaments

Starting from the pair (a, ab) , after eight steps we again obtain the same pair (a, ab). Hence after every 8 ornaments, the numbers will repeat.
Therfore, if the wreath size is $n_1 = 2024$, which is divisible by 8, there will be the numbers $a, ab, b, (1-a)b, (1-a), (1-a)(1-b), (1-b), a(1-b)$ repeated 253 times. We can compute the sum of 8 consecutive ornaments:

$$\begin{aligned}
&a + ab + b + (1-a)b + (1-a) + (1-a)(1-b) + (1-b) + a(1-b)\\
&= a + ab + b + b - ab + 1 - a + 1 + ab - a - b + 1 - b + a - ab\\
&= 3.
\end{aligned}$$

Since the considered arrangement was arbitrary, for $n_1 = 2024$ the sum is uniquely determined as $3 \cdot 253 = 759$.

If the wreath size is $n_2 = 2026$, which has remainder 2 when divided by eight, in the periodic pattern, the number on ornament 1 must coincide with the number on ornament 3 and the number on ornament 2 with the number onornament 4 and we get that $a = b$ and $ab = (1-a)b$.
Substituting $a = b$ in the second equation we get $a^2 = (1-a)a$, so $0 = (1-2a)a$. Since a number carried by an ornament is never zero, we get $a = b = 1/2$. Therefore the only

fully glowing arrangement for n_2 is the one described in the beginning. The sum is uniquely determined as $1013 \cdot 1/2 + 1013 \cdot 1/4 = 759.75$.

Hence, for both wreath sizes, fully glowing arrangements exist, the sum of the numbers is uniquely determined, and the two sums add up to 1518.75.

Illustration: Friederike Hofmann

18 Christmas Tree Farm

Author: Tim Kunt (Zuse-Institute Berlin)

Challenge

On the wish lists that are sent to Santa Claus, it is not uncommon for a Christmas tree to be requested. Therefore, Anna the gardener elf is responsible for growing them. Since the climatic conditions at the North Pole make outdoor cultivation impossible, Anna is provided with a room in Santa's workshop that has been built specifically for this purpose. Its square floor plan is divided into 7×7 small tiles, with a water basin located exactly in the center.
Each of the 49 small tiles must be used either for a Christmas tree or for water supply. Every Christmas-tree tile must directly border at least one water tile (i.e., share an edge with it). All water tiles must be connected (i.e., from any water tile one must be able to reach any other water tile by a sequence of water tiles in which each pair of consecutive tiles shares an edge). The central tile is a water tile.
What is the maximum number of Christmas trees Anna can plant at the same time under these conditions?

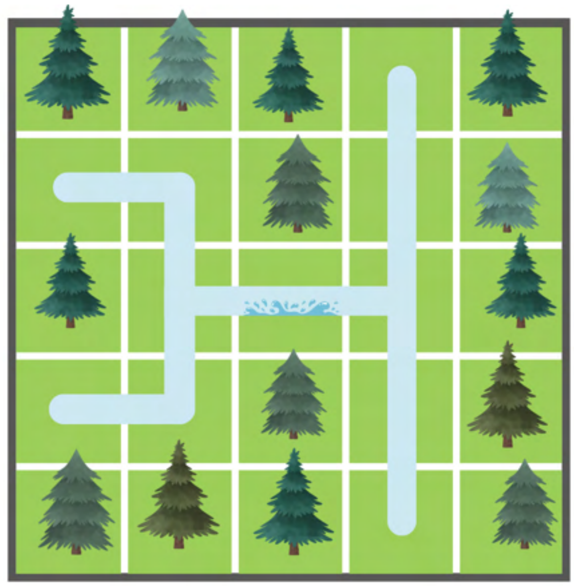


Figure 44: Planting of a 5×5 room under the given conditions.

Note: Figure 44 shows an example of how the maximal number of Christmas trees can be planted in a 5×5 room under the given conditions.

Possible Answers:

1. less than 20
2. 20
3. 22
4. 23
5. 24
6. 26
7. 27
8. 28
9. 30
10. more than 30

Solution

The correct answer is: 8.

As shown in Figure 45, a planting of 28 Christmas trees is possible. Thus, only the answers $8, 9$ or 10 can be correct.
To rule out 9 and 10, we now show that, under the given conditions, it is not possible to plant 30 or more Christmas trees. To do this, we prove that every valid planting of the area must contain at least 20 water tiles. We will carry out the proof in 3 steps.

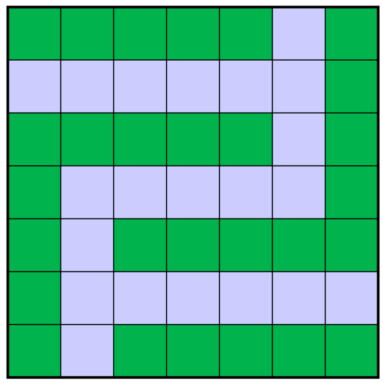


Figure 45: A planting containing 21 water tiles and 28 Christmas trees

Step 1: Water tiles at the boundary

We call a tile *supplied* if it is a water tile or if it is directly next to a water tile, either horizontally or vertically. In a valid planting, all 49 tiles must be supplied.
Therefore, the 4 corners in particular must be supplied. Since all water tiles are required to be connected, one of the two adjacent [8] tiles must be a water tile, regardless of whether the corner itself is one or not. Thus, there are at least 4 water tiles at the edge.

We will now show that, in the case of exactly 4 or 5 water tiles at the edge, it is impossible to achieve a planting with fewer than 21 water tiles:
We already know that one of the two tiles adjacent to a corner tile must be a water tile. This circumstance is illustrated by blue bars on the left side in Figure 46. Now consider the tiles that are colored red. The respective adjacent edge tile must be supplied. This means that either this edge tile itself, one of the two edge tiles adjacent to it or the red-colored tile must be a water tile.
If there are only 4 water tiles at the boundary, it follows that all red-colored tiles must be water tiles. Furthermore, for the water tiles to be connected, the 4 diagonally neighboring tiles of the corners must also be water tiles. This situation is shown in Figure 46 on the right side. In this case, there are 13 water tiles and 9 water clusters that are disconnected. To connect these clusters, at least 8 additional water tiles are required. Consequently, a minimum of 21 water tiles is needed.

[8] A tile is called *adjacent* to another tile if they share a side.

Figure 46: The general case is shown on the left, while the situation with 4 water tiles at the boundary is depicted on the right

If there are exactly 5 water tiles at the edge, it follows from the observation above that at least 3 of the red-colored tiles must be water tiles.
Assume that exactly 3 of the red-colored tiles are water tiles. Due to reasons of symmetry, we can assume that the top red-colored tile is not a water tile. This forces the adjacent edge tile to be supplied, meaning one of its neighbors on the edge must be a water tile. Due to symmetry, we can assume that it is the one on the left. This situation is shown on the left side in Figure 47. To ensure the connectivity of the water tiles, the diagonally neighboring tiles of the corner tiles, except for the yellow-colored one, must be water tiles.

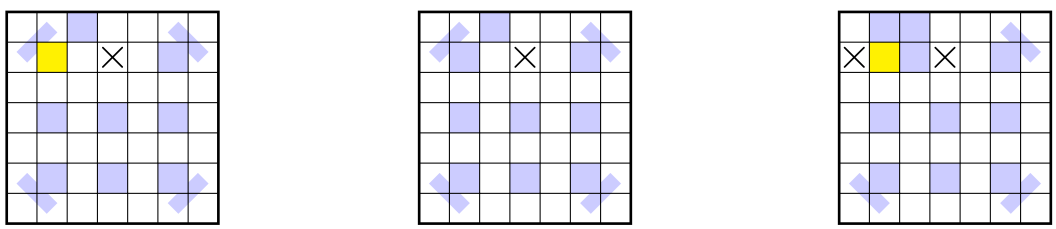


Figure 47: The situation with 5 water tiles at the edge on the left side, when 3 of the red-colored tiles are water tiles. The situations depending on whether the yellow-colored tile is a water tile on the right side.

We will now perform a case distinction based on whether the yellow tile in Figure 47 is a water tile.
Case 1: If the yellow-colored tile is a water tile, the layout appears as shown in the center of Figure 47. There are 13 water tiles and, depending on which of the two tiles adjacent to the upper-left corner is a water tile, there are either 8 or 9 disconnected clusters of water tiles. In the case where the water tile is located directly below the upper left corner, there are 9 disconnected areas, and at least 8 additional water tiles would be needed to connect them, which would already result in 21 water tiles. In the case where the water tile is located directly to the right of the upper left corner, at least 7 additional water tiles are required to connect these. However, in this case, if we connect them with only 7 water tiles, there is no water tile that connects two clusters while simultaneously supplying the tile marked with a cross (which is not a water tile). Thus, at least one more water tile is needed, and therefore a minimum of 21 water tiles is required.
Case 2: If the yellow-colored tile is not a water tile, the situation is necessarily as shown on the right in Figure 47. The edge tile marked with a cross is not a water tile, since we only

have 5 water tiles at the edge, and it remains unsupplied unless the yellow tile becomes a water tile, which leads to a contradiction. Therefore, this case cannot exist.

