20 Years of Math Calendar: The Most Beautiful Problems and Solutions

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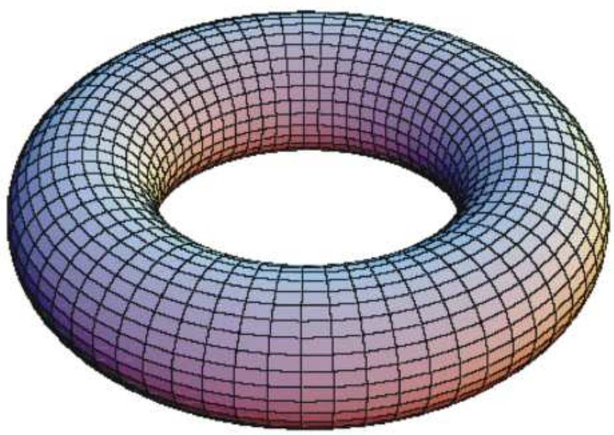
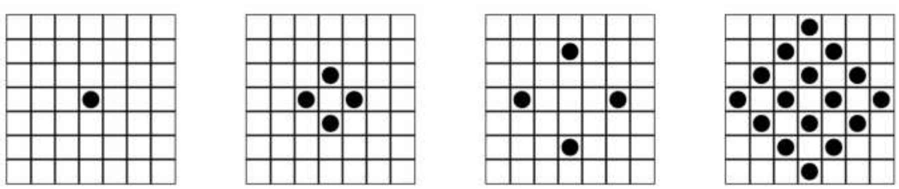
2004 News from Zellularien

Challenge

The planet "Zellularien" is different from Earth in that it is not a spherical world, but a "curly-shaped" one, resembling a car tire (see picture). Since its inhabitants are practical, they have divided their globe into small cells, each along the edges of 1024 lines of longitude and 1024 lines of latitude, so that each of the $1024 \cdot 1024 = 1048576$ inhabitants has exactly four neighbors: one to the North, one to the South, one to the East, and one to the West. If you were to draw an atlas of Zellularien on graph paper, a square on the left edge of the paper would have the cell at the same height on the right edge as its Western neighbor.



Unfortunately, a Zellularian has brought back an extremely dangerous and voracious "Kroll" from a visit to a neighboring planet. These otherwise cute little creatures multiply rapidly, consume large amounts, and even devour each other. Within a month, one Kroll in a cell becomes exactly four, which then migrate in the four neighbouring directions and leave the cell empty. If they encounter other Krolls, two Krolls will devour each other, meaning that either only one or no Kroll can inhabit a cell—until it multiplies again, and the Kroll and its three offspring migrate in the four directions.
The Kroll populations after the first four months thus appear as follows (empty cells are not marked, cells with one Kroll are marked with a dot):



In the second month, for instance, one Kroll from every neighbour of the center migrates to the center. However, since four Krolls remain there, they all devour each other, leaving the central cell empty.

Question: Will the Zellularians ever be freed from the Kroll plague?
Possible answers:

1. Yes, after 10 months
2. Yes, after 511 months
3. Yes, after 512 months
4. Yes, after 1023 months
5. Yes, after 1024 months
6. Yes, after 2048 months
7. Yes, after 1048576 months
8. No, never

Hint: Try working through this on a smaller grid of $4 \cdot 4$, $8 \cdot 8$ or $16 \cdot 16$ squares. Pay special attention to the Kroll distribution after 3, 7, and 15 months. It is important that the length and width of Zellularien are powers of two ($2^{10} = 1024$) and that the edges are connected, meaning that a Kroll on the far left has an offspring that migrates to the far right.

Solution

The correct answer is: 3 "Yes, after 512 months".

It's best to consider the following (with pencil and paper for simulation). You only need to think about the first four generations:

After three steps, you get a checkerboard pattern filled with Krolls. After four steps, you then have four isolated Krolls, each at distance four from the center (on the map). What happens after another three steps?
Since the Krolls can each spread out a maximum of one cell per step, these four cannot influence each other, so they all behave just like the original Kroll-—after three more steps, four copies of the checkerboard-patterned rhombus thus emerge. Since they fit exactly side by side, a large rhombus of double "side length" (the number of Krolls on the edge, with corners counted) is created. One more step further, at time step eight, only the Krolls at the corners remain, four of them, each at a distance of eight from the center.
From now on, it should be clear how the game continues: After each $2^k - 1$ steps, a rhombus of "side length" 2^k. One step further results in four Krolls, each at a distance of 2^k from the center.
Now it gets interesting: Since the Kroll world is exactly $1024 = 2^{10}$ squares "big" and curved like a ring, after $2^{10-1} - 1 = 511$ steps, a truly enormous rhombus of "side length" 2^9 is formed, which covers the planet almost entirely, leaving only a small edge of one square. One step later, only the four Krolls at the corners survive (each at a distance of 2^9 from the center), but they are positioned exactly on top of each other in the remaining

edge ($2 \cdot 2^9 = 2^{10} = 1024$), as the planet is "ring-shaped" and closed. Therefore, the four Krolls devour each other, and the Krolls are gone after $2^9 = 512$ steps!

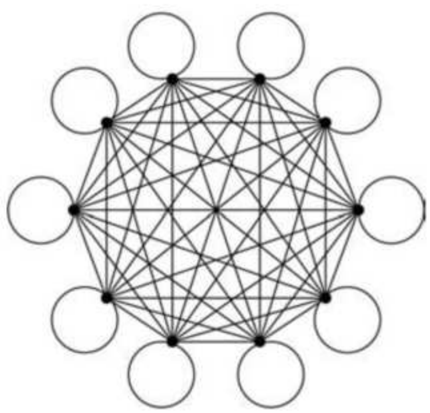
2005 Domino Snakes

Authors: Frank Lutz, Brigitte Lutz-Westphal
Project: G5*, G6

Challenge

Still so long until Christmas! Sebastian is passing the time by creating domino snakes. He lays the domino pieces in a line so that adjacent pieces touch each other with the same number:

In his first attempt, some pieces are left over. He wonders if he could have used these pieces if he had started more cleverly. He plays with a set of 9-dominoes, which contains exactly one piece for all possible pairs of numbers from 0 to 9.
What is the least number of unused pieces when constructing such a domino snake?
Hint: A Christmas star hangs in the window of Sebastian's room. It points the way to the solution! There is a mysterious relationship between the star and the dominoes. Furthermore, Sebastian just drew a "House of Nikolaus."



Possible answers:

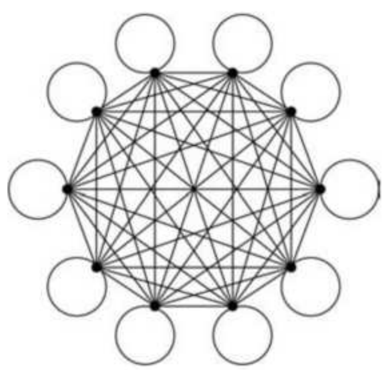
1. None; all pieces can be laid in a snake if done cleverly.
2. At least one piece always remains.

3. At least two pieces always remain.
4. At least three pieces always remain.
5. At least four pieces always remain.
6. At least five pieces always remain.
7. At least six pieces always remain.
8. At least seven pieces always remain.
9. At least eight pieces always remain.
10. At least nine pieces always remain.

This task comes from discrete mathematics and is related to planning optimal paths. Topics in discrete mathematics are very well suited for mathematics education. In two MATHEON projects
(Project G5*, http://www.math.tuberlin.de/westphal/projekt/ and Project G6, http://www.math.tu-berlin.de/didaktik/tikiindex.php?page=Matheon+G6), concepts and materials for teaching combinatorial optimization are being developed.

Solution

The correct answer is: 5
The 9-domino game has 55 pieces, which can be understood by counting, or calculated using the formula $\frac{(n+1)\cdot n}{2}$ (n is the number of different digits, which in this case is 10). The Christmas star (which is a graph) has 55 edges, so there is an edge for every piece. The nodes represent each of the numbers from 0 to 9. For example, if an edge connects nodes 1 and 5, the corresponding domino piece is the one with the number pair $\{1, 5\}$.



To lay a snake means to search for a path in the graph without repeating edges (since there is only one of each piece). The House of Nikolaus also looks for such a path. There, one can draw all edges in one stroke. Laying all dominoes in a snake would mean being able to

traverse all edges of the graph in a single path. This is not possible with the 9 dominoes because the corresponding graph only has nodes of "odd degree" (i.e., with an odd number of edge ends). Nodes with an odd degree have the property that, when traversing the edges, one eventually gets stuck at that node and has no edge available to leave the node.
In the House of Nikolaus, there are only two nodes of odd degree, so one can start or end there. In the domino graph, one can also choose two nodes with odd degree to be the start and end of the snake. However, there will still be 8 nodes of odd degree remaining. Now, if one removes one edge end from each of these nodes (which amounts to a total of 4 edges running between two of these odd nodes), the graph is "repaired," and a path without edge repetitions can be found, starting at one remaining odd degree node and ending at the other. The 4 removed edges correspond to 4 domino pieces. Of course, one can also remove more edges to make all but two nodes even degree, ensuring that at least 4 pieces always remain.
The sought path is called an "Euler path." Euler paths play a role in planning routes for waste collection or mail delivery, as well as in technical manufacturing.

2006 The Elven City

Authors: Gregor Wünsch, Janina Brenner, Christian Liebchen
Projects: B 15, B 16

Challenge

The most important helpers of Santa Claus, the elves, have a problem! Their land has been flooded. Nevertheless, the 64 elven families keep their spirits up. Since elves traditionally build their houses on pillars above the ground, fortunately, everything has remained dry. They quickly build a boat for each family and also want to build bridges so that every house is reachable from every other house along a path of bridges. So far, so good!
However, since their land is now in the middle of the sea after the flood, the clever elves think the following: *"We should build the bridges so that every part of our land can be reached from the sea by boat."* This means that no area should be enclosed by a circle of bridges. And so the elves start building. Just a few days later, the connection of all houses with bridges is complete. The elves' solution is illustrated in Figure 1.

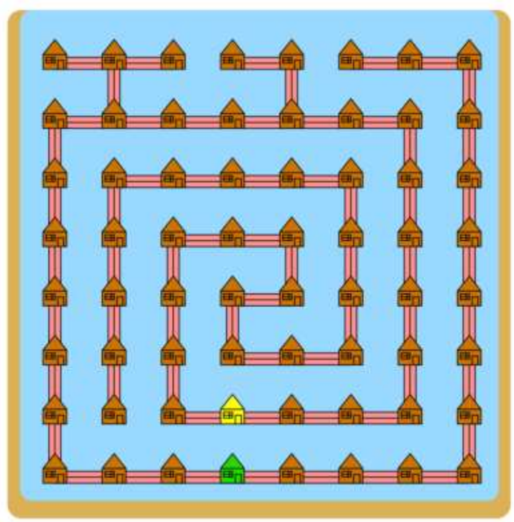


Figure 1: The Elven Kingdom with 64 families

But just as they finish, the elves realize they have overlooked something important: For the elven children, who are not yet allowed to drive the boats alone, the bridges are the only way to reach their friends' houses. With the current construction, many of these paths are very, very long!
Most elven children had made friends with their neighbors' children before the flood. And now, for example, the path from Elfriede in the green house to Elvira in the yellow house, who were previously directly adjacent, goes over 23 bridges. The elves don't like this! They decide to reposition the bridges again, this time better:
For elven families that live directly next to each other horizontally or vertically, the average distance between their houses, measured in the number of bridges, should be minimized.

Two houses connected by a bridge have a distance of 1, for example. The distances between all other families, such as those that are "diagonal neighbors" or not neighbors at all, should not be taken into account.
In a smaller elven city, the bridge construction might look like that in Figure 2. The relevant distance values yield a sum of 22.

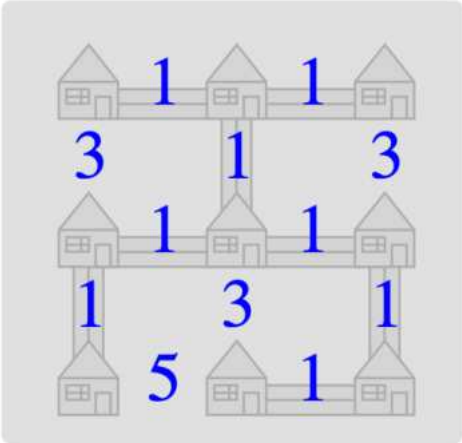


Figure 2: A small example.

Can you help the elves find the best way to construct the bridges?

Warning: Bridges can only be built horizontally or vertically between neighboring houses; diagonally adjacent houses cannot be directly connected. It's easy to realize that minimizing the average distance or the sum of distances is equivalent. What is the smallest of the sums of distances below that the elves can achieve?

Possible answers:

1. 273
2. 294
3. 280
4. The elves' first solution was already optimal.
5. 251
6. 210
7. 336

8. 230
9. 276
10. 217

Hint: The given construction is not optimal! Does minimizing the distances of the houses to the open sea by boat help?

Project:

The search for the described structures is an exciting research topic (spanning trees in grid graphs that induce a minimal strictly fundamental circular base). Currently, it is not even known whether one can expect that there will be an efficient method to construct the best such tree for a grid of any (but fixed) size. On the other hand, such circular bases have proven very helpful in the past for calculating schedule plans with minimal network-wide transfer waiting times, see Matheon Project B15: The shorter the sum of all "paths on bridges," the fewer options need to be considered in the search for the best schedule.
Now you might argue that traffic networks generally look very different from those in the Elven land. However, you might want to take a look at
http://www.mta.nyc.ny.us/nyct/maps/subwaymap.pdf, where you can discover similar structures for certain areas.

Solution

The correct answer is: 9
If one heeds the tip given in the task and first tries to build a network of bridges that allows for the shortest paths from the interior of the city to the open sea via boat, one arrives at the solution illustrated in Figure 3. This has a value of 280. Thus, we can already exclude the two inferior answers 2 (294) and 7 (336). Similarly, answer 4 is, of course, worse, yielding a total sum of 1028!
If we were dealing with a smaller elven city, say with $6 \times 6 = 36$ families, a corresponding solution would have already been optimal. However, here one can still find a better solution through some clever thinking. We have provided two options in Figures 4 and 5; both yield a total distance sum of 276. One can only arrive at these solutions through clever experimentation or reasoning. It is quite astonishing that there are ways to improve the result found in Figure 3 by 4 length units!
Now, how can one demonstrate that answer 9 with a value of 276 is the best answer and that there may not be a better bridge construction?

First, one can assure oneself that the sum of the path lengths must always result in an even number. To do this, one first clarifies the following: There are exactly $112, (= 2 \cdot 7 \cdot 8)$ pairs of neighboring houses, half of which are horizontal neighbors and the other half vertical neighbors. This means that the total length of any bridge construction consists of 112 summands. Furthermore, it can be observed that every path between neighboring houses along the bridges has an odd length. To see this, let's imagine walking from any elven house

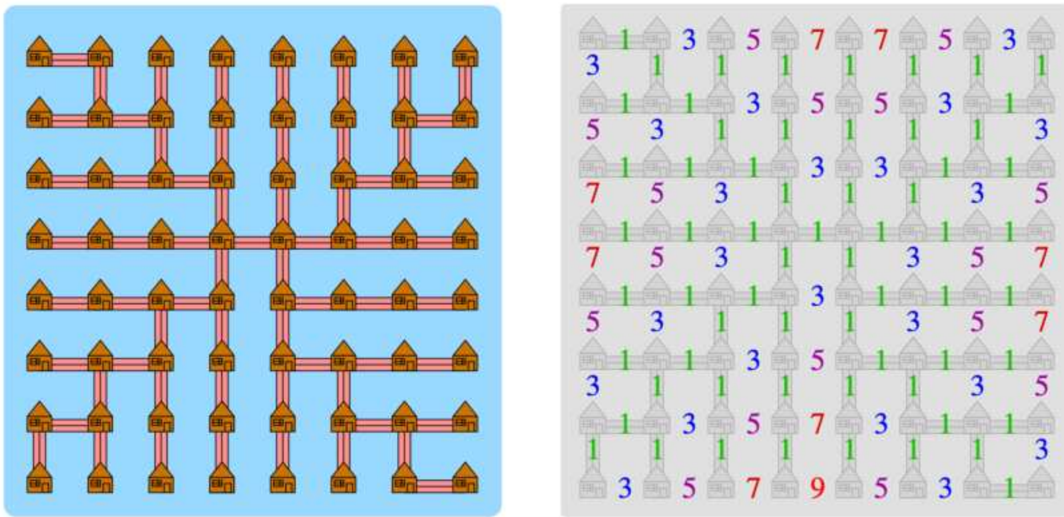


Figure 3: A solution with the shortest water paths to the open sea. Value: $63 \cdot 1 + 24 \cdot 3 + 16 \cdot 5 + 8 \cdot 7 + 1 \cdot 9 = 280$.

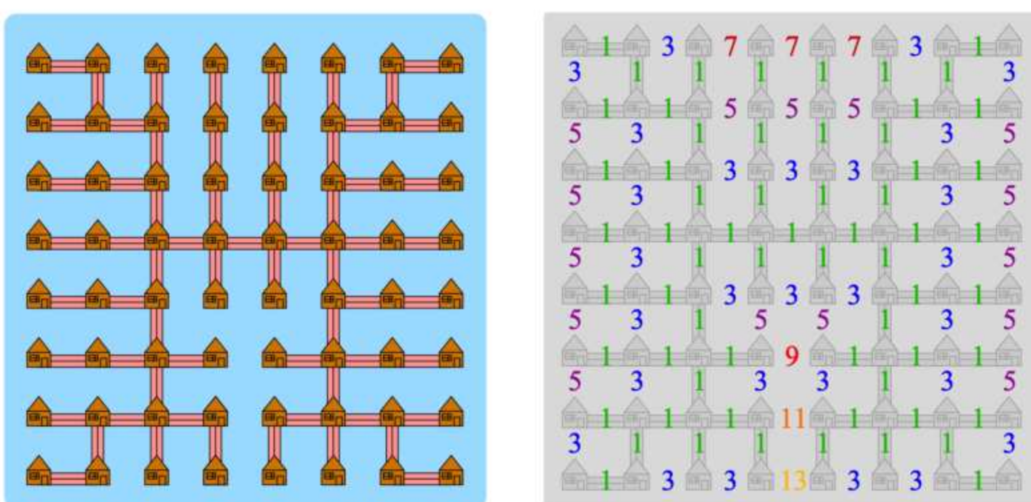


Figure 4: A bridge construction with a value of $63{\cdot}1{+}28{\cdot}3{+}15{\cdot}5{+}3{\cdot}7{+}1{\cdot}9{+}1{\cdot}11{+}1{\cdot}13 = 276$.

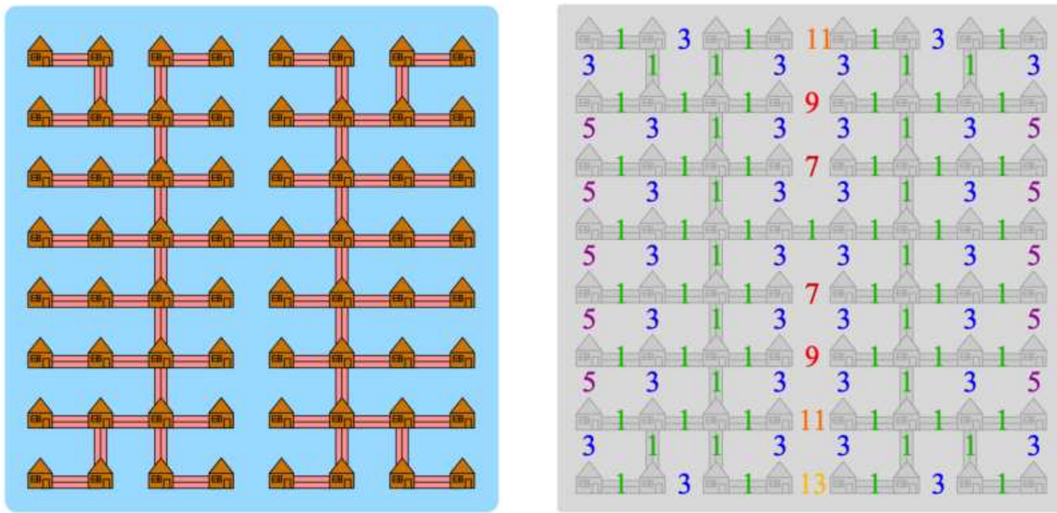


Figure 5: An alternative bridge construction with a value of $63 \cdot 1 + 32 \cdot 3 + 10 \cdot 5 + 2 \cdot 7 + 2 \cdot 9 + 2 \cdot 11 + 1 \cdot 13 = 276$.

H_1 to its neighboring house H_2 directly to the right. Since the two houses are vertically aligned, the number of bridges walked upwards must equal the number of bridges walked downwards. This means that the number of vertical bridges (up or down) on the path between H_1 and H_2 is always even. On the other hand, because H_2 is to the right of H_1, there must be exactly one more bridge walked to the right than to the left. Thus, the number of horizontal bridges along the path between the two houses is odd. Therefore, the total number of bridges on the path is odd ("even + odd = odd"). A very similar argument can be made for different arrangements of the neighboring houses (e.g., "one above the other"). In total, the total length of any bridge construction must thus be an even number ("even · odd = even"). These considerations allow us to exclude answers 1, 5, and 10, as all three have odd values.

Thus, we are left with answers 6 (210), 8 (230), and 9 (276, which corresponds to our found solution). To exclude the two smaller solutions, we mathematicians determine what are called "lower bounds." This means we make considerations that allow us to show that a solution can never be better than a certain value. Below, we describe how one can think of 232 as a lower bound.

First, one should clarify once again what we want to achieve. In this case, we want to construct a bridge layout such that every water cell is still accessible from the open sea, but the total sum of the distances between previously neighboring houses is as small as possible. It is, of course, good if each individual distance is small. However, one quickly realizes that not all former neighboring houses can have a distance of one, meaning they cannot be directly connected. Otherwise, one would not be able to travel to the city by boat at all, and there would be many small cut-off water cells.

No matter how the bridges are distributed, one can build at most 63 bridges without constructing a circle of bridges that would cut off a piece of water from the outside world.

This can be reasoned as follows: to connect two houses, one needs exactly one bridge. With each additional bridge, one can connect exactly one new house to the already connected houses. It is not possible to connect two new houses at once, since a bridge always runs between two houses, one of which is already connected. If one were to not connect any new house, but extend from a connected house, one would connect an already connected house a second time and thus create a circle! Therefore, we can calculate exactly how many bridges can be built: one bridge for each house except the first, so $64 - 1 = 63$ bridges. Thus, it is already clear that exactly 63 previously neighboring families will also be direct neighbors afterwards.
From this, we can already calculate a first lower bound. If only 63 paths can have a length of one, and all paths have odd lengths, it follows that the remaining $112 - 63 = 49$ path lengths must each be at least 3. Thus, the total sum is at least

$$63 \cdot 1 + 49 \cdot 3 = 210.$$

However, not all 49 remaining distances can only be 3 bridges long. To create a distance of 3, three bridges must encircle one of the "unit squares," as shown in Figure 6. Since each bridge belongs to at most two unit squares, it can lie on at most two "three-paths." Moreover, each three-path consists of three bridges. With 63 bridges, one can thus construct at most $(63 * 2)/3 = 42$ three-distances. However, each bridge would have to be involved in two of these squares. Since one will eventually hit the edge and there at least one edge can only be used for one three-path, we can confidently assume that there are at most 41 three-distances.

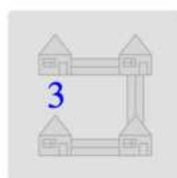


Figure 6: A distance of three bridges always surrounds a "unit square."

This, in turn, means that there are at least $112 - 63 - 41 = 8$ distances that are longer than 3, thus at least 5. We obtain a new lower bound of

$$63 \cdot 1 + 41 \cdot 3 + 8 \cdot 5 = 226.$$

At the same time, we know that there must also be a boat route from the water area in the very center to the open sea. At the point where this boat route exits the city, it must go between two formerly neighboring families who now have to circumvent the central water area to visit each other. This path is at least 9 bridges long! Likewise, the distance between the penultimate pair of houses that the boat route cuts through is at least 7. If the connection of the central water area to the sea were to look different, the distances would be even larger. So we have at least one distance of at least 7, and at least one of at least 9. Together, we now have the required lower bound:

$$63 \cdot 1 + 41 \cdot 3 + 6 \cdot 5 + 1 \cdot 7 + 1 \cdot 9 = 232.$$

Thus, we can also exclude answers 6 and 8, whose values of 210 and 230 can never be reached, as we have reasoned here. Therefore, answer 9 is the best possible indicated solution.