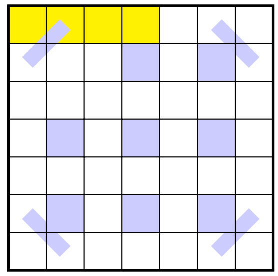


Figure 48: The situation with 5 water tiles at the edge when all 4 red-colored tiles are water tiles.

It remains to consider the situation where there are exactly 5 water tiles at the boundary and all 4 red tiles are water tiles. This case is shown in Figure 48. Due to symmetry, we can assume that one of the yellow tiles is the fifth water tile at the edge. Again, 3 of the tiles that are diagonally adjacent to the corners are necessarily water tiles.
In this case, there are 13 water tiles and at least 9 disconnected clusters of water tiles. To connect these, at least 8 additional water tiles are required. Thus, a minimum of 21 water tiles is needed.

Before addressing in Step 3 the cases with more than five boundary tiles, we next examine plantings in which the configuration of water tiles exhibits 4 so-called bends.

Step 2: Bends

We define the configuration shown in Figure 49, including all its reflected and rotated versions, as a *bend*.

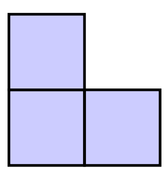


Figure 49: Illustration of a bend

Since the water tiles must be connected, we can visualize the process by starting with the center water tile and adding subsequent water tiles one by one in such a way that the water tiles always remain connected.
Now, we want to count the number of added water tiles that result in (at least) one bend.
Due to symmetry, we can assume that the second water tile is placed above the central one.
To supply the two corners on the left side of the room, at least one water tile must be added in a way that creates a bend to the left side. However, this can supply at most one of the two left corners. Therefore, an additional water tile must either be added to the central column to create a second bend to the left side, resulting in a U-shaped water-tile configuration open

to the left (*Case 1*, see Figure 50), or we can reach both left corners by using two bends in which the water tiles form a U-shape open upward or downward (*Case 2*, see Figure 51).
The same argument can be applied to the right corners. Therefore, the total number of water tiles that produce a bend must be at least 4.

We will now demonstrate that if this number is exactly 4, it is impossible to achieve a valid planting with fewer than 21 water tiles:

If we have Case 1, assuming symmetry, the left corners are supplied as shown in one of the grids in Figure 50. If, on the other hand, we are in Case 2, we can assume, due to symmetry, that the supply of the left corners occurs as in one of the grids in Figure 51.
The supply of the corners on the right side follows analogously to the left side. These possibilities involve reflections across the vertical axis of symmetry or the center point of the room.
Suppose we reach the two left and the two right corners by a combination of Cases 1 and 2. It is easy to see that if Case 1 occurs for the left or the right corners, not all tiles can be supplied without an additional bend as some of the tiles in the middle row will not be supplied yet. However, if, Case 2 occurs on both sides, there are inevitably at least 21 water tiles.

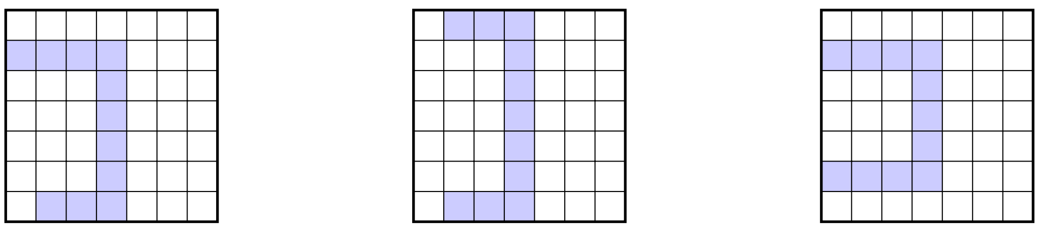


Figure 50: Supply of the left corners in Case 1, where water tiles form a U-shape open to the left.

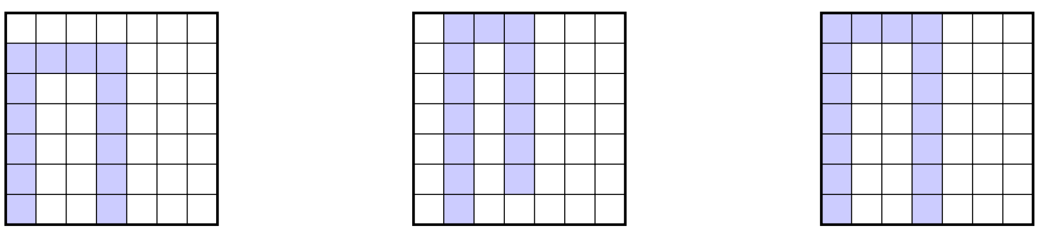


Figure 51: Supply of the left corners in Case 2, where water tiles form a U-shape open upward or downward.

The final scenario to consider is one where bends proceed from the central column to only one side (for instance, the right side, due to symmetry). In this instance, the water supply follows a spiral pattern, as exemplified in Figure 52. One can easily verify that in all such cases, at least 21 water tiles are required.
Step 3: Analysis of the remaining cases

In this final step, we examine how the total number of supplied tiles changes as we add water tiles one by one, following the method described previously:
Starting with only the central water tile, this tile itself and its 4 immediate neighbors are

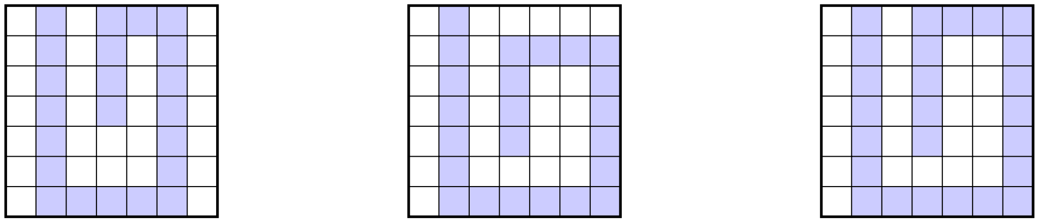


Figure 52: Examples of spiral-shaped water supply.

supplied - making a total of 5 supplied tiles. It can be easily observed that each additional water tile we add increases the total number of supplied tiles by at most 3.
However, if the addition of a water tile creates (at least) one bend, or if the added water tile is a boundary tile, the total number of supplied tiles increases by at most 2.
If a tile is both a boundary tile and a bend, the number of supplied tiles increases by at most 1.

Let w represent the total number of water tiles, r the number of water tiles at the boundary, and d the number of added tiles that create a bend. Based on the observations above, the maximum number of tiles that can be supplied is given by the formula:

$$5 + 3(w-1) - r - d \geq 49.$$

To find the minimum number of water tiles required, we rearrange the inequality for w:

$$w \geq \frac{47 + d + r}{3}$$

Following the conclusions from Step 1 and Step 2, we must now consider the cases where $r \geq 6$ and $d \geq 5$. Applying these values to the inequality, we find

$$w \geq \frac{58}{3} > 19.$$

This means that in these cases, just as we intended to show, there must be at least 20 water tiles.

Consequently, every valid planting of the room must contain at least 20 water tiles, which allows for a maximum of 29 Christmas trees. This confirms that answer 8 is correct.

Remark: Using the arguments from Step 1 (connecting disconnected clusters and symmetry considerations), it is also possible to show for 6 and 7 edge water tiles that at least 21 water tiles are required. By then excluding the only remaining case where $47 + d + r = 60$ (namely $r = 8$ and $d = 5$), it holds in all other cases that $w > 20$. This ultimately proves that any correct planting of the room can contain a maximum of 28 Christmas trees.