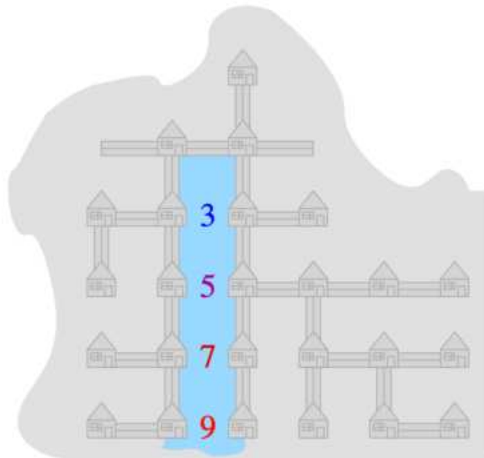


Figure 7: The exit from the central water area enforces long paths.

Note: This problem is particularly close to the current research work of Matheon employees among all math calendar problems. As you have already read in the problem statement, finding "as short as possible bridge constructions," or in technical terms, circular bases, is important for creating good subway timetables. Therefore, researchers at the Technical University of Berlin are currently investigating this question intensively. They have encountered the astonishing fact that on right-angled grids (like the elven city), minimal circular bases have a simple structure as long as the grid has up to 6×6 nodes, but on larger grids, completely different constructions are suddenly better. For larger grids, we cannot yet prove what the (with respect to path length sum) smallest possible bridge constructions are! As you can see, there are still many questions and problems open in mathematics. Some can be described very easily, yet finding a solution is very difficult! Perhaps you can help us. Good luck!

2007 The Weights

Author: Falk Ebert
Project: D13

Challenge

The loader master Valentin has a problem: His weights are gone. And this just before Christmas. He has to ensure that Santa Claus's sleigh complies with night flight regulations, which state that a sleigh over inhabited areas must not weigh more than 220kg. Santa Claus alone weighs exactly 100kg, but the gifts vary each year. Thus, Valentin has to weigh each gift individually. Standardly, the gifts always have full kilogram weights (the remainder is always filled with packaging materials and sweets). For weighing, Valentin uses an ancient balance scale, where weights are placed on one side and the gift to be weighed on the other side. However, since he is missing the weights, he wants to borrow a few of Gernot's gift-building gnomes as weights. Fortunately, these also only have full kilogram weights (the remainder is always filled with sweets). The gnomes have all possible weights from 1 kg (Microelectronics Gnome) to 120 kg (Scooter Gnome). Of course, Gernot does not want to send 120 gnomes, but as few as possible. Valentin can place several gnomes together on the weight side of the scale to obtain different counterweights. Then he has a brilliant idea: he can place his weights on both sides of the scale, i.e., also on the side with the gift.

How many weighting gnomes does Valentin need to weigh each gift (1 - 120kg) exactly, if he can place the weights on one or both sides of the scale?

Possible answers:

1. 6 weights or 6 weights
2. 5 weights or 7 weights
3. 8 weights or 7 weights
4. 9 weights or 4 weights
5. 7 weights or 4 weights
6. 7 weights or 6 weights
7. 7 weights or 7 weights
8. 6 weights or 5 weights
9. 6 weights or 4 weights
10. 7 weights or 5 weights

Note: Quibbles about the fact that weights can be placed at different distances from the pivot point are not taken into account by the scale!

Project Reference:

The author is primarily concerned within project D13 with the question of how to efficiently solve equations that describe complicated circuits on a computer. Additionally, he gives mathematical lectures to students and is currently contemplating what he can give his family for Christmas - thus, there is only an indirect project reference.

Solution

The correct answer is: 10
First, let's consider the case where only one side of the scale can be occupied with weights. Each weight has the state 'on the scale' or 'down'. We assign the value $\alpha = 1$ to the state 'on' and the value $\alpha = 0$ to the state 'down'. If the weights are numbered, and the i-th gnome has weight g_i, then their total weight is given by

$$G = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \ldots$$

If the weights g_i are known, it suffices to know the sequence $\alpha_1\alpha_2\alpha_3\ldots$ to calculate the total weight. This means that we must represent all numbers between 1 and 120 by a sequence of zeros and ones. This fact is also known as the binary system [1] where the weights $g_i = 2^{i-1}$. Thus, with weights of 1 kg, 2 kg, 4 kg, 8 kg, 16 kg, 32 kg, and 64 kg, all weights between 1 kg and 127 kg can be expressed.
We proceed similarly when weights can be on both sides of the scale:
This time we allow the states

- 'right' (weight is on the side of the gift $\equiv -1$);
- 'left' (weight is not on the side of the gift $\equiv 1$);
- 'down' (weight is not on the scale $\equiv 0$)
  to be available.

The availability of 3 states suggests that we should use the ternary system instead of the binary one, i.e., the number representation in base 3.
With the powers $3^0; 3^1; 3^2; 3^3; 3^4$, all numbers between 1_{10} and 242_{10} can be represented in the ternary system.[2]

[1] In the decimal system, each number is represented as a sum of powers of the number 10. E.g. (the number in the index denotes the base):

$$127_{10} = 7 * 10^0 + 2 * 10^1 + 1 * 10^2$$

Similarly for the binary system:

$$127_{10} = 1 * 2^0 + 1 * 2^1 + 1 * 2^2 + 1 * 2^3 + 1 * 2^4 + 1 * 2^5 = 111111_2$$
$$126_{10} = 0 * 2^0 + 1 * 2^1 + 1 * 2^2 + 1 * 2^3 + 1 * 2^4 + 1 * 2^5 = 111110_2$$

[2] However, for this, the weights would need to be present in duplicate, which is not the case here (e.g. $242_{10} = 2 * 81 + 2 * 27 + 2 * 9 + 2 * 3 + 0 * 1 = 22220_3$ - however, the weights can be placed on both sides of the scale, and you also have the weight of the gifts).

On the other hand, any given sequence $\alpha_1\alpha_2\alpha_3\ldots$ with $\alpha_i \in \{-1; 0; 1\}$ represents the distribution of the weights on the scale. If we add 1 to each sequence member, we can consider the resulting number sequence as a ternary number.

Example:
60 kg is to be balanced. The distribution of the weights 81 kg, 27 kg, 9 kg, 3 kg, 1 kg is known:
1 -1 1 -1 0 (Distribution of the weights in descending order from the left)

This is represented by adding 1 to the ternary number:
2002011 $\left(1 * 3^0 + 0 * 3^1 + 2 * 3^2 + 0 * 3^3 + 2 * 3^4 = 181_{10}\right)$ alternatively: $20201_3 = 181_{10}$
To deduce the distribution of the weights from this equation/number representation, there are 2 ways:
a) in the ternary system:
Subtract 1 from each position of the sequence.
b) More interesting is the case in the decimal system (in which the weights of the gifts are also specified):
Here, the reduction of the place value by 1 (as in the ternary system) is replaced by the subtraction of a power of 3:
So we need to calculate: $181 - 81 - 27 - 9 - 3 - 1 (= 60)$.

This also means that for a given weight in decimal representation, you only need to add the powers of 3 to determine the ternary representation and thus the distribution of the weights.
Ex.:
Suppose the weight of the gift is 75 kg. The distribution of the weights is sought. From the above considerations, we have:
$75 + 1 + 3 + 9 + 27 + 81 = 196_{10}$
The representation of 196 in the ternary system is:
$196_{10} = 2 * 81 + 1 * 27 + 0 * 9 + 2 * 3 + 1 * 1 = 21021_3$

In this representation, each sequence member is reduced by 1 to deduce the distribution of the gnomes:
This results in the sequence $1; 0; -1; 1; 0$. This means:
On the side of the gift is the weight 9 kg.
The weights 1 kg and 27 kg are not used.
On the other side of the scale are the weights 81 kg and 3 kg.

Thus, it has been shown through examples that the 5 weights 1 kg, 3 kg, 9 kg, 27 kg, and 81 kg are sufficient to weigh any gift.

To justify that 7 or 5 weights are also the minimum, we rely on the principle of coding. We already know that the weights of the gifts to be measured can be encoded by an ordered sequence of weights. This provides in the first case the binary representation and in the second case the equivalent form in the ternary representation. In each case, the sequence looks like $a_n, a_{n-1}, \ldots, a_2, a_1$. Each a_i can take one of k states. Such a sequence of n

members can thus encode exactly k^n states. We want to be able to weigh all gifts in the integer range from 1 kg to 120 kg. Therefore, this sequence must be able to take at least 120 states. In the first case, $k = 2$, and $2^7 = 128$ provides enough such states, while $2^6 = 64$ is insufficient. In the second case, $k = 3$, and $3^5 = 243$ is sufficient, but $3^4 = 81$ is not.
Answer 10 is correct.

Note:
The specified base weights are not unique. In the first case, for example, the distribution $1\text{ kg}, 2\text{ kg}, 4\text{ kg}, 8\text{ kg}, 15\text{ kg}, 30\text{ kg}, 60\text{ kg}$ would also be possible. In the second case, the range is smaller, but one has the option of choosing $2\text{ kg}, 3\text{ kg}, 9\text{ kg}, 27\text{ kg}$, and 81 kg or $1\text{ kg}, 3\text{ kg}, 9\text{ kg}, 27\text{ kg}$, and 80 kg. However, this does not change the number of weights needed.

2008 The Most Beautiful Reindeer of All!

Author: Dirk Becherer
Project: E8 and E9

Challenge

As every year at Christmas, Santa Claus and his little angel must choose a reindeer from the Christmas magazine. And of course, they would love to get the most beautiful reindeer.

"Oh dear, what a complicated selection procedure this year," grumbles Santa Claus. "But why?" asks the little angel. Santa explains, "The elves in the magazine will present us with various reindeer one by one, from which we may choose one. However, the reindeer insist that we must decide immediately whether to take the respective reindeer or decline it to take one of the following. Once a reindeer is declined, it is no longer available. And by the latest, we must take the tenth reindeer, because then the elves will no longer be interested." "Sounds exciting. What's the problem?" says the angel. "Well," grumbles Santa Claus, "the reindeer are presented to us in random order, and I have no prior knowledge of the quality of this year's reindeer offer. Since we can't see all the reindeer beforehand, we cannot know if the currently offered animal is the most beautiful or if perhaps a better one will come later. If we decide to take the currently offered animal, it might turn out that a more beautiful one comes later. On the other hand, if we decide to view more animals, we might miss out on the most beautiful reindeer."

They are now considering how to best ensure that they have the greatest probability of receiving the most beautiful of the ten reindeer. "I can't think of anything good. If we don't know which will be the most beautiful, we might as well take the first one," says Santa Claus. "No, there's a better way," exclaims the angel, "if we first look at a few specimens! Because in doing so, we learn something about the quality of the animals." "And how long do you want to do that?" asks Santa Claus.
"If we wait until the last animal, we will learn the most, right?" "True," smiles the angel, "but then we wouldn't have much to gain because we could only decide on the last animal." "Hmm. So the answer probably lies in the golden mean?" ponders Santa Claus. "Maybe not entirely," explains the angel, "Let's first look at X reindeer and then choose the next one from the remaining $10 - X$ animals that is better than the best of the first X animals (or the last one if no better comes)." "And do you already know for which X the probability will be the highest that we will select the most beautiful reindeer this way?" asks Santa Claus hopefully. "Yes," laughs the angel, "we even have a chance of almost 40 percent if we look at the optimal number X^* of animals. The optimal X^* is ... "

Possible answers:

1. $X^* = 0$ (i.e., the first reindeer is taken.)
2. $X^* = 1$ (i.e., after one reindeer has been evaluated, the next one that is better than the first is taken.)

3. $X^* = 2$
4. $X^* = 3$
5. $X^* = 4$
6. $X^* = 5$
7. $X^* = 6$
8. $X^* = 7$
9. $X^* = 8$
10. $X^* = 9$ (i.e., the last reindeer is taken.)

Project Reference

This problem is an example of a stochastic optimization problem from the family of optimal stopping problems. Stochastic optimization problems occur in the application area E (Finance), for instance, in the hedging and quantification of financial risks and portfolio optimization. For the evaluation and hedging of so-called American options, where an exercise right should be exercised at an optimal time, optimal stopping problems need to be solved, for example.