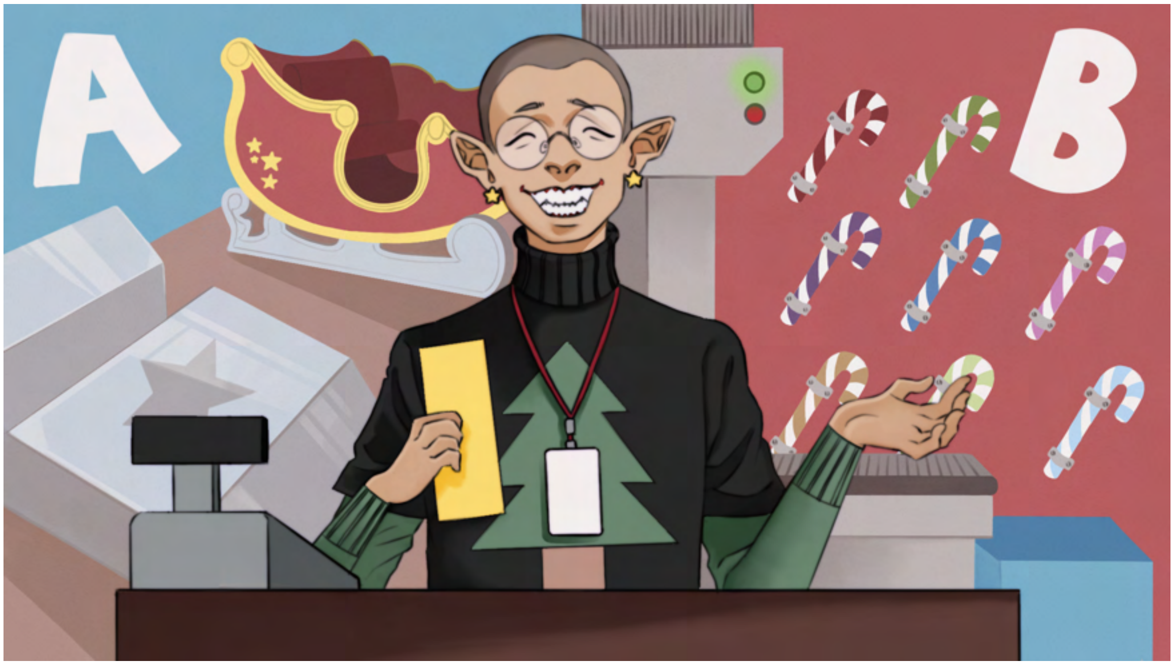


Illustration: Julia Nurit Schönnagel

19 The North Pole Museum

Author: Lucas van Kreveld

Challenge

Excitement is in the air at the North Pole Museum of Christmas Technology. For the very first time, the museum is opening its doors to the public — and since a huge crowd is expected, it will remain open around the clock for the first three days!
To showcase the ingenious sleigh innovations developed over the centuries, Santa established the first major exhibition space, Hall A: Here shine the greatest triumphs — from the classic wooden sleigh powered by reindeer to the short-lived jet-powered model, and the legendary — but unfortunately completely uncontrollable — AI prototype. Next door, in Hall B, the Candy Cane Engineering Hall, visitors discover how candy canes are manufactured, shaped, and designed for delicious delight.

The museum has three doors: one connecting the outside world to Hall A, one connecting the outside world to Hall B, and one connecting Hall A to Hall B. Visitors arrive at the popular museum at a rate of exactly 1 visitor per minute starting at the opening hour of the museum and continuing throughout the three days. All visitors want to visit both halls, but may take one of two different routes through the museum. Route 1 first visits Hall A, then Hall B, and then leaves, and route 2 first visits Hall B, then Hall A, and then leaves.
Arriving visitors alternate between these two routes: The first visitor starts with Hall A, then after 1 minute the next visitor starts with Hall B, then after 1 minute the next visitor starts with Hall A, and so on. Each visitor stays at Hall A for exactly 15 minutes, and at Hall B

for exactly 45 minutes (because of all the delicious candy cane samples).

Elf Alma is the 2025th visitor. Let k be the number of other visitors she sees during her visit to the museum. Here, a visitor is seen if at any time they are in the same hall with her, or meet while transitioning from one hall to the other. What is the units digit of k?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Solution

The correct answer is: 4.

The correct total number is **104**. One way to see this is to distinguish between visitors who choose *Route 1* and those who choose *Route 2*.

We first note that two visitors can never meet while changing between the halls at the same time. A visitor on Route 1 changes halls an odd number of minutes after arrival (15 min), and a visitor on Route 2 also changes after an odd number of minutes (45 min). Since Route-1 and Route-2 visitors arrive at times of opposite parity[9], their respective switching times also have opposite parity and therefore can never coincide.

Visitors on Route 1.
Among the visitors who take Route 1, Alma meets exactly those who arrive within 45 minutes before or after her own arrival. By assumption, these are precisely the visitors whom she encounters together in Hall B.
This gives 44 visitors. If we set Alma's arrival time to be 0, then these are the visitors who arrive at the times

$$-44, -42, \dots, -4, -2, 2, 4, \dots, 42, 44.$$

Visitors on Route 2.
Among the visitors who take Route 2, Alma meets exactly those who arrive within 60 minutes before or after her own arrival. The description of the routes implies that all these visitors are in the museum simultaneously with Alma and therefore at some point in the same hall.
This gives 60 visitors. If Alma's arrival time is again set to 0, then these are the visitors who arrive at the times

$$-59, -57, \dots, -3, -1, 1, 3, \dots, 57, 59.$$

Altogether, Alma meets

$$44 + 60 = 104$$

other visitors during her museum visit.

[9]Two integers have the same parity if both are even or both are odd.

Illustration: Zyanya Santuario

20 Too Many Sheep to Count

Author: Matthew Maat (Universiteit Twente)
Project: Combining algorithms for parity games & linear programming

Challenge

It seems like any other night. A small group of shepherds is sitting by a fire. The last rays of sunshine slowly get replaced by the light of a bright star in the middle of the sky. Unaware of what they would witness later that night, the men try to fall asleep.

As every shepherd knows, the best way to fall asleep is to count sheep. However, as they do this every night, they want to have some fun and try to arrange the sheep in a vertical *sheep-conga*, which we will define iteratively.

A vertical 1-sheep conga consists of just one sheep. Larger vertical sheep-congas can be constructed from smaller ones. In the following construction, sheep are idealized and represented as circles. For any integer $n > 1$, a vertical n-sheep-conga consists of one vertical $(n-1)$-sheep-conga and two identical horizontal $\lceil \frac{n}{2} \rceil$-sheep-congas,[10] where a horizontal sheep-conga is simply a vertical sheep-conga rotated by 90 degrees. The two horizontal sheep-congas are placed above and below the vertical sheep-conga and centered on the same vertical line, so that the final figure has both a horizontal and a vertical axis of symmetry.

[10] $\lceil x \rceil$ means the smallest integer that is at least as big as x, for example $\lceil 6.5 \rceil = 7 = \lceil 7 \rceil$

In Figure 53 you can see some examples of vertical sheep-congas. The brown sheep represent the vertical $(n-1)$-sheep-conga and the white sheep represent the horizontal $\lceil \frac{n}{2} \rceil$-sheep-conga of the respective n-sheep-conga.

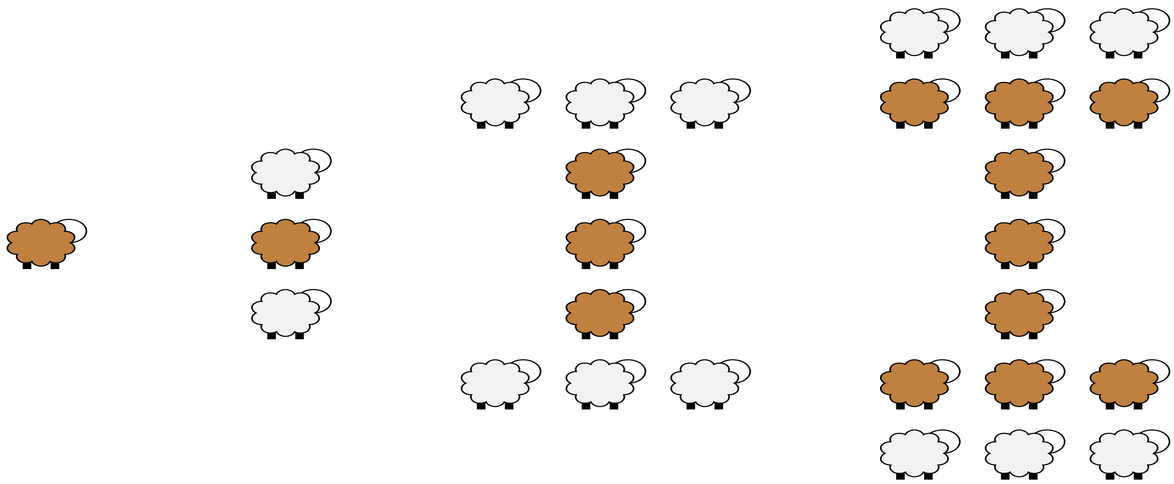


Figure 53: From left to right: a 1-sheep-conga, a 2-sheep-conga, a 3-sheep-conga and a 4-sheep-conga.

If C is the number of sheep in a (2025^5)-sheep-conga, which of the following statements is true?

A hint can be found below the answer choices.

(*The possible answers are on the next page.*)

Possible Answers:

1. $C \leq 2025^5$
2. $2025^5 < C \leq 2025^{20.25}$
3. $2025^{20.25} < C \leq 10^{100}$
4. $10^{100} < C \leq 10^{202.5}$
5. $10^{202.5} < C \leq 2.025^{2025}$
6. $2.025^{2025} < C \leq 5.202^{2025}$
7. $5.202^{2025} < C \leq 20.25^{2025}$
8. $20.25^{2025} < C \leq 2025!$
9. $2025! < C \leq \left(2025^5\right)^{2025^5}$
10. $C > \left(2025^5\right)^{2025^5}$

Hint: Think about powers of two.

Project Reference:

The function f, which assigns to each n the total number of sheep in an n-sheep conga, behaves similarly to the function $2^{(\log n)^2}$. In general, if a function f grows approximately like $2^{(\log n)^c}$ for some constant $c \geq 1$, we say that it has *quasi-polynomial growth*. The running time of the fastest known algorithms for finding optimal strategies in parity games also grows quasi-polynomially.

Solution

The correct answer is: 5.