Solution

The correct answer is: 4
In a slightly different formulation, this problem is known as '*the secretary problem*'. We discuss three different possible solution approaches.

1. Let $\xi(X)$ denote the probability of selecting the most beautiful reindeer from the $n = 10$ if we first look at X specimens. Then,

$$\xi(X) = \sum_{k=X+1}^{n} P[k\text{-th reindeer is the most beautiful}] \cdot P[k\text{-th reindeer chosen, if } k\text{-th is the most beautifu}$$

Here, P denotes the probability (**P**robability) of an event, and the second factor in the summands is a so-called conditional probability. Since each of the n positions for the most beautiful reindeer is equally likely, we have

$$P[k\text{-th reindeer is the most beautiful}] = \frac{1}{n}.$$

Given that the overall most beautiful reindeer is in the k-th position, it will be chosen if and only if the most beautiful reindeer from the first $k - 1$ specimens comes at a position among the first X animals. The probability for this is $\frac{X}{k-1}$. Thus, we have

$$\xi(X) = \frac{X}{n} \sum_{k=X+1}^{n} \frac{1}{k-1}.$$

By calculating, it follows that $\xi(X)$ is maximal at $n = 10$ for $X = 3$.

2. We could also have the computer approximate the probabilities $\xi(X)$ through so-called Monte Carlo simulations, in order to then compare which X provides the highest success probability. With a sufficiently large number of simulations, the approximation errors are small, and the simulation method reliably provides the correct answer.

3. But how can we see that the optimal solution to the stopping problem indeed has the structure 'X animals to observe and then take the next best one', and that there isn't some other, better strategy?

   Let $p(y, k)$ denote the probability that we will select the most beautiful reindeer with the optimal strategy when we are already at the k-th reindeer, and this reindeer is the most beautiful so far ($y = 1$) or not the most beautiful so far ($y = 0$). Our goal now is to calculate $p(1, 1)$ and determine the strategy that achieves the optimal success probability. The method we will use is known as *dynamic programming*. It is based on the idea that a strategy is optimal if it makes an optimal decision for the current step (to take or reject the current reindeer) and is also optimal for future steps.

   First, some preliminary considerations: If the k-th specimen is the most beautiful among the first k reindeer, then $\frac{k}{n}$ is the probability that it is also the overall most beautiful. If it is not the most beautiful so far, it is clear that it cannot be the overall most beautiful. So, it only makes sense to choose a reindeer that is at least the most beautiful so far. It is also easy to see that $\frac{1}{k+1}$ is the probability that the $(k + 1)$-th reindeer is more beautiful than all previous ones. The probability that it is not more beautiful is $\frac{k}{k+1}$.

   For $k = n$, it is clear that $p(y, k)$ equals one for $y = 1$, and zero for $y = 0$. If we already know $p(y, k + 1)$ for $y = 0, 1$, then we can determine $p(y, k)$ for $k < n$ as follows:

$$p(1,k) = \max\left\{\frac{k}{n}, \frac{1}{k+1}p(1,k+1) + \frac{k}{k+1}p(0,k+1)\right\},$$
$$p(0,k) = \frac{1}{k+1}p(1,k+1) + \frac{k}{k+1}p(0,k+1).$$

   The maximum corresponds to the decision at position k between the two possible alternatives: to take the currently most beautiful reindeer or to continue. This decision is made optimally in such a way that the success probability is maximized. Using the above algorithm, we not only obtain $p(1, 1) = 0.3987$, but also the optimal strategy. The optimal time to decide on a currently most beautiful ($y = 1$) reindeer is the first k for which the maximum equals $\frac{k}{n}$. When we have calculated all $p(y, k)$, comparing $p(1, k)$ with $\frac{k}{n}$ shows that the strategy from the 4th answer option is optimal.

2009 The Application

Authors: Volker Mehrmann, Falk Ebert

Challenge

Have you ever wondered how an old man at the North Pole manages to keep a business running under the most adverse climatic conditions? And not just any business, but a globally operating consortium that gifts billions of people every year. How are wages paid, where do the raw materials come from, and what about the energy costs? It is tempting to suspect criminal activities behind this, where the round man with the sleigh and the great giveaways is just a facade. But we can reassure you. Everything is absolutely legal. Behind the whole gifting enterprise is an even higher authority, namely the so-called December Festivity Benefactors, or simply DFB. They provide all the money that Santa Claus then sensibly converts into gifts. The DFB gladly grants the money—you just have to ask them nicely. And asking specifically means: writing an application that meticulously explains what the money will be spent on. The sheer magnitude of this application is the reason why Christmas only happens once a year. The gifts can be built quickly, but the application takes up most of the year.
Now, there are 10 individual groups in each of the 6 areas: Gingerbread Sciences, Logistics, Gift Production, Optical Enhancement of Gifts, Christmas Financing, and Computer Games. Additionally, there are 4 groups from the "Additional" category, which, for example, take care of training young Christmas elves. Each of these groups must write their own report, explaining how important they are for Santa, what great gifts they build, and why they want so much money. Ultimately, these partial reports are combined to form the application. Unfortunately, experience shows that each partial report contains about 10 typographical errors. Since the DFB reduces funding for such errors, the application is meticulously checked for errors one last time at the end. This is, of course, done electronically and online in the Winternet. For this purpose, 3 spelling elves look at the application independently of one another. Each finds a number of errors that is uniformly distributed between 0 and the total number of existing errors. (This includes both finding 0 errors and finding all errors. All elves have the same chances.) Then, each elf simply overwrites the

last version of the application with their corrected version. This results in the correction version of the elf who found the most errors, and thus took the longest, becoming the preliminary final version. This, of course, rubs off on the pride of the other two spelling elves, who have now worked completely in vain. So the preliminary final version is subjected to the same process again. All three check independently again, and the version in which the most errors are found is declared the interim winner. And to ensure that all three have a chance to be the interim winner, there is also a third round of the correction competition. Elfish competitiveness or not—after that, it's over! The last correction version is submitted to the DFB. The only question is: How many errors are, on average, still present in the final version?

Possible answers:

1. none at all; they are all already eliminated after the second round
2. 0-2
3. 5-6
4. 9-12
5. 25-35
6. exactly 42
7. $50 - 60$
8. about 80
9. about 120
10. about 160

Example: *The 3 elves check another text, which contains 10 errors, in the same way. In the first round, they independently find 3, 5, and 6 errors. Thus, after the first round, there are still 4 errors in the text. In the next round, they find 1, 2, and 4 errors, leaving 0 errors in the text after the second round. In the third round, all find 0 errors, and thus the text is error-free after 3 rounds.*

Project Reference:

The application process is a simplified version of what MATHEON undergoes every 4 years (fortunately not every year!). Sometimes, the correction process also runs like in the task, albeit rather unintentionally.

Solution

Correct Solution: Answer 4: 9-12 Errors

The first consideration should be: 6 areas with 10 groups each and 4 additional groups make a total of 64. If each introduces 10 errors into the application, that amounts to 640 in total. We first try to determine how many errors remain in the application after one round of correction. There are various approaches to solve this, all requiring different basic knowledge but yielding almost identical solutions.

Intuitive Solution

If there were only one corrector, then their corrected version would automatically be the best. The proportion of errors found lies somewhere between 0 and 1. There is no reason to assume that it should be closer to 0 or closer to 1 in any way. Accordingly, intuition suggests that a corrector, on average, finds half of the errors. How does this look with n correctors? Each of them again finds a certain proportion of errors, and we are interested in how many can be maximally found. We denote the found error proportions as p_1 to p_n and assume, for simplicity, that $p_1 \leq p_2 \leq \ldots \leq p_n$. Again, there is no reason why any range between 0 and 1 should be more likely for the found proportions. Analogous to the case with only one corrector, we can now assume that p_1 is midway between 0 and p_2, p_2 is midway between p_1 and p_3, and so on. Consequently, the proportions are, on average, arranged such that the intervals $[0, p_1], [p_1, p_2], \ldots [p_n, 1]$ all have the same length because, with equally likely p_i values, there is no reason why one of the intervals should be shorter or longer. In total, there are $n+1$ such intervals, and thus p_n, on average, is given by

$$p_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

This also fits well with the expected result for $n = 1$ corrector. In the case of 3 correctors, approximately $\frac{3}{4}$ of the errors are found in each round. Therefore, about $\frac{1}{4}$ of the errors remain in each round. After 3 rounds, that results in $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$. With originally 640 errors, there are still 10 remaining.

With Discrete Probability Distributions

Preliminary consideration: Suppose there are no errors in the application, then each corrector has exactly one possibility to find a certain number—namely, 0. If we represent the errors found by the three correctors as triples, there is exactly one triple: $(0, 0, 0)$. As soon as there is one error, each of the correctors has two equally likely chances, namely either to find 0 or 1 error. Now there are $2^3 = 8$ possibilities to represent the found errors as triples, namely $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$. Exactly one of these triples results in a maximum of 0 found errors, while the remaining $2^3 - 1 = 7$ yield 1 as the maximum. If we now assume there are 2 errors in the application, each corrector can find 0, 1, or 2, hence has 3 possibilities. In total, there are now $3^3 = 27$ triples that represent the found 2 or fewer errors. One of them, $(0, 0, 0)$, yields 0 as the maximum. The other 7 we found in the last consideration are, of course, among the possibilities. These $1 + \left(2^3 - 1\right) = 2^3$ possibilities thus yield no 2 found errors. This means that the remaining

$3^3 - 2^3 = 19$ triples each contain at least one 2, leading to a maximum of 2. If we continue this thought, it means that for n errors in the application, each corrector has $n+1$ possibilities to find errors. In total, there are $(n+1)^3$ possibilities to form triples. Exactly one of these results in 0 maximum found errors, 7 lead to one found error, and 19 to two errors. In general, there are $(k+1)^3 - k^3$ possibilities for exactly k errors to be found as the maximum. We denote by X_i, for $i = 1, 2, 3$ as the random variable that indicates how many errors each corrector finds. It holds that $P(X_i = k) = \frac{1}{n+1}$ because each value between 0 and the total number is equally likely. With the above considerations, we have that

$$P(\max(X_1, X_2, X_3) = k) = \frac{(k+1)^3 - k^3}{(n+1)^3}.$$

This allows us to determine the expected value of the found errors.

$$\begin{aligned} E(\max(X_1, X_2, X_3)) &= \sum_{k=0}^{n} k \cdot P(\max(X_1, X_2, X_3) = k) \\ &= \sum_{k=0}^{n} k \frac{(k+1)^3 - k^3}{(n+1)^3}. \end{aligned}$$

This term can be simplified to

$$\begin{aligned} E(\max(X_1, X_2, X_3)) &= \frac{1}{(n+1)^3} \sum_{k=0}^{n} k^4 + 3k^3 + 3k^2 + k - k^4 \\ &= \frac{1}{(n+1)^3} \left(3\sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k \right). \end{aligned}$$

The terms for the power sums can be quickly found—after all, we have the internet.

$$E(\max(X_1, X_2, X_3)) = 3\frac{(n(n+1))^2}{4(n+1)^3} + 3\frac{(2n+1)(n+1)n.}{6(n+1)^3} + \frac{n(n+1)}{2(n+1)^3}$$

This term can be simplified with some calculation to

$$E(\max(X_1, X_2, X_3)) = \left(\frac{3}{4} + \frac{1}{4(n+1)} \right) n$$

Again, it shows that about 3/4 (and a little more) of the errors are, on average, found in one round. And if you ignore this small deviation from 3/4, you can use the same argument as in the intuitive solution approach and ultimately arrive at 10 remaining errors. If you want to take the small deviation into account, then the calculation of the expected value over 3 rounds becomes significantly more complicated, as you then have to consider all combinations of found errors over 3 rounds.

2010 Corrupt UEFA

Author: Gregor Heyne (MATHEON)

Challenge

Secretary Sigismund (once again):

Santa Claus is still missing. By now, this has quite an impact on the morale at work. This morning, a two-year-old dispute has flared up again. In March 2008, the following happened. During the lunch break, the Christmas elves and the boss followed the radio broadcast of the draw for the quarter-finals in the Champions League. The following four matches were drawn:

1. Arsenal - Liverpool,
2. AS Roma - Manchester United,
3. Schalke - Barcelona,
4. Fenerbahce Istanbul - Chelsea.

In the evening, Santa Claus found out that about an hour and a half before the draw, a member of an Internet forum of FC Liverpool spread a message claiming: *There are rumors that the draw has been manipulated. He does not believe it, but if it is true, the following four matches are it.* He then named exactly the matches that were drawn later, in the correct order of the opponents (which determines the home advantage). Additionally, the author of this message also correctly predicted the simultaneously drawn semi-finals, including the order of the teams (matches are numbered, and then numbers are drawn).

This strange event[3] was vividly discussed in various internet forums for days under the title: Is UEFA corrupt? A laconic response from UEFA to many angry emails in the media stated: *he was just lucky and guessed the matches.*
The Christmas elves also began to discuss excitedly how high the probability is of guessing the correct match pairings. Santa promised the elf who determines the correct answer an extra week of vacation. So far, no one has been able to provide a convincing answer. And now that Santa is gone and some boredom is setting in, the discussion has come up again. Which of the probabilities mentioned by the elves is correct—or closest to the correct value? Then they could put the matter to rest and get back to work.
Possible answers:

1. 1/967680
2. 1/1000000
3. 1/40320
4. 1/568
5. 1/483840
6. 1/10
7. 1/20160
8. 1/4032
9. 1/80640
10. 1/90740

Note: To prevent potential confusion among football novices:

- The match *Arsenal - Liverpool* is a different match than *Liverpool - Arsenal*, as it is important who plays at whose home stadium.
- Whether the match *Arsenal - Liverpool* is drawn before or after *AS Roma - Manchester United* does not matter. The main thing is that the match takes place.
- For the semi-final matches: Numbers are thrown into the pot in the order in which the matches take place. For the first semi-final, 2 numbers are drawn (the order is important), and for the second semi-final, the same applies.
- Whether semi-final match 1 is drawn before semi-final match 2 or vice versa is irrelevant, as one just wants to know who plays against whom.

[3] This is a true event. See also: http: //www.liverpooldailypost.co.uk/liverpool-fc/liverpool-fc-news/2008/03/14/daily-post-forums-swamped-by-champions-league-claim-64375-20624765/

Solution

Correct answer: 7 1/20160
The probability of guessing the quarter-final pairings correctly can be determined as follows:
Exp. 1: draw Team 1 for Match 1: 8 possibilities
Exp. 2: draw Team 2 for Match 1: 7 possibilities
Exp. 3: draw Team 1 for Match 2: 6 possibilities
Exp. 4: draw Team 2 for Match 2: 5 possibilities
Exp. 5: draw Team 1 for Match 3: 4 possibilities
Exp. 6: draw Team 2 for Match 3: 3 possibilities
Exp. 7: draw Team 1 for Match 4: 2 possibilities
Exp. 8: draw Team 2 for Match 4: 1 possibility.

According to the general counting principle, there are thus 8! possibilities to draw the 4 matches, taking the order into account. The order of the 4 matches does not matter in the draw. There are 4! possible permutations of the 4 matches. Therefore, there are a total of $\frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$ different results of the quarter-final draw.

For the additional drawing of the semi-finals, the following procedure is followed. Quarter-final matches are numbered from 1 to 4.
Exp. 1: draw Team 1 for Match 1: 4 possibilities
Exp. 2: draw Team 2 for Match 1: 3 possibilities
Exp. 3: draw Team 1 for Match 2: 2 possibilities
Exp. 4: draw Team 2 for Match 2: 1 possibility.

Thus, there are 4! possible drawings of the matches, considering the order of the draw. Since the home advantage is also relevant here, the order of the teams in the matches is important. Only the order of the two semi-final pairings is irrelevant. Therefore, there are $\frac{4!}{2!} = 12$ possible outcomes of the semi-final draw.

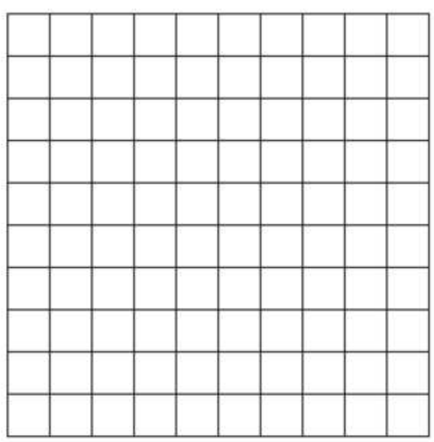
Combining the two correct predictions, the author of the email had a chance of 1 to $1680 \cdot 12 = 20160$ to guess the correct results of the draw.

2011 Gift Wrapping

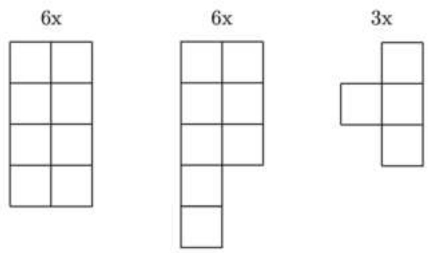
Author: Marco Sarich

Challenge

"Oh no!" screams from the gnome warehouse. The wrapping paper is dramatically running out. Only one square sheet remains available for the eager helpers.



This must be enough for 15 gifts, which, however, consume different amounts of material and require various cuts.



The gnomes begin to argue about how the wrapping paper should be divided best. Soon, initial speculations arise that the paper may not be sufficient for all gifts. Other voices, however, hold an entirely different opinion. There is agreement only on one thing: as many gifts as possible should be wrapped. But how many gifts can be wrapped at most by cleverly dividing the paper?

Possible answers:

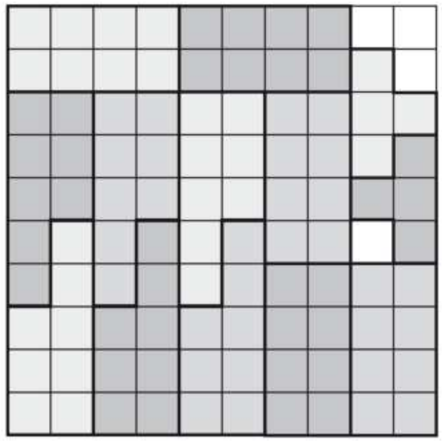
1. 15
2. 14
3. 13
4. 12
5. 11
6. 10
7. 9
8. 8
9. 7
10. 6

Solution

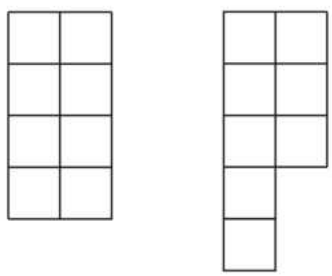
Correct answer: 3
A solution for 13 gifts is given, for example, by the following cutting pattern.
The question, however, is why this solution is also optimal, meaning that no more gifts can be wrapped.
1) It is not possible to wrap all gifts since the area of the wrapping paper consists of only 100 squares, while the 15 gifts would require a total of 108 squares, as

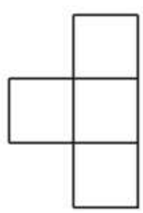
$$6 \cdot 8 + 6 \cdot 8 + 3 \cdot 4 = 108.$$

2) Now one must consider whether it might still be possible to wrap 14 gifts. For this to work at all, it is clear that one of the gifts requiring 8 squares of wrapping material must be



left out. The remaining gifts would then require only 100 squares of paper: exactly as much as the gnomes still have available. However, it is still not possible to wrap the remaining gifts. If one imagines the paper marked in a checkerboard pattern, there are exactly as many white squares as black squares. The two cutting patterns also have this property, always having 4 white and 4 black squares. However, with the last cutting pattern, this is different. In every paper cut, 3 of the 4 squares would be the same color. Additionally, three gifts must be wrapped with this pattern. This means that among these three cutting patterns, there cannot be an equal number of black and white squares. Thus, one can wrap 13 gifts.





2012 Gnome Democracy

Author: Gerhard Woeginger

Challenge

The 15 gnomes Atto, Bilbo, Chico, Dondo, Espo, Frodo, Gumbo, Harpo, Izzo, Jacco, Kuffo, Loco, Mirko, Nemmo, and Onno want to share 100 cookies among themselves. This immediately leads to problems: First, due to the crumbs, none of the cookies can be divided. Second, 100 is not evenly divisible by 15. Third, there is disagreement.

The gnomes eventually agree on a thoroughly democratic approach. The alphabetically last gnome in the group is appointed as the division leader (not to be confused with a department head) and proposes a division of the 100 cookies to all participating gnomes, including himself. Then a vote is held on his proposal.

- If there is one or no dissenting votes, then the proposal is accepted. The cookies are divided accordingly, and there are no further votes.
- However, if there are two or more dissenting votes, then the division leader is removed and heavily punished for his poor proposal. He loses any right to the cookies, may not participate in further votes, and instead must eat a terribly sour green apple. The procedure is then repeated with a new division leader – and one less gnome.

The gnomes take their democracy very seriously; votes are always conducted honestly, and there are neither collusions nor secret agreements, let alone conspiracies. The goals of the 15 gnomes during the votes are simple to describe: The primary goal of each gnome is not to have to bite into a sour apple himself. A nearly equally important goal is to receive as many cookies as possible. Finally, the gnomes find it quite amusing when another has to eat a sour apple. Thus, if a gnome has to choose between two situations that yield the same number of cookies and in which he himself does not have to eat a sour apple, he will always choose the situation where more green apples are eaten.

The extremely clever and thoughtful Onno is appointed as the first division leader since he is the alphabetically last gnome. Onno thinks long and hard and then makes the best possible division proposal for himself.

Which of the following statements is true for Onno's division proposal?

Possible answers:

1. Onno gets 2 cookies, and every other gnome gets 7 cookies.
2. Onno gets 3 cookies, and Izzo gets 8 cookies.
3. Onno gets 4 cookies, and Espo gets 11 cookies.
4. Onno gets 5 cookies, and Loco gets 8 cookies.
5. Onno gets 6 cookies, and Dondo gets 8 cookies.
6. Onno gets 7 cookies, and Kuffo gets 7 cookies.
7. Onno gets 8 cookies, and Frodo gets 3 cookies.
8. Onno gets 9 cookies, and Atto gets 13 cookies.
9. Onno gets 10 cookies, and Harpo gets 8 cookies.
10. Onno gets no cookie and must definitely bite into the sour apple.

Solution

The correct answer is: 8

This problem is best approached from the back (only 2 gnomes left) to the front (all 15 gnomes present). For each natural number n with $2 \leq n \leq 15$, we will show that there is a unique division proposal V_n for the situation with n gnomes that gives the best possible outcome for the division leader. So, if any of the remaining gnomes wants to evaluate a specific division proposal X, he only needs to compare X with the proposal V_{n-1}. If X is better for him than V_{n-1}, he votes for X, and otherwise he votes against X.

We start with the case of $n = 2$ remaining gnomes, Atto and Bilbo. Since division leader Bilbo can win any vote with at most one dissenting vote, he will secure all 100 cookies and make the best possible division proposal V_2 with $A = 0$ and $B = 100$.

In the case of $n = 3$ gnomes (Atto, Bilbo, Chico), division leader Chico must secure the vote of Atto or Bilbo. Bilbo will definitely vote against Chico's proposal V_3 since proposal V_2 assigns all 100 cookies to him. Atto prefers any proposal with $A = 1$ over proposal V_2 with $A = 0$. Atto will vote against any proposal with $A = 0$, since proposal V_2 also guarantees him $A = 0$ cookies, and he enjoys seeing Chico bite into the sour apple. Thus, Chico makes the proposal V_3 with $A = 1$, $B = 0$, $C = 99$ and wins the vote.

In the case of four gnomes (Atto, Bilbo, Chico, Dondo), division leader Dondo must secure the votes of two other gnomes by offering them a better deal than proposal V_3. Atto's vote costs 2 cookies, Bilbo's vote costs 1 cookie, and Chico's vote costs 100 cookies. Thus, Dondo makes the proposal V_4 with $A = 2$, $B = 1$, $C = 0$, $D = 97$ and wins the vote.

In the case of five gnomes (Atto, Bilbo, Chico, Dondo, Espo), Espo must secure the votes of three other gnomes. Atto's vote costs 3 cookies, Bilbo's vote costs 2 cookies, Chico's vote costs 1 cookie, and Dondo's vote costs 98 cookies. Thus, Espo makes the proposal V_5 with $A = 3$, $B = 2$, $C = 1$, $D = 0$, $E = 94$ and wins the vote.