Let $f(n)$ denote the number of sheep in an n-sheep-conga. We have a recursive relation for f:

$$\begin{aligned} f(1) &= 1 \\ f(n) &= f(n-1) + 2f\left(\left\lceil \frac{n}{2} \right\rceil\right) \text{ for all } n \geq 2 \end{aligned}$$

Note that f is *increasing*, meaning that $f(n+1) > f(n)$ for all positive integers n. We can use this property to estimate the value of $f(2025^5)$. As mentioned in the hint, it is easier to work with powers of two. We have $2^{54} < 2025^5 < 2^{55}$, and therefore know that

$$f(2^{54}) < f(2025^5) < f(2^{55})$$

First, we want to estimate $f(2^{55})$ from above. By repeatedly applying the recursive relation to the first summand $2^{55} - 1$ times, we obtain

$$\begin{aligned} f(2^{55}) &= f(2^{55}-1) + 2f(2^{54}) \\ &= f(2^{55}-2) + 4f(2^{54}) \\ &= f(2^{55}-3) + \big(4f(2^{54}) + 2f(2^{54}-1)\big) \\ &= \ldots \\ &= f(1) + \big(4f(2^{54}) + 4f(2^{54}-1) + \ldots + 4f(2) + 2f(1)\big) \\ &< 2^{56} \cdot f(2^{54}), \end{aligned}$$

where the last inequality follows from the fact that there are $2^{56} - 1$ terms, each at most $f(2^{54})$. In exactly the same way we can prove that

$$f(2^{54}) \leq 2^{55} \cdot f(2^{53}), \quad f(2^{53}) \leq 2^{54} \cdot f(2^{52}),$$

and so on. This gives us

$$f(2^{55}) < 2^{56} \cdot f(2^{54}) < 2^{56} \cdot 2^{55} \cdot f(2^{53}) < \ldots < 2^{56} \cdot 2^{55} \cdot \ldots \cdot 2^2 \cdot f(2^0) = 2^{1595}.$$

Here, the last equality follows from the laws of exponents and the fact that the sum of all natural numbers up to a number n is given by $\frac{1}{2}n(n+1)$. Therefore,

$$f(2025^5) < f(2^{55}) \leq 2^{1595} < 2.025^{2025}.$$

Next, we consider $f(2^{54})$ and want to find a good lower bound. By applying the recursive relation (to the first summand) 2^{53} times in a row , we obtain

$$\begin{aligned} f(2^{54}) &= f(2^{54}-1) + 2f(2^{53}) \\ &= f(2^{54}-2) + 4f(2^{53}) \\ &= f(2^{54}-3) + \big(4f(2^{53}) + 2f(2^{53}-1)\big) \\ &= \ldots \\ &= f(2^{53}) + \big(4f(2^{53}) + 4f(2^{53}-1) + \ldots + 4f(2^{52}+1)\big) \\ &> 2^{54} \cdot f(2^{52}), \end{aligned}$$

where in the last inequality we use the fact that there are $2^{54}+1$ terms, all of which are greater than $f(2^{52})$. In the same way as before, we also find that

$$f(2^{52}) > 2^{52} \cdot f(2^{50}),$$

and so on. This gives

$$f(2^{54}) > 2^{54} \cdot f(2^{52}) > 2^{54} \cdot 2^{52} \cdot f(2^{50}) > \ldots > 2^{54} \cdot 2^{52} \cdot \ldots \cdot 2^{2} \cdot f(2^{0}) = 2^{756}.$$

Again, the last equality follows from the laws of exponents and the formula for the sum of all natural numbers up to a number n (in this case $n = 27$), multiplied by two. Now we see that

$$10^{x} < 2^{y}$$

is equivalent to

$$\frac{x}{y} < \frac{\log 2}{\log 10} \approx 0.3 \quad \text{or} \quad x < 0.3y.$$

Hence,

$$f(2025^{5}) > f(2^{54}) > 2^{756} > 10^{756 \cdot 0.3} > 10^{202.5}.$$

Alternatively, one can easily compute, that $2^{10} > 10^{3}$ and thus

$$f(2025^{5}) > f(2^{54}) > 2^{756} > 2^{750} > 10^{3 \cdot 75} > 10^{202.5}.$$

Illustration: Friederike Hofmann

21 The Snow Elf's Christmastime Quest

Author: Prof. Dr. Daniel Gembris (Duale Hochschule Sachsen in Dresden)

Challenge

The diligent elf Willi has the important task of clearing the sports field of snow for the big Christmas party of the elves. He proceeds systematically: he divides the sports field into equally sized squares and clears one square completely before starting the next. The rest of the sports field, which cannot be divided into squares, he cleans afterwards. But it doesn't stop there - he wants to use the snow to create magnificent snowballs! Thanks to Willi's years of experience in forming snowballs, each of his snowballs turns out perfectly and has an exact diameter of 1.5 m.

The following information is given:

Information about the snow

- The sports field is equally covered with 5 cm of fresh snow.
- The snow has a density of 50 kg/m^3.
- Forming the snow into a snowball compresses it, increasing its density to 100 kg/m^3.

Willi's snow clearing strategy for one square

- **Division of the area and choice of lane width**

  The square with side length a is divided into parallel clearing lanes, all of which are parallel to one side of the square. The width b of a lane corresponds to the width of the snow shovel. This width is chosen such that Willi pushes at most 5 kg of snow at a time while clearing a lane. Willi therefore calculated the lane width according to his maximum pushing force.

  The lanes lie directly next to each other and together cover the entire area of the square. If the side length a is not an integer multiple of b, the last lane is narrower. In this case, it must be ensured that, when clearing this narrower lane, the snow of the surrounding squares remains untouched.

- **Movement while clearing a lane**

  Willi starts at one corner of the square. To position himself to clear the first lane he moves along the edge of the square by a distance of $\frac{b}{2}$, so that the snow shovel lies centered on the first lane. From this position, he clears the lane by pushing the snow of that lane in a straight line over the entire length a in the pushing direction up to the edge of the square.

  While clearing, the snow completely remains within the current lane. No snow is left behind and none ends up on neighboring lanes or squares. In the following considerations, the snow pushed to the edge is assumed to lie along the corresponding side of the square.

- **Change between to lanes**

  After a lane has been completely cleared, Willi first walks back along the same lane to the starting side of the square. He then moves perpendicular to the pushing direction by exactly one lane width b and positions himself at the beginning of the next lane. There, he repeats the clearing process following the same pattern.

  This procedure is continued lane by lane until all lanes of the square have been cleared. If the last lane is narrower, the lateral offset is adjusted according to its width.

- **Snow collection point and end of snow pushing**

  The snow collection point of a square is the corner that lies diagonally opposite the starting corner. After clearing the last lane, Willi does not walk back. Instead, he moves directly to the collection point. (Movements along the edge of the square are carried out independently of the snow located there.)

  There, Willi replaces the snow shovel with a bucket. The snow that has been pushed together along the edge of the square is then transported away and carried back to the snow collection point.

Information about the snow-transport

- Generally, Willi carries the snow in 5-kg portions; only during the final transport may the amount of snow collected be smaller.

- In order not to walk unnecessarily far before he has collected 5 kg of snow, he positions himself along the side of the square so that he can collect snow from both sides. Therefore, he moves along the side of the square that he pushed the snow to until he has passed 2.5 kg of snow (or has reached the end of the square). Willi then scoops 2.5 kg of snow from each side of his position, or whatever remains at the end of the square, and carries it to the snow collection point.

Willi's speed and time assumptions

- He moves at a constant speed of 1 m/s - both with and without a bucket or snow shovel.
- The time required to put down and pick up the bucket and shovel, to fill and empty the bucket, as well as to form the snowballs, is not taken into account.

Your Challenge

1. **Determine the side length** a of the square that is needed for a snowball.
2. **Determine the width** b **of the snow shovel** that Willi uses.
3. **Calculate the time** $t_{\textbf{total}}$ **Willi needs** to transport the snow from a single square to the respective snow collection point, i.e. the time to push the snow with the snow shovel as well as the time for the transport with the bucket. The activity is complete when Willi ends up standing at the collection point.

Give all results in the following units and with the following rounding:

- a in meters, rounded to one decimal place.
- b in meters, rounded to two decimal places.
- t_{total} in minutes, rounded to one decimal place.

Possible Answers:

1. $a = 8.4\,\text{m}$, $b = 0.22\,\text{m}$, $t_{\text{total}} = 10.1\,\text{min}$
2. $a = 8.4\,\text{m}$, $b = 0.22\,\text{m}$, $t_{\text{total}} = 13.4\,\text{min}$
3. $a = 8.4\,\text{m}$, $b = 0.22\,\text{m}$, $t_{\text{total}} = 15.2\,\text{min}$
4. $a = 8.4\,\text{m}$, $b = 0.24\,\text{m}$, $t_{\text{total}} = 10.1\,\text{min}$
5. $a = 8.4\,\text{m}$, $b = 0.24\,\text{m}$, $t_{\text{total}} = 15.2\,\text{min}$
6. $a = 9.0\,\text{m}$, $b = 0.22\,\text{m}$, $t_{\text{total}} = 10.1\,\text{min}$
7. $a = 9.0\,\text{m}$, $b = 0.22\,\text{m}$, $t_{\text{total}} = 13.4\,\text{min}$
8. $a = 9.0\,\text{m}$, $b = 0.24\,\text{m}$, $t_{\text{total}} = 10.1\,\text{min}$
9. $a = 9.0\,\text{m}$, $b = 0.24\,\text{m}$, $t_{\text{total}} = 13.4\,\text{min}$
10. $a = 9.0\,\text{m}$, $b = 0.24\,\text{m}$, $t_{\text{total}} = 15.2\,\text{min}$

Solution

The correct answer is: 5.

Step 1: Calculation of the mass of a snowball

The snowball has a diameter of 1.5 m. Therefore, its radius is:

$$r = \frac{1.5}{2}\,\mathrm{m} = 0.75\,\mathrm{m}\ .$$

The volume of the sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (0.75)^3\,\mathrm{m}^3 = \frac{9}{16}\pi\,\mathrm{m}^3 \approx 1.767\,\mathrm{m}^3\ .$$

The snowball consists of compacted snow with a density of $100\,\mathrm{kg/m}^3$. The mass of the snowball is therefore:

$$m_{\text{sphere}} = V \cdot \text{density} = \frac{9}{16}\pi\,\mathrm{m}^3 \cdot 100\,\mathrm{kg/m}^3 = \frac{225}{4}\pi\,\mathrm{kg} \approx 176.7\,\mathrm{kg}\ .$$

Step 2: Calculation of the amount of snow needed for the snowball

The fresh snow on the sports field has a density of $50\,\mathrm{kg/m}^3$ and a height of $5\,\mathrm{cm} = 0.05\,\mathrm{m}$. The volume of $\frac{225}{4}\pi\,\mathrm{kg}$ of fresh snow is therefore:

$$V_{\text{fresh snow}} = \frac{m_{\text{sphere}}}{\text{density}} = \frac{\frac{225}{4}\pi}{50}\,\mathrm{m}^3 = \frac{9}{8}\pi\,\mathrm{m}^3 \approx 3.534\,\mathrm{m}^3\ .$$

Given the height of 0.05 m, this volume corresponds to a base area of:

$$A = \frac{V_{\text{fresh snow}}}{\text{height}} = \frac{\frac{9}{8}\pi}{0.05}\,\mathrm{m}^2 = \frac{45}{2}\pi\,\mathrm{m}^2 \approx 70.68\,\mathrm{m}^2\ .$$

The side length of the square from which the snow is cleared to form one snowball is:

$$a = \sqrt{A} = \sqrt{\frac{45}{2}\pi}\,\mathrm{m} \approx 8.4\,\mathrm{m}\ .$$

Step 3: Calculation of the snow shovel width

The mass of snow per square meter is:

$$\text{mass per m}^2 = \text{density} \cdot \text{height} = 50\,\mathrm{kg/m}^3 \cdot 0.05\,\mathrm{m} = 2.5\,\mathrm{kg/m}^2\ .$$

The mass of snow that Willi pushes per lane is proportional to the area of the lane:

$$\text{snow mass per lane} = a \cdot b \cdot \text{mass per m}^2\ .$$

Since Willi can push at most 5 kg of snow, we obtain:

$$a \cdot b \cdot 2.5\,\mathrm{kg/m}^2 = 5\,\mathrm{kg}\ ,$$

$$b = \frac{5}{a \cdot 2.5}\,\mathrm{m} = \frac{5}{\sqrt{22.5\pi}\cdot 2.5}\,\mathrm{m} \approx 0.238\,\mathrm{m}\ .$$

Thus, the width of the snow shovel is:

$$b \approx 0.238\,\mathrm{m} \quad \text{(approximately 24 cm).}$$

Step 4: Calculation of the transport time

(a) Number of lanes
The number of lanes is determined by the side length of the square and the width of the snow shovel:

$$n = \frac{a}{b} = \frac{\sqrt{22.5\pi}}{\frac{5}{\sqrt{22.5\pi}\cdot 2.5}} = \frac{22.5\pi}{2} \approx 35.34 \implies n = 36.$$

Thus, Willi must clear 35 full lanes and one narrower lane.

(b) Snow clearing time
Willi clears the snow in parallel lanes. For each lane, he walks back and forth, meaning that he walks the length of each lane twice. The total length of all lanes is therefore:

$$\text{total length} = 2 \cdot n \cdot a = 2 \cdot 36 \cdot \sqrt{22.5\pi}\,\text{m} \approx 605.3\,\text{m}.$$

At a speed of $1\,\text{m/s}$, the clearing time is:

$$t_{\text{clearing}} = \frac{\text{total length}}{v} = \frac{72 \cdot \sqrt{22.5\pi}}{1}\,\text{s} \approx 605.3\,\text{s}.$$

The movement in the perpendicular direction when switching from one lane to the next increases the distance by $a = \sqrt{22.5\pi}\,\text{m}$, since Willi starts in one corner and ends in the diagonally opposite corner, thus covering the full side length of the square once. However, he does not need to walk back along the last cleared lane, which reduces the distance by $8.4\,\text{m}$. These changes in the total distance, and hence in the clearing time, cancel each other out.

(c) Snow transport
Willi carries the snow from each lane to the snow collection point using a massless bucket in portions of $5\,\text{kg}$. To reach the center of the i-th lane from the snow collection point, Willi must cover a distance of

$$d_i = (i - 0.5) \cdot b\ .$$

The time required for one transport trip (there and back) for the i-th lane is:

$$t_i = 2 \cdot d_i \cdot \frac{1}{v} = 2 \cdot (i - 0.5) \cdot b \cdot \frac{1}{v}\ .$$

The total transport time is the sum of the times for all lanes:

$$t_{\text{transport}} = \sum_{i=1}^{n} 2 \cdot (i - 0.5) \cdot \frac{b}{1\,\text{m}}\,\text{s}\ .$$

The factor $2b$ can be factored out of the sum:

$$t_{\text{transport}} = 2 \cdot b \cdot \sum_{i=1}^{n} (i - 0.5)\,\frac{\text{s}}{\text{m}}\ .$$

The sum of the first n natural numbers is:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} (i - 0.5) = \frac{n(n+1)}{2} - \frac{n}{2}\ .$$

Although the 36th lane contains less snow than the previous 35 lanes, this does not affect the time needed to transport its snow to the collection point. Substituting $n = 36$:

$$\sum_{i=1}^{n} i = \frac{36 \cdot 37}{2} = 666, \quad \sum_{i=1}^{n} 0.5 = \frac{36}{2} = 18 \ ,$$

$$\sum_{i=1}^{n} (i - 0.5) = 666 - 18 = 648 \ .$$

Thus, the transport time is:

$$t_{\text{transport}} = 2 \cdot \frac{5}{\sqrt{22.5\pi} \cdot 2.5} \cdot 648\,\text{s} \approx 308.3\,\text{s} \ .$$

Final results

1. **Side length of the square:** $8.4\,\text{m}$.
2. **Width of the snow shovel:** $0.238\,\text{m}$ (approximately 24 cm).
3. **Transport time:** $t_{\text{transport}} \approx 308.3\,\text{s}$.
4. **Clearing time:** $t_{\text{clearing}} \approx 605.3\,\text{s}$.
5. **Total time:**

$$t_{\text{total}} = t_{\text{clearing}} + t_{\text{transport}} \approx 605.3\,\text{s} + 308.3\,\text{s} = 913.6\,\text{s} \approx 15.23\,\text{minutes}.$$

Illustration: Zyanya Santuario

22 Weakly and Strongly Pulling Reindeer

Author: Max Klimm (TU Berlin)

Project: AA3-18
Evolution Processes for Populations and Economic Agents

Challenge

Santa sports a sleigh drawn by two reindeer that are selected at random from a large population of reindeer at the North Pole. Every reindeer chosen for sleigh duty is of one of two types: it either pulls hard or it barely pulls. If both reindeer pull hard, the sleigh flies gracefully across the sky, everybody is content, and both reindeer receive 4 piles of hay each. If one reindeer pulls hard while the other barely pulls, the sleigh still flies well, but it is very exhausting for the hard-pulling reindeer, who is too tired to eat and, hence, receives no hay while the other reindeer, as a freerider, receives 5 piles of hay. If both reindeer barely pull, the sleigh moves very slowly, Santa is dissatisfied, and the reindeer receive only the minimum amount of 1 pile of hay each.