And so forth. The following table lists the best possible proposals V_n for the division leader. The last row shows that Onno can secure exactly 9 cookies with proposal V_{15} and that V_{15} assigns 13 cookies to Atto.

2013 Hats

Author: Gerhard Woeginger

Challenge

Santa Claus has invited 126 intelligent elves to a cozy afternoon with coffee and cake. As the elves enter the hall, each of them is swiftly given a new elf hat from behind. It happens so quickly that none of them can see the color of their own hat. Santa opens the meeting with a short speech.
"My dear intelligent elves! We want to start this afternoon with a little thinking game. None of you knows the color of your own hat, and each of you can see the hats of all the other 125 elves. The goal of this game is to find out the color of your own hat as quickly as possible by pure reasoning. I will ring my big Christmas bell every five minutes. If anyone has figured out the color of their own hat, they must immediately leave the hall at the next ringing. In the next room, they will be served a cup of coffee and a large piece of Sachertorte."
Just as Santa is about to go to the bell, elf Atto has an important question: *"Is it really possible for each of us to determine the color of our hats through logical thinking? For example, if each of us had a different hat color, then no one could figure out their color by thinking. Then the game would not be winnable for us!"* Santa replies somewhat gruffly: *"If, could, would!!! Of course, each of you can win this game! I have carefully chosen the hat colors so that each of you can indeed deduce your color through reasoning during the game."* And then the elves begin to think. And Santa starts ringing the bell.

- At the first ringing, Atto and nine other elves leave the hall.
- At the second ringing, all elves with buttercup yellow, egg yolk yellow, primrose yellow, and sunflower yellow hats leave the hall.

- At the third ringing, all elves with crimson hats leave; at the fourth ringing, all with cactus green hats; at the fifth ringing, all with aquamarine blue hats; at the sixth ringing, all with golden orange hats; at the seventh ringing, all with amber brown hats; and at the eighth ringing, all with shell gray hats.
- At the ninth, tenth, eleventh, and twelfth ringing, no one leaves the hall.
- At the thirteenth ringing, all elves with snow white and all elves with ebony black hats leave.

And so it continues. At the N-th ringing of Santa, the last group of elves finally leaves the hall. Santa has rung the bell a total of seven times in between without anyone leaving the hall (and during these seven times, the ninth, tenth, eleventh, and twelfth ring have already been counted). Our question now is: What is the value of N?
Possible answers:

1. $N = 17$
2. $N = 18$
3. $N = 19$
4. $N = 20$
5. $N = 21$
6. $N = 22$
7. $N = 23$
8. $N = 24$
9. $N = 25$
10. $N = 26$

Solution

The correct answer is: 2

First, let's consider the situation before the first ringing of Santa's bell. Elf Atto looks around and notices that Bilbo is the only elf wearing a tomato red hat. Santa has said that every elf can deduce their own hat color through reasoning. How on earth is Bilbo supposed to determine the color of his hat? If Bilbo does not see a single other tomato red hat, he cannot be sure whether his own hat is fire red, brick red, tomato red, or perhaps some entirely different color. No, Bilbo must see at least one tomato red hat on another elf. Since Atto cannot find another tomato red hat, Atto concludes that the tomato red hat seen by Bilbo must be on Atto's own head. Atto stands up and leaves the hall at the first ringing. Bilbo, of course, reasons the situation in exactly the same symmetric way:

he sees Atto as the only one with a tomato red hat and concludes that his own hat must also be tomato red. Together with Atto, Bilbo, four more pairs of elves leave at the first ringing: Alpo and Barbo with poppy red hats, Akko and Bebbo with rose red hats, Archo and Bodo with sour cherry red hats, and Ando and Buzzo with vermilion hats.

And what happens before the second ringing? Chico looks around the hall and notices that Coco and Cambo are the only elves with buttercup yellow hats. Why hasn't Coco left the hall at the first ringing? If Coco saw just one other elf with a buttercup yellow hat, he could have concluded (analogous to Atto's and Bilbo's reasoning) that his own hat is buttercup yellow and left at the first ringing. Since Coco did not do that, he must see another elf with a buttercup yellow hat, and that elf can only be Chico himself. Therefore, Chico leaves the hall at the second ringing. Along with him go Coco and Cambo (who reasoned the situation symmetrically) and three more groups of three elves with egg yolk yellow, primrose yellow, and sunflower yellow hats.

What happens before the third ringing? Dondo sees that Darko, Dezzo, and Duffo are the only other elves with crimson hats in the hall. Why hasn't Darko left the hall at the second ringing? If Darko saw only two other elves with crimson hats, he could have concluded (analogous to Chico's reasoning above) that his own hat is crimson and left at the second ringing. Since Darko did not do that, he must see another elf with a crimson hat, and that elf must be Dondo. Therefore, Dondo leaves the hall, and with him go Darko, Dezzo, and Duffo. And so on and so forth. If at the start of the game there are exactly k elves with the same fixed hat color in the hall, this group will leave the hall together at the $(k-1)$-th ringing of Santa. The following table lists the number of elves who left the hall by the thirteenth ringing.

Round	Comment	Number
1st ringing	five pairs of elves: 5×2	10
2nd ringing	four shades of flower yellow: 4×3	12
3rd ringing	crimson: 1×4	4
4th ringing	cactus green: 1×5	5
5th ringing	aquamarine blue: 1×6	6
6th ringing	golden orange: 1×7	7
7th ringing	amber brown: 1×8	8
8th ringing	shell gray: 1×9	9
9th ringing	no one: 0×10	-
10th ringing	no one: 0×11	-
11th ringing	no one: 0×12	-
12th ringing	no one: 0×13	-
13th ringing	white, black: 2×14	28

After the thirteenth ringing, exactly 89 of the total 126 elves have left the hall, leaving 37 elves in the hall. Since each remaining hat color must appear at least 15 times, either all 37 remaining elves have the same hat color, or there are two hat colors for $15 + 22$ or $16 + 21$ or $17 + 20$ or $18 + 19$ elves. However, since there are only seven rounds in total where no one leaves the hall, the table can only be completed in the following way.

Round	Comment	Number
14th ringing	no one: 0×15	-
15th ringing	no one: 0×16	-
16th ringing	no one: 0×17	-
17th ringing	sandstone color: 1×18	18
18th ringing	honey color: 1×19	19

In total, there are exactly $N = 18$ rounds. Therefore, answer #2 is correct.

2014 Loss of Beard

Task Presenter: Max von Kleist (FU Berlin)

Challenge

Seven days until Christmas, and Santa Claus has suddenly lost his beard. From his magnificent beard, consisting of 999 beard hairs, only a single hair remains. Santa cannot make an appearance like this! It's chaos at the North Pole! - What to do? Under *normal beard growth*, 3 beard hairs would grow back each day, but this will not be enough to have 999 beard hairs by Christmas day.
Fortunately, one of the Christmas elves has dubious connections and managed to procure an untested magical remedy. According to the instructions, while the remedy suppresses *normal beard growth*, it simultaneously triples the number of existing beard hairs with each use. It is also important to note that the remedy cannot be taken more than once a day, or there is a risk of transforming into an Easter bunny! Therefore, it is crucial to apply the remedy precisely so that Santa is not disfigured (too few/too many beard hairs or an Easter bunny). To ensure Christmas can be celebrated as usual, the elves must now devise the perfect treatment plan: On which days must Santa use the magical remedy to present a magnificent beard with exactly 999 hairs on Christmas day (in 7 days)?

Possible answers:

1. There is no solution. Santa should overdose and turn into an Easter bunny and hide

chocolates.

2. Santa should take the magical remedy daily.
3. Santa should start taking the magical remedy from the fourth day.
4. Santa should start taking the magical remedy from the third day.
5. Santa should take the magical remedy on all days except the first and the last.
6. Santa should start taking the magical remedy on the second day and then every second day.
7. Santa should start taking the magical remedy on the first day and then every second day.
8. Santa should take the magical remedy on the 1st, 3rd, 5th, 6th, and 7th days.
9. Santa should take the magical remedy on the 2nd, 3rd, 4th, 6th, and 7th days.
10. Santa should take the magical remedy on the 2nd, 3rd, 4th, 5th, 6th, and 7th days.

Project reference:

The problem describes an optimal control problem, specifically a switched system. Optimal control problems are applicable in all engineering disciplines; for example, in controlling so-called autonomous systems, for internet protocols, etc. In the research group "Systems Pharmacology & Disease Control," we develop methods to calculate optimal treatment strategies. The systems to be solved are usually so large that not all possible solutions can be computed. Numerical methods help to narrow down the search space of possible solutions. We have not yet found a remedy for hair loss, but we are working on it :).

Solution

The correct solution is: 9

Santa must take the remedy on days 2, 3, 4, 6, and 7. A simple calculation is sufficient. The following rules apply:

$$x_{k+1} = \begin{cases} x_k + 3 & \text{, if no remedy is taken} \\ x_k \cdot 3 & \text{, if the remedy is taken} \end{cases}$$

where x_k describes the number of beard hairs at time k, with the initial condition $x_0 = 1$ for $k = 0$.

In this example, there are theoretically $2^7 = 128$ possible solutions. One could determine all possible solutions and select the correct one. In a (significantly) larger search space, one could use methods that narrow down the search space. Typically, one starts with the second boundary condition, namely the end state $x_N = 999$ (where $N = 7$). With the transition rules

$$x_{k-1} = \begin{cases} x_k - 3 & \text{, if no remedy is taken} \\ x_k/3 & \text{, if the remedy is taken} \end{cases}$$

this is the case in our scenario:

$$999/3 \Rightarrow 333/3 \Rightarrow 111 - 3 \Rightarrow 108/3 \Rightarrow 36/3 \Rightarrow 12/3 \Rightarrow 4 - 3 \Rightarrow 1$$

In general, one solves the problem for $N - 1$ by checking with a linear program whether the last decision, e.g., not taking the magical remedy, can be part of an optimal solution. This process is iterated (on a computer) for each possible control (taking/not taking) until reaching the initial state.

2015 Snowflake

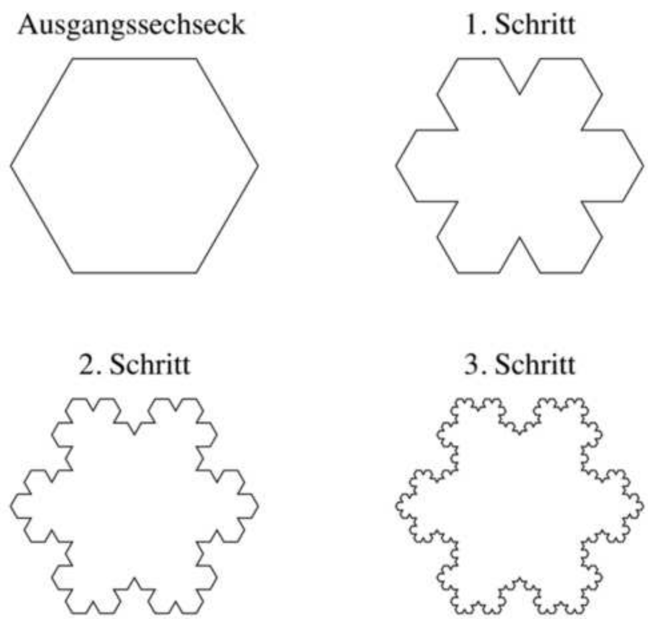
Author: Oleh Omelchenko (Weierstrass Institute)

Challenge

To completely paint the area of a hexagonal ornament for the Christmas tree, the elves need exactly six tubes of golden paint. But what a shock! This time, one tube got lost, and the elves don't know what to do. One elf suggests making a beautiful snowflake out of the hexagon. To do this, he does the following: He cuts out a small equilateral triangle in the middle of each side of the hexagon. The sides of the triangles are as long as one-third of a side of the original hexagon. Thus, he obtains a new figure that also has equal sides. This figure is shown in Step 1 of the illustration. Then he repeats the process in a similar manner and cuts out an even smaller triangle on each side of the figure, which has one-third of the current side length. This figure is shown in Step 2 of the illustration.
How many times must the elf perform this process at a minimum to be able to completely paint a snowflake with his remaining five tubes of paint?