We are interested in understanding how the proportion of the hard-pulling reindeer in the population evolves using what is called *replicator dynamics.* Denoting this proportion at time t by $x(t)$, we have that a reindeer randomly drawn at time t is hard-pulling with probability $x(t)$, and barely pulling with probability $1 - x(t)$. With this observation, it is easy to compute that a hard-pulling reindeer can expect

$$f_{\mathrm{h}}(t) = 4 \cdot x(t) + 0 \cdot (1 - x(t))$$

piles of hay when recruited for sleigh duty at time t. A similar equation can be obtained for the function $f_{\mathrm{b}}(t)$ describing the expected number of piles a barely pulling reindeer receives when recruited.
The central assumption in replicator dynamics is that the increase in the proportion of a type of reindeer is proportional to the excess in hay compared to an average reindeer from the population, if recruited for sleigh duty. More specifically, for the instantaneous rate of change $x'(t)$ (cf. remark 2) of the proportion of hard-pulling reindeer at time t we have the equation

$$x'(t) = x(t)\Bigg(f_{\mathrm{h}}(t) - \big(x(t) f_{\mathrm{h}}(t) + (1 - x(t)) f_{\mathrm{b}}(t)\big)\Bigg). \tag{5}$$

Which of the following statements is true?

Remark 1: For a given point in time t_0 and a value c there is exactly one function $x(t)$ satisfying $x(t_0) = c$ as well as equation 5.

Remark 2: The *instantaneous rate of change* (also called *derivative*) can be interpreted as follows:

- If $x'(t) = 0$ at all times t in a certain interval, then $x(t)$ is constant in this interval.
- If $x'(t) > 0$, there exists a time interval around t where $x(t)$ is increasing.
- Similarly, if $x'(t) < 0$, there exists a time interval around t where $x(t)$ is decreasing.

For a linear function described by $f(t) = at + b$ for real numbers a, b, at all times t, the instantaneous rate of change $f'(t)$ is equal to a, i.e. equal to the slope of the function.

(*The possible answers are on the next page.*)

Possible Answers:

1. The formula for the expected number of piles of hay a barely pulling reindeer receives at time t when recruited is

$$f_{\mathrm{b}}(t) = 5 \cdot (1 - x(t)) + 1 \cdot x(t).$$

2. When at time 0, all reindeer are hard-pulling, at some time t, not all reindeer will be hard-pulling.
3. When at time 0, all reindeer are barely pulling, it will stay like this forever.
4. When at time 0, there is an even split of hard-pulling and barely pulling reindeer, it will stay like this forever.
5. The proportion of hard-pulling reindeer at time 1 is always the same, no matter what the initial proportion of hard-pulling reindeer at time 0 is.
6. For some proportion of hard-pulling reindeer at time 0, the proportion of hard-pulling reindeer is increasing within a certain period of time.
7. Santa can choose an initial proportion $x(0)$ of hard-pulling reindeer time 0 with $0 \leq x(0) \leq 1/2$, so that the proportion of hard-pulling reindeer stays above $1/4$ for all times.
8. Suppose Santa changes his feeding practice so that in every scenario every reindeer receives twice as much hay as before. This does not influence the dynamic of the proportion, no matter the initial proportion at time 0.
9. Suppose Santa changes his feeding practice so that every reindeer receives one additional pile of hay. This influences the dynamic of the proportion for some initial proportion.
10. Suppose Santa changes his feeding practice so that every reindeer always obtains 1 pile of hay no matter the sleighing performance. Then, for all initial proportions at time 0, after some time, all reindeer will be barely pulling.

Project Reference:

The replicator dynamics is an example of a question studied in the field of evolutionary game theory. This area studies dynamical systems that describe both the dynamics of evolutionary forces that describe the development of different genotypes or phenotypes of a species, and the dynamics of different behaviors by economic actors. The main objects of study of the project AA3-18 are mathematical processes that describe both how different genotypes or different economic behavioral patterns emerge over time. This research area, where similar mathematical models are used to describe seemingly unrelated processes in two disconnected application domains in biology and economics, is an interesting example of the power of mathematics, aiming for a general understanding of fundamental processes that may have completely different interpretations, depending on the application in question.

Solution

The correct answer is: 3.

Before giving detailed solutions, it is helpful to compute $f_b(t)$ for each t:
At time t, a barely pulling reindeer, which is recruited for sleigh duty, encounters a hard-pulling reindeer with probability $x(t)$, which leads to 5 piles of hay, and a barely pulling reindeer with probability $1-x(t)$, which leads to 1 pile of hay. In conclusion, the expected number of piles of hay a barely pulling reindeer receives when recruited is

$$f_{\mathrm{b}}(t) = 5 \cdot x(t) + 1 \cdot (1 - x(t)).$$

Now, we can substitute the expressions for $f_{\mathrm{b}}(t)$ and $f_{\mathrm{h}}(t)$ in equation 5 and conclude that the replicator dynamics is characterized by the (differential) equation

$$\begin{aligned} x'(t) = x(t)\Big(&4x(t) + 0(1-x(t)) \\ &- x(t)\Big(4x(t) + 0(1-x(t))\Big) - (1-x(t))\Big((5x(t) + 1(1-x(t))\Big)\Big) \\ = x(t)\Big(&4x(t) - 4x(t)^2 - 5x(t) + 5x(t)^2 - 1 + 2x(t) - x(t)^2\Big) \\ = x(t)\Big(&x(t) - 1\Big) \\ = x(t)&^2 - x(t) \end{aligned}$$

for all $t \geq 0$.

Now we can evaluate the answers one by one:

1. *This answer is wrong.* Above, we have computed the expected number of piles of hay a barely pulling reindeer receives at time t when recruited for sleigh duty to be

$$f_{\mathrm{b}}(t) = 5 \cdot x(t) + 1 \cdot (1 - x(t)).$$

2. *This answer is wrong.* When at time 0 all reindeer are hard-pulling, we have $x(0) = 1$. Moreover, if at any time t_0 all reindeer are hard-pulling, with equation 5, we then have

$$x(t_0) = 1, \quad x'(t_0) = 0.$$

Especially, $x(t) \equiv 1$ is a solution to $x(0) = 1$ and equation 5. By remark 1, this is the only solution and therefore the proportion of hard-pulling reindeer is 1 for all times.

3. *This answer is correct.* When at time 0 all reindeer are barely pulling, we have $x(0) = 0$. Moreover, if at any time t_0 all reindeer are barely pulling, with equation 5, we then have

$$x(t_0) = 0, \quad x'(t_0) = 0.$$

Especially, $x(t) \equiv 0$ is a solution to $x(0) = 0$ and equation 5. By remark 1, this is the only solution and therefore the proportion of hard-pulling reindeer is 0 for all times.

4. *This answer is wrong.* We have $x(0) = 1/2$. With the equation above, we obtain

$$x'(0) = \frac{1}{2}\left(\frac{1}{2} - 1\right) = -\frac{1}{4},$$

so the proportion of hard-pulling reindeer will decrease.

5. *This answer is wrong.* We have computed before that $x(1) = 0$ when $x(0) = 0$. Next, consider the case that $x(0) = 1/2$, that is, that half of the reindeer are hard-pulling at time 0. Above we showed that

$$x'(t) = x(t)^2 - x(t).$$

The right-hand side of this equation is a quadratic expression in $x(t)$ that takes its minimum when $x(t) = 1/2$ and this minimum is $-1/4$. Moreover, the roots are at 0 and 1. Because of $0 \leq x(t) \leq 1$ for all t, we can conclude that $x'(t) \in [-1/4, 0]$[11]. This implies that the proportion of hard-pulling reindeer can only decrease, but not with a larger rate than 1/4. Hence, we can conclude that, when $x(0) = 1/2$, $x(1) \in [1/4, 1/2]$ and especially $x(1) \neq 0$.

6. *This answer is wrong.* As argued in the evaluation of the previous answer, we have $x'(t) \in [-1/4, 0]$ for all $t \geq 0$, so the population of hard-pulling reindeer can never increase.

7. *This answer is wrong.* Suppose there is an initial proportion $z \in [0, 1/2]$ for which this is the case. Then, since for the resulting dynamics we have that $x(t) \geq 1/4$ for all $t \geq 0$, $x(t)$ is non-increasing, and $x(0) = z \leq 1/2$, we can conclude that $x(t) \in [1/4, 1/2]$ for all $t \geq 0$, and hence, $x'(t) \in [-1/4, -3/16]$. Thus, $x(t)$ decreases with a rate of at least 3/16, and hence, we have $x(t) \leq 1/8$ for $t \geq 2$, a contradiction to the assumption that $x(t) \geq 1/4$ for all $t \geq 0$.

8. *This answer is wrong.* Under the new feeding practice, we have

$$\begin{aligned} f_{\mathrm{h}}(t) &= 8x(t) \quad \text{and} \\ f_{\mathrm{b}}(t) &= 10x(t) + 2 \cdot (1 - x(t)). \end{aligned}$$

With the same computations as before, we obtain

$$\begin{aligned} x'(t) &= x(t)\Big(8x(t) - x(t) \cdot 8x(t) - (1 - x(t))\Big(10x(t) + 2(1 - x(t))\Big)\Big) \\ &= x(t)\Big(8x(t) - 8x(t)^2 - 10x(t) + 10x(t)^2 - 2 + 4x(t) - 2x(t)^2\Big) \\ &= 2x(t)\Big(x(t) - 1\Big) \\ &= 2(x(t)^2 - x(t)). \end{aligned}$$

This defines a different dynamic where the proportions change at double speed compared to the original one.

[11] $[a, b]$ denotes the set of all real numbers x with $a \leq x \leq b$.

9. *This answer is wrong.* Under the new feeding practice, we have

$$f_{\mathrm{h}}(t) = 5x(t) + 1 \cdot (1 - x(t)) \quad \text{and}$$
$$f_{\mathrm{b}}(t) = 6x(t) + 2 \cdot (1 - x(t)).$$

With the same computations as before, we obtain

$$\begin{aligned} x'(t) &= x(t)\Big(5x(t) + 1(1 - x(t)) \\ &\quad - x(t)\Big(5x(t) + 1(1 - x(t))\Big) - (1 - x(t))\Big(6x(t) + 2(1 - x(t))\Big)\Big) \\ &= x(t)\Big(4x(t) + 1 \\ &\quad - 5x(t)^2 - x(t) + x(t)^2 - 6x(t) + 6x(t)^2 - 2 + 4x(t) - 2x(t)^2\Big) \\ &= x(t)\Big(x(t) - 1\Big). \end{aligned}$$

Thus, the dynamics are the same as before. (The intuitive reason is that the dynamics is based on the differences of the payoff compared to the average, so that constant shifts cancel out.)

10. *This answer is wrong.* It is straightforward to compute that $x'(t) = 0$ for all $t \geq 0$, so the intitial proportion $x(0)$ of hard-pulling reindeer will be kept for all times. In particular, if we start with $x(0) = 1/2$, we will have $x(t) = 1/2$ for all $t \geq 0$.

Illustration: Julia Nurit Schönnagel

23 Mr. Johnson's Broken Christmas Ornaments

Authors: Zoe Geiselmann and Kevin Kühn
Project: EF-LI-Opt-1

Challenge

Mr. Johnson is a Christmas enthusiast who has five favorite Christmas tree ornaments that he built himself! Sadly, last year, two of them were smashed by his cat Rhombi. This year, to save Christmas Eve, he has to rebuild the two broken ones.

Luckily, he still knows where to buy the nets he needs to craft these ornaments. The nets are flat versions of the ornaments. He has to fold and glue them in the correct places to create three-dimensional objects. He buys the same pack of five nets that he had bought years ago, from which he can craft the five different ornaments that brought him so much joy - but he still has three whole ornaments that look brand new!

In Figure 54 you can see Mr. Johnson's remaining whole ornaments and in Figure 55 the five nets (A-E) he just bought. Help the old man keep track of his ornaments: Which two of the nets create ornaments that are **not** among those that he already has?

(*The possible answers are on the next page.*)

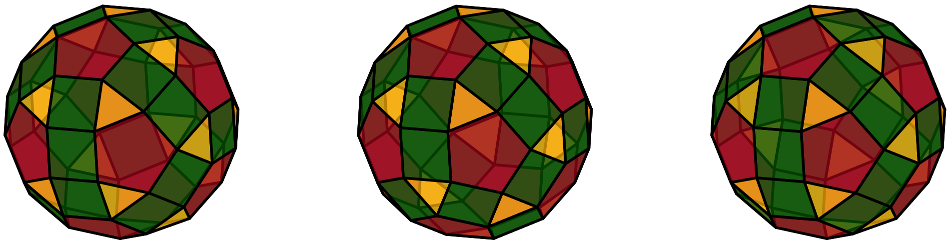
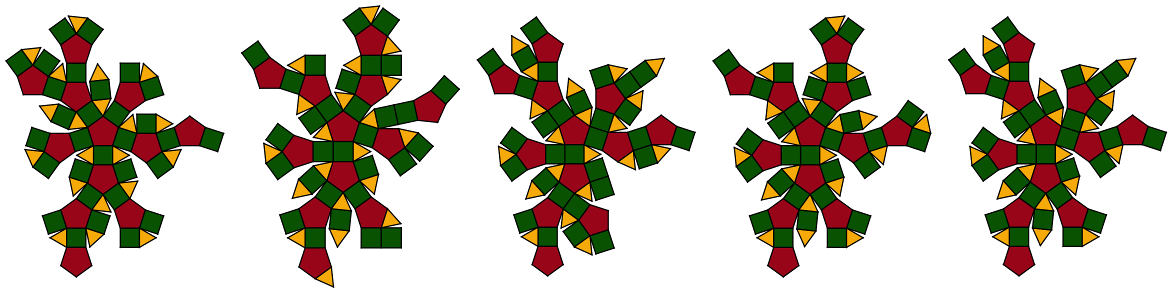


Figure 54: The three remaining intact Christmas ornaments.



A B C D E

Figure 55: The five nets (A–E) used to construct the Christmas ornaments.

Possible Answers:

1. A and B
2. B and E
3. A and C
4. C and D
5. B and D
6. A and E
7. A and D
8. B and C
9. D and E
10. C and E

Project reference:

In the project EF-LI-Opt-1, algebro-geometric objects are studied with combinatorial and discrete methods. Polytopes naturally arise in this intersection of disciplines, three of which are depicted in the challenge.

Solution

The correct answer is: 7.

The solids (called *polytopes*) all consist of 20 triangles, 30 squares, and 12 pentagons. The ones corresponding to the nets A and D are missing. In net A, there are no squares that share a line which is surely the case for all of the polytopes shown in Figure 54.
Three aligned squares exist only in the nets C and E, hence they must correspond to the first and last polytope(s). Only the nets B and D are candidates for the middle polytope. At the bottom of net D, after gluing corresponding sides together, one can see the configuration of triangles and squares as shown in Figure 56.

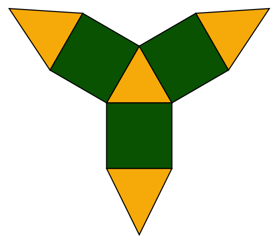


Figure 56: Configuration of triangles and squares.

This configuration does not occur in the middle ornament, therefore B must be the corresponding net.

You can see the final ornaments in Figure 57, after Mr. Johnson has rebuilt the polytopes.

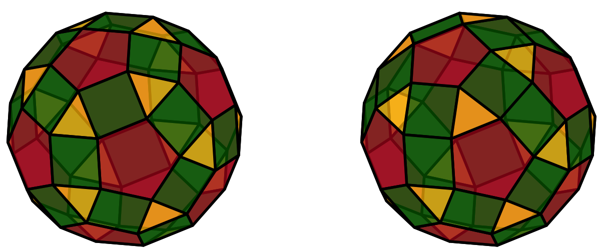


Figure 57: The final ornaments created by the nets A and D.

Bonus task for those that like Christmas polytopes as much as Mr. Johnson: Deduce which ornaments correspond to C and E.

The polytopes from this challenge are among the so called *Archimedean* and *Johnson* solids, related to the *Platonic* solids you may have encountered in school. These are fascinating classes of polytopes that are still being studied by researchers today! For those curious about polytopes, we recommend this article.
A similar game version of puzzles like the one above can be found on www.matchthenet.de.

Illustration: Mar Curcó Iranzo

24 Spiral Poem Competition

Author: Mar Curcó Iranzo

Challenge

It's the 24th of December, and to celebrate Christmas Eve the elves of Santa's workshop are organizing a special poem competition. The rules of the competition are as follows.

Rule 1: If the poem has n stanzas, then each stanza must comprise n lines.

Rule 2: Each word ending a line in a stanza needs to be different from all other line-ending words within the stanza.

Rule 3: The lines of each stanza must end with the same words as the lines in the previous stanza, following the order given by the **spiral permutation**, which is described below.

For example, if we are writing a poem with 5 stanzas, the first stanza will have 5 lines, each of which will end in a different word. As in figure 58, we call the ending words by W_1, W_2, W_3, W_4, W_5, in order.
If we now draw a spiral starting from the last word W_5 as indicated in figure 59, the order of the spiral meeting the words determines the order of appearance of the words in the next stanza. That is, stanza number two should have lines with ending words W_5, W_1, W_4, W_2, W_3, in this order. Next, the order of the ending words of stanza number three will be determined by drawing the spiral permutation with the end words of stanza number two. This iteration is represented in figure 60.

Figure 58: Schematic listing of the five ending words.

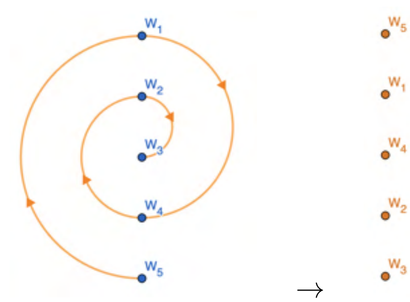


Figure 59: The spiral permutation.

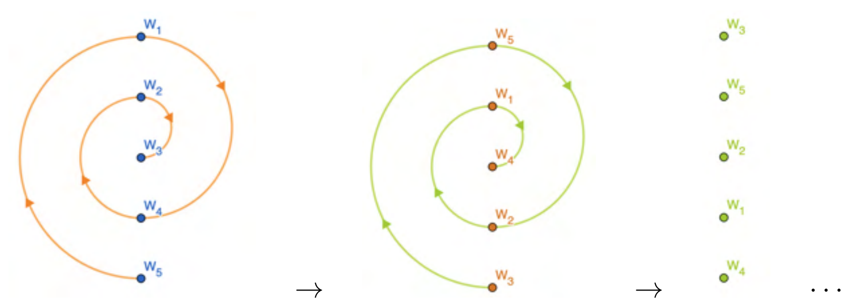


Figure 60: Iteration of the spiral permutation.