Possible answers:

1. It is enough to only perform Step 1
2. The elf must go to Step 2
3. The elf must go to Step 3
4. The elf must go to Step 4
5. The elf must go to Step 5
6. The elf must go to Step 6



7. The elf must go to Step 7
8. The elf must go to Step 8
9. The elf must go to Step 9
10. The elf must make infinitely many cuts

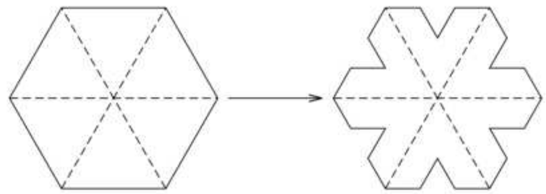
Project reference:

If we repeat the procedure described above infinitely many times, we obtain a strange figure known as a *fractal*. Its special properties include a high degree of self-similarity and, remarkably, infinite boundary length despite having a finite area. In mathematics, fractal structures are encountered in the study of dynamical systems that generate complex behavior, called *strange attractors*. But fractals can also be found in nature. Typical examples from biology are the fractal structures in the cultivation of green cauliflower Romanesco and ferns. Coastlines and the trajectories of Brownian motion also show a great similarity to fractals.

Solution

The correct answer is: 3

To find the correct answer, we first need to calculate the area of the snowflake at each step. We assume without loss of generality that the area of the original hexagon is equal to 1. Then we calculate how large the pieces are that the elf cuts out in the first step. For this purpose, it is useful to divide the original hexagon into six equilateral triangles.
In the first step, each triangle loses a part that is also an equilateral triangle but with three times shorter side lengths. This means that the area of one cut-out triangle is 9 times



smaller than the area of the original triangle. Therefore, after the first step, the elf gets a snowflake with an area

$$S_1 = 1 - \frac{1}{9}.$$

In the second step, the elf must cut out even smaller triangles that have areas 9 times smaller than in the first step. However, the snowflake now has 4 times more sides. Thus, the area after the second step changes as follows:

$$S_2 = S_1 - 4 \cdot \frac{1}{9} \cdot \frac{1}{9}.$$

It can be seen that the elf must cut out new triangles each time that are 9 times smaller than before and that the number of triangles increases by 4 compared to the previous step. Based on this rule, we can write a formula for the area of the snowflake after the n-th step:

$$S_n = 1 - \frac{1}{9} \sum_{k=0}^{n-1} \left(\frac{4}{9}\right)^k.$$

The sum on the right side is called a geometric series, and there is a simple way to calculate it:

$$\left(1 - \frac{4}{9}\right) \sum_{k=0}^{n-1} \left(\frac{4}{9}\right)^k = 1 - \frac{4}{9} + \frac{4}{9} - \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^2 - \cdots - \left(\frac{4}{9}\right)^{n-1} + \left(\frac{4}{9}\right)^{n-1} - \left(\frac{4}{9}\right)^n = 1 - \left(\frac{4}{9}\right)^n.$$

So we get

$$S_n = 1 - \frac{1}{9} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}} = 1 - \frac{1}{5}\left(1 - \left(\frac{4}{9}\right)^n\right) = \frac{4}{5} - \frac{1}{5} \cdot \left(\frac{4}{9}\right)^n.$$

The elf is missing one out of six tubes of paint. Therefore, he must cut out triangles until $S_n > 5/6$. This condition can also be formulated differently:

$$\left(\frac{4}{9}\right)^n > \frac{1}{6}.$$

Now we calculate:

$$\left(\frac{4}{9}\right)^1 > \frac{1}{6}, \quad \left(\frac{4}{9}\right)^2 > \frac{1}{6}, \quad \text{but} \quad \left(\frac{4}{9}\right)^3 < \frac{1}{6}.$$

After the third step, the five remaining tubes of paint are sufficient to paint the entire figure.

2016 Christmas Mystery

Author: Axel Flinth
Project: AG Kutyniok

Challenge

During the labor-intensive Advent season, Santa's reindeer liked to relax in the evening with a glass of mulled wine. Accordingly, they had a huge storage where they kept plenty of the sweet drink.

One night, the reindeer observed two—exactly two—Santa's elves stealing something from the storage. When Santa learned of this, he immediately initiated an investigation to clarify the criminal offense. He quickly identified a small group of elves to whom the thieves surely belonged.

However, the elves in this group were very good friends and didn't want Santa to find out who had committed the crime. They would prefer that Santa didn't learn who stole what. Nevertheless, Santa is obligated by Christmas law to conduct this investigation and question the elves about the theft. And a law states that elves may not lie.

To make it as difficult as possible for Santa, the elves came up with the following: Santa is indeed entitled to question the elves, but Christmas law does not specify exactly how this should happen. Therefore, the elves will only respond in pairs, stating only how much they stole together. Since the elves are also very busy during the Advent season, Santa had to decide a day in advance which pairs he intended to question so that the elves could sensibly schedule the interrogations into their daily routines. Thus, he could not make the choice of pairs dependent on the answers he had already received. Santa is considering a good strategic plan for the three cases where the group of suspected elves consists of i) exactly three, ii) exactly four, and iii) exactly five elves.

This leads to the following question:
Investigate for all three cases how many questions Santa must ask at a minimum to find out who stole how much. (The latter is important to repay the debts to the reindeer fairly.)

The result A is given in the following form: $A = \big(\text{Case i), Case ii), Case iii)}\big)$.

Possible answers:

1. $A = (1, 2, 3)$
2. $A = (1, 2, 4)$
3. $A = (2, 2, 3)$
4. $A = (2, 3, 4)$
5. $A = (2, 3, 5)$
6. $A = (2, 4, 5)$
7. $A = (3, 3, 4)$
8. $A = (3, 4, 5)$
9. $A = (3, 4, 6)$
10. $A = (4, 5, 5)$

Project reference:

In many applications of mathematics, solutions to systems of equations where only a few of the involved variables are non-zero (i.e., sparse) are of particular importance. Such solutions can often be determined with significantly fewer equations than general solutions—that is the main message of the so-called *Compressed Sensing* theory. Remarks:

Solution

Answer 7: The number of questions Santa must ask for the three cases is: $A = (3, 3, 4)$.

First, we notice that we must ask each elf at least once ("embedded in a pair"). Otherwise, the information Santa receives would be completely independent of the elves that were not questioned. Now let's consider the solutions for the three cases, where three, four, or five elves are under suspicion.

Case 1: Three elves. If we only select one pair, we cannot question all the elves. Thus, questioning only one pair is ruled out. Two pairs are also not enough: Let's call the pairs we question (A, B) and (B, C)—note that the pairs must share at least one elf in each case—then the two configurations "A steals 0, B steals 2, and C steals 1" and "A steals 2, B steals 0, and C steals 3" (which are both possible) will produce the same "measurements." However, three pairs are enough: If we choose $(A, B), (A, C)$, and (B, C), we will get a linear system of equations in each case that is uniquely solvable—accordingly, we can reconstruct the amounts stolen.

Case 2: Four elves. Again, one pair is not enough. We can exclude two pairs in this case as follows: If the first pair questioned is (A, B), the second pair must be (C, D), because otherwise D would never be questioned. However, we will only learn about the sum of the stolen amounts for each pair, and we will never be able to reconstruct the exact amounts for each individual elf.

Thus, we must question at least three pairs; however, that is sufficient: The strategy $(A, B), (A, C)$, and (A, D) will always succeed. This can be seen as follows: If A has stolen nothing, at least one of the sums must equal zero. If all measurements are non-zero, we can conclude that A is guilty. If we then investigate which of the three sums differs from the other two, we can determine who the other culprit is and then calculate the exact stolen amounts.

However, if one sum is equal to zero, we can immediately conclude that both elves in the corresponding pair are innocent. The others are thus guilty, and we can directly read off how much each of them stole (since A has stolen nothing in this case).

Case 3: Five elves. Here, both one pair and two pairs can be excluded using the "not all questioned" argument. Three pairs are also not enough: Since we must question all elves, we will obtain, up to permutations, the pairs $(A, B), (C, D)$, and (A, E). In the case that only the middle measurement is non-zero, we can never backtrack to find out how much C and D stole.

However, four pairs are sufficient: $(A, B), (A, C), (A, D)$, and (A, E) works. The argument is similar to the above situation: If all sums are non-zero, A must be guilty, and we can determine who the other culprit is through comparisons. In the case that there is a zero measurement, A is innocent, which implies that we can easily read off the culprits and the amounts they stole from the non-zero measurements.

Thus, the correct answer is: $(7) : \mathrm{A} = (3, 3, 4)$.

2017 Concert

Author: Onno Boxma (TU Eindhoven)

Challenge

This evening 26 gnomes are visiting the Christmas concert. They have tickets for the 26 seats in row 1, which are numbered 1, 2, ..., 26. Every gnome has the number of his seat printed on his entrance ticket. The gnomes enter the concert hall in alphabetical order: first Atto, then Bilbo, then Chico, then Dondo, and so on, with Ziggo being the last.

Atto has lost his ticket and completely randomly (more precisely, uniformly distributed) takes a seat in the first row. Bilbo, Chico, and Dondo have also lost their tickets and take seats completely randomly among the remaining empty seats. The next 22 gnomes still have their tickets and behave as follows: they first go to the seat for which they have a ticket. If that seat is still free, they take it; otherwise, they randomly take one of the remaining empty seats in the first row.

Let p denote the probability that Ziggo gets the seat for which he has a ticket. Which of the following statements about p is correct?

Possible answers:

1. $p \leq 0.002$
2. $0.002 < p \leq 0.004$
3. $0.004 < p \leq 0.008$
4. $0.008 < p \leq 0.016$
5. $0.016 < p \leq 0.032$
6. $0.032 < p \leq 0.064$
7. $0.064 < p \leq 0.128$
8. $0.128 < p \leq 0.256$
9. $0.256 < p \leq 0.512$
10. $0.512 < p$

Solution

The correct answer is: 8.

We show two possible solution methods.

First approach: Denote the seat numbers of Atto, Bilbo, Chico, Dondo, and Ziggo by A, B, C, D, and Z. We make two observations:

- Ziggo can only be sitting in one of the five seats A, B, C, D, or Z. Each other seat will at the latest be occupied by the person who had the corresponding seat number.
- When one of the first 25 gnomes randomly chooses a seat, his decision makes no distinction between the five seats A, B, C, D, and Z.

This implies that Ziggo will randomly get one of the five seats A, B, C, D, or Z. Hence, the probability that Ziggo gets his own seat is $p = 1/5$.

Second approach: We generalize the problem to $n = 26$ gnomes and consider an arbitrary number $n \geq 5$ gnomes. The first four gnomes randomly take a seat, as before, and the remaining $n - 4$ gnomes behave as indicated above. Let $P(n)$ denote the probability that the last gnome gets his own seat. It is easily seen that $P(5) = \frac{1}{5}$: the first four gnomes randomly occupy four seats, and the fifth gnome has equal probabilities $1/5$ of being seated in any of the five seats.
Now, let $n \geq 6$. Consider the point at which Espo, the fifth gnome, enters the concert hall. We distinguish two cases:

- In the first case, Espo's seat is free. He takes it, and plays no further role in the remainder of the reasoning. We might as well remove Espo and his seat from the story, reducing it to a situation with $n - 1$ gnomes. Hence, the probability that the last gnome gets his own seat then is $P(n - 1)$.
- In the second case, Espo's seat is taken by one of the first four gnomes—say, by Atto. Now five seats are taken: Espo's and four random seats. We can now let Atto and Espo switch seats and again remove Espo and his seat. This reduces the problem to one in which the first four gnomes randomly took seats from a set of $n - 1$ seats, and $P(n-1)$ is again the probability that the last (of the $n-1$) gnome gets his own seat.