For a poem with n stanzas – and n lines, with n end words – the spiral permutation Q_n can be described mathematically as follows: If in a certain stanza, which is not the last, a word ends line m, then in the next stanza it ends line

$$Q_n(m) = \begin{cases} 2m & \text{if } m \leq n/2 \\ 2n - 2m + 1 & \text{if } m > n/2. \end{cases}$$

Rule 4: The order of the ending words must be different in every stanza of the poem.

Notice that, with this last rule, we cannot write a poem with n stanzas that satisfies the rules of the competition for every number n. For example, for $n = 4$, we would have the situation in figure 61. That is, after three iterations, we would end up with the same word order as in

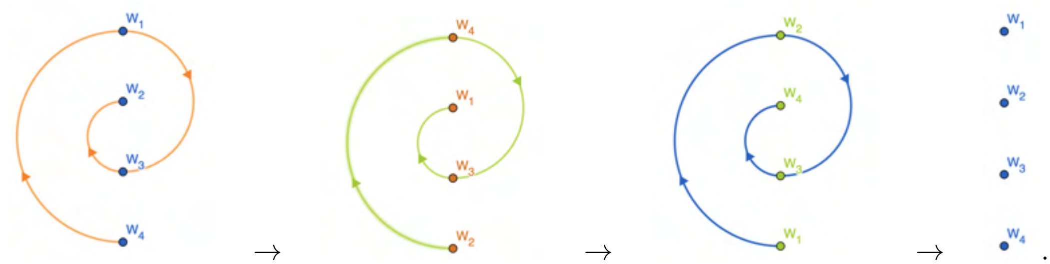


Figure 61: Counterexample to rule 4 for $n = 4$.

the first stanza. This would contradict the last rule of the competition.

If the poems of the competition have a maximum of 10 stanzas, what are all the possible numbers of stanzas that the poems can have?

Possible Answers:

1. $1, 2, 3, 5, 6$ and 7.
2. $1, 2, 3, 5, 6$ and 8.
3. $1, 2, 3, 5, 6$ and 9.
4. $1, 2, 3, 5, 6$ and 10.
5. $1, 2, 3, 5, 7$ and 9.
6. $1, 2, 3, 5, 8$ and 10.
7. $1, 2, 5, 6, 7$ and 8.
8. $1, 2, 5, 6, 7$ and 9.
9. $1, 2, 5, 6, 7$ and 10.
10. $1, 2, 5, 6, 8$ and 10.

Extra: We asked one of the elves to write a poem fulfilling all the rules of the competition as an example. See below:

Snow falls hard on the tall pine tree
Santa trips because he slips on milk
He asks the elves for help with the lost gift
Mrs. Claus calls them with a loud bell
And all eyes turn to Pip, the shy elf

Santa says, “Can you help us, Pip the elf?”
Pip nods and points at Santa’s map by the tree
The map shows numbers and arrows like a bell
Santa reads the map clue: “Find the cup of milk”
Rudolph finds it fast and wins a tiny gift

The milk clue contains a math puzzle gift
Can it be solved with the help of Pip the elf?
He sets down the riddle while Rudolph drinks the milk
The puzzle says, “Next clue under the tree”
Mrs. Claus laughs and rings her sweet bell

The clue reads “Footprints will bring you to the bell”
They are close to finding the lost gift!
Pip cheers and claps by the bright glowing tree
“Rudolph, stop licking milk!” shouts the shy elf
Santa looks around and sees footprints of milk

They rush outside behind the trace of milk
The prints lead them toward the chiming bell
Mrs. Claus claps, impressed with the shy elf
And with a soft smile, hands Pip the lost gift
Finally the gift is back, beside the glowing tree

Solution

The correct answer is: 3.

All the possible numbers of stanzas that the poems in the competition can have are $2, 3, 5, 6$ and 9. We will prove this by ruling out the other numbers and showing that it is in fact possible for a poem to have $2, 3, 5, 6$ or 9 stanzas.

First, a poem with one stanza fulfills all four rules. Furthermore, it was shown in the challenge that a spiral poem cannot have 4 stanzas.

For $n = 7$, the spiral permutation is drawn in figure 62 We can check that $Q_7(1) = 2, Q_7(2) = 4, Q_7(4) = 7$ and $Q_7(7) = 1$. On the other hand, we have $Q_7(3) = 6$ and $Q_7(6) = 3$. Furthermore, it is $Q_7(5) = 5$. This means that, following the spiral, after 4 iterations, all the words would be back to their initial position. Hence, stanza number 5 would have the same order of end words as the first stanza, contradicting the last rule of the competition.

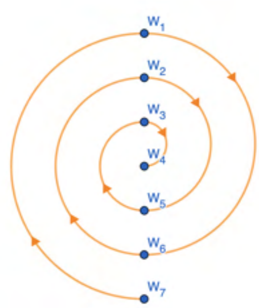


Figure 62: The spiral permutation for $n = 7$

For $n = 8$, the spiral permutation is drawn in figure 63 We can check that $Q_8(8) = 1$, $Q_8(1) = 2$, $Q_8(2) = 4$ and $Q_8(4) = 8$. On the other hand, we have $Q_8(7) = 3$, $Q_8(3) = 6$, $Q_8(6) = 5$ and $Q_8(5) = 7$. This means that, following the spiral, after 4 iterations, all the words would be back to their initial position. Hence, stanza number 5 would have the same order of end words as the first stanza, contradicting the last rule of the competition.

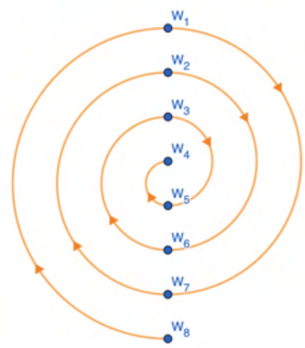


Figure 63: The spiral permutation for $n = 8$

For $n = 10$, we have something similar: the corresponding spiral permutation is drawn in figure 64. We can check that $Q_{10}(1) = 2, Q_{10}(2) = 4, Q_{10}(4) = 8, Q_{10}(8) = 5, Q_{10}(5) = 10$ and $Q_{10}(10) = 1$. On the other hand, we have $Q_{10}(3) = 6, Q_{10}(6) = 9$ and $Q_{10}(9) = 3$. Furthermore, it is $Q_{10}(7) = 7$. This means that, following the spiral, after 6 iterations, all the words would be back to their initial position. Hence, stanza number 7 would have the same order of end words as the first stanza, contradicting the last rule of the competition.

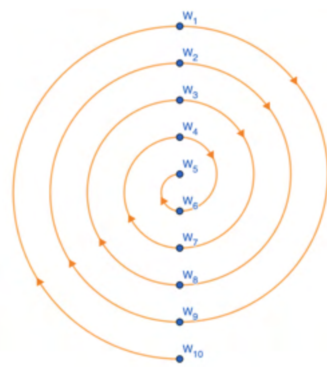


Figure 64: The spiral permutation for $n = 10$

Now, all numbers except of $2, 3, 5, 6$ and 9 are ruled out. We will check that it is in fact possible for a poem in our competition to have $2, 3, 5, 6$ or 9 stanzas:
The case $n = 5$ is clear, because of the example given in the challenge.
To check the other numbers, we track in which verse W_1 is positioned within the stanzas by repeatedly applying the spiral permutation:
For $n = 2$ we have $Q_2(1) = 2$ and $Q_2(2) = 1$.
For $n = 3$ we have $Q_3(1) = 2, Q_3(2) = 3$ and $Q_3(3) = 1$.
For $n = 6$ we have $Q_6(1) = 2, Q_6(2) = 4, Q_6(4) = 5, Q_6(5) = 3, Q_6(3) = 6$ and $Q_6(6) = 1$.
Finally, for $n = 9$ we have $Q_9(1) = 2, Q_9(2) = 4, Q_9(4) = 8, Q_9(8) = 3, Q_9(3) = 6, Q_9(6) = 7, Q_9(7) = 5, Q_9(5) = 9$ and $Q_9(9) = 1$.
We can see that for these values of n, the end word W_1 is positioned differently in each stanza. Hence, following the spiral, we get n stanzas with different order of end words.

Remark: In general, the numbers n satisfying the rules of the competition are the so called *Quenau-Arnaut numbers*. The origin of these numbers can be found in a medieval poem by the troubadour Arnaut Daniel. His poem, called "Lo ferm voler" was a sextine that was constructed according to the rules of our competition: a poem of 6 stanzas, 6 lines per stanza, and each stanza has 6 end words that permute according to the spiral permutation into the next one. The mathematician Raymond Queneau generalized Arnaut's construction to arbitrary numbers n and asked which n satisfy the property given by Rule 4. For a survey on Queneau-Arnaut numbers and their relation to other areas of mathematics and other fields, you can check "How a medieval troubadour became a mathematical figure" from M.P. Saclolo.