The first case occurs with some probability q, and the second with probability $1-q$. Hence,

$$P(n) = qP(n - 1) + (1 - q)P(n - 1) = P(n - 1).$$

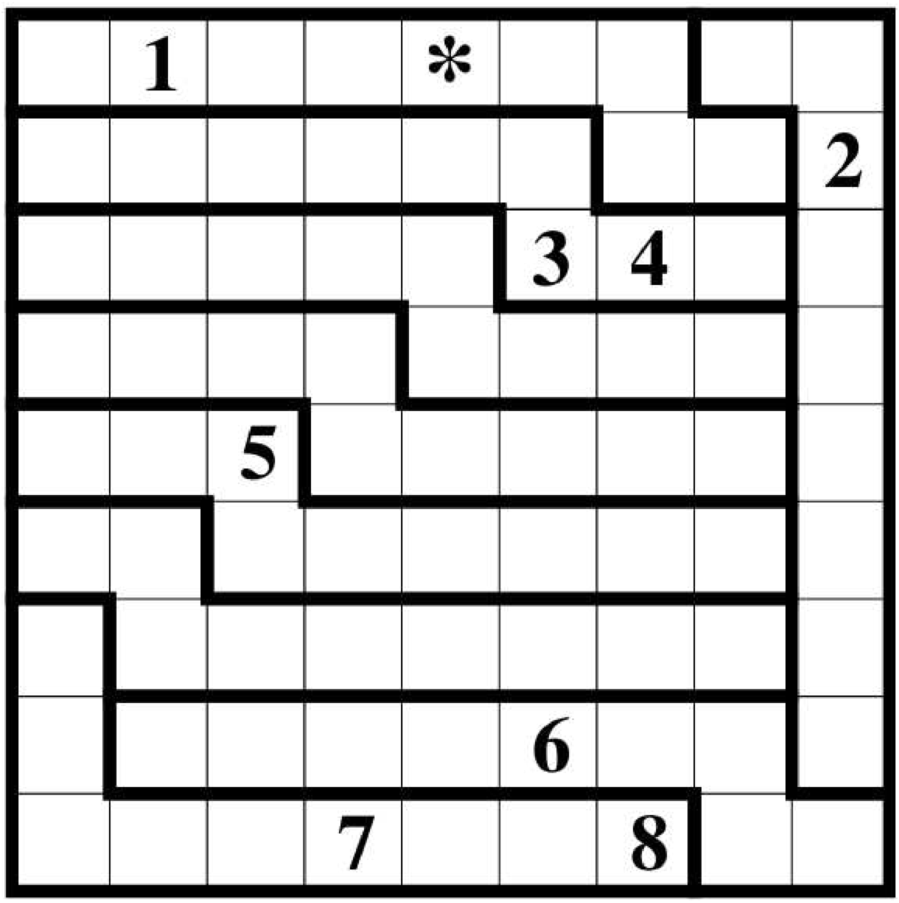
By induction, we conclude that $P(n) = P(5)$ for all $n \geq 5$.

2018 Kudosu

Author: Cor Hurkens (TU Eindhoven)

Challenge

Kudosu is a variant of the well-known Sudoku puzzle. The 81 cells in the following 9×9 grid are to be filled with the digits $1, 2, \ldots, 9$, such that each row and each column contains each of the nine digits exactly once. Furthermore, each of the nine marked regions must also contain each of the nine digits exactly once.



We want to know from you: Which digit goes in the cell with the star?

Illustration: Friederike Hofmann

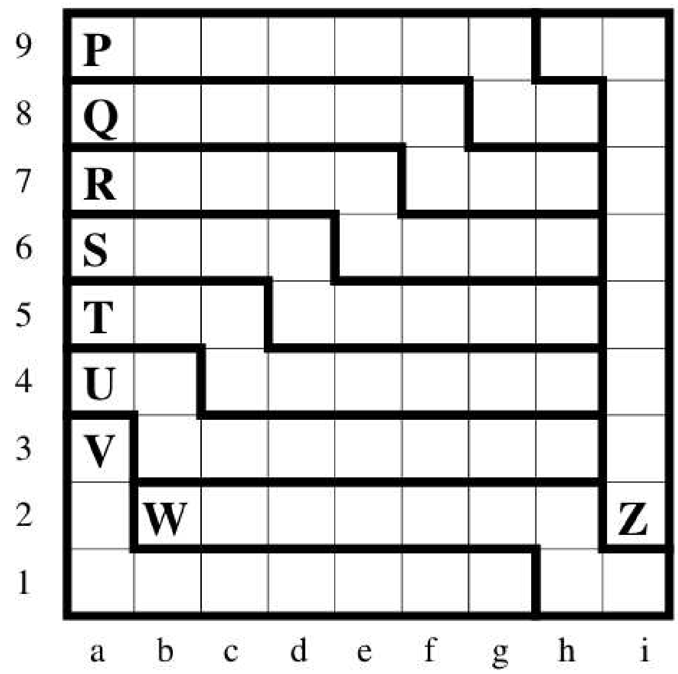
Possible answers:

1. The digit 1 goes in the cell with the star.
2. The digit 2 goes in the cell with the star.
3. The digit 3 goes in the cell with the star.
4. The digit 4 goes in the cell with the star.
5. The digit 5 goes in the cell with the star.
6. The digit 6 goes in the cell with the star.
7. The digit 7 goes in the cell with the star.
8. The digit 8 goes in the cell with the star.
9. The digit 9 goes in the cell with the star.
10. The content of the cell with the star is not uniquely determined.

Solution

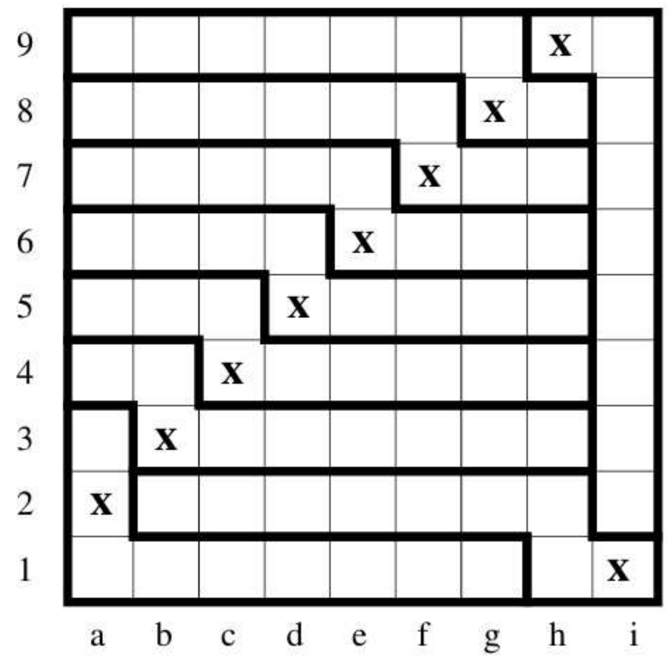
The correct answer is: 9.

We number the rows from 1 to 9 and the columns from a to i, naming the nine regions as $P, Q, R, S, T, U, V, W, Z$ as shown in the following figure:

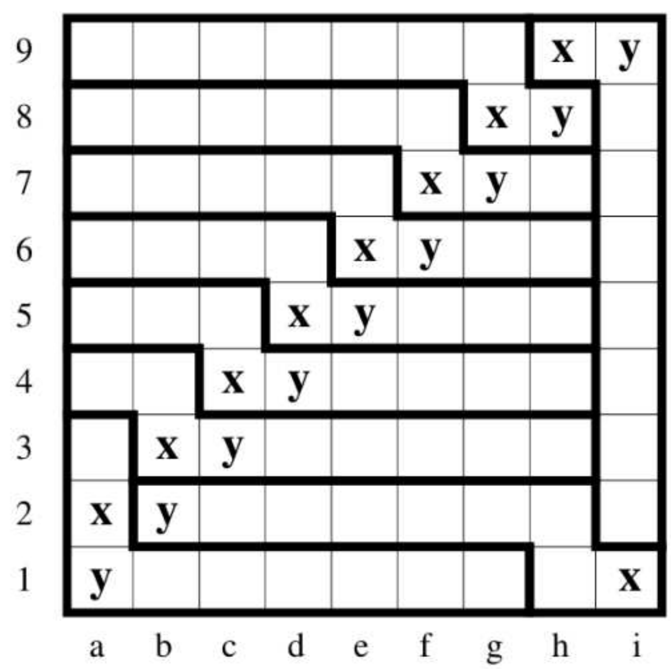


Now we consider the entry x in the cell $i1$ (which belongs to the region W). This entry must also appear in the region Z; since x only appears once in column i, the cell $h9$ must contain this entry x. The entry x must also appear in the region P; it is easy to see that only the cell $g8$ is suitable for this. Subsequently, we can continue diagonally in a similar manner:

- In region Q, entry x is in cell $f7$,
- in region R in cell $e6$,
- in region S in cell $d5$,
- in region T in cell $c4$,
- in region U in cell $b3$, and
- in region V in cell $a2$:



Next, we consider the entry y in cell $i9$. This entry must also appear in region P, and only cell $h8$ is suitable for this. Through similar reasoning, it can be seen that the seven cells $g7$, $f6$, $e5$, $d4$, $c3$, $b2$, $a1$ must also contain entry y:



Analogous arguments along other diagonals show that each entry must match the entry directly above it to the right. This results in the following unique solution to the Kudosu puzzle:

9	2	1	7	6	9	8	5	3	4
8	1	7	6	9	8	5	3	4	2
7	7	6	9	8	5	3	4	2	1
6	6	9	8	5	3	4	2	1	7
5	9	8	5	3	4	2	1	7	6
4	8	5	3	4	2	1	7	6	9
3	5	3	4	2	1	7	6	9	8
2	3	4	2	1	7	6	9	8	5
1	4	2	1	7	6	9	8	5	3
	a	b	c	d	e	f	g	h	i

2019 Organizing Gingerbread

Author: Falk Ebert (Herder-Gymnasium Berlin)

Challenge

It smells delicious in the Christmas bakery "Tasty Pastry" at the North Pole, where the world's best gingerbread, cookies, and Christmas stollen are baked.

Packing elf Paul is assigned to pack the most tasteful gingerbread (which are almost cuboidal, 5 cm wide and 15 cm long[4]) into big boxes. Each of the normed boxes is 40 cm wide and 40 cm long. Paul always packs layers of several gingerbread with parchment paper between the layers. It is up to him to decide how to arrange the biscuits in each layer. However, it is a matter of honor for Paul to arrange the gingerbread as efficiently as possible, i.e., such that there are no other arrangements that fit more gingerbread.

He notices one fact right away: in every cake layer there is always a small empty space left which he cannot fill, regardless of how he arranges the gingerbread. After packing several boxes, he observes that when packing most efficiently, the same phenomenon occurs in each layer.

Which phenomenon does Paul notice?

Artwork: Frauke Jansen

Possible answers:

[4] Since the height of gingerbreads is not important for the solution, it is not stated here.

1. There is exactly one possibility for the position of the empty region.
2. There are exactly four possible positions for the empty region.
3. There are exactly eight possible positions for the empty region.
4. There are exactly 16 possible positions for the empty region.
5. There are exactly 64 possible positions for the empty region.
6. The empty space always touches the boundary of the box.
7. There is always a gingerbread-long distance (15 cm) between the empty region and a side of the box.
8. The empty region is never connected.
9. The empty region always has a quadratic shape with side length 10 cm.
10. The empty region is always **L**-shaped.

Solution

The correct answer is: 2.
We can partition the base of the box into 64 small squares of size 5 cm x 5 cm. The gingerbreads can be partitioned into three small squares of size 5 cm x 5 cm as well (see Figure 8).

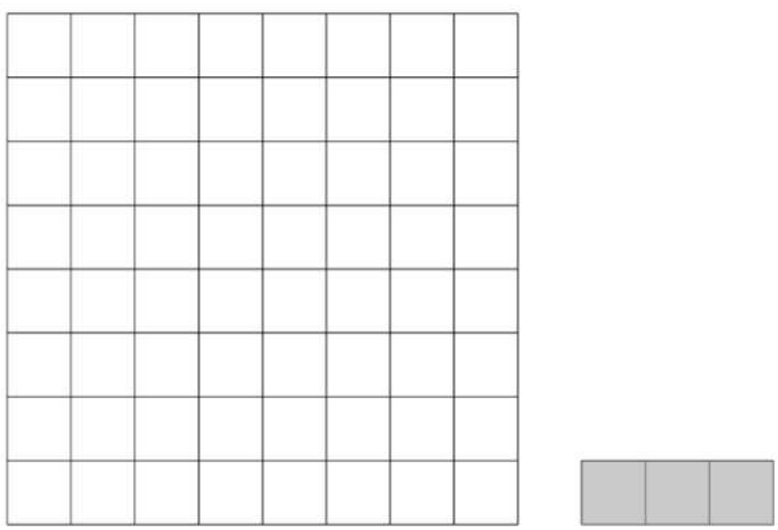


Figure 8: On the left: base of the box. On the right: a gingerbread.

We notice immediately (as Paul does) that 64 leaves a remainder of 1 when divided by 3, i.e., $64 = 21 \cdot 3 + 1$. Therefore, we can fit a maximum of 21 gingerbreads into one layer. An example of how this can be achieved is shown in Figure 9.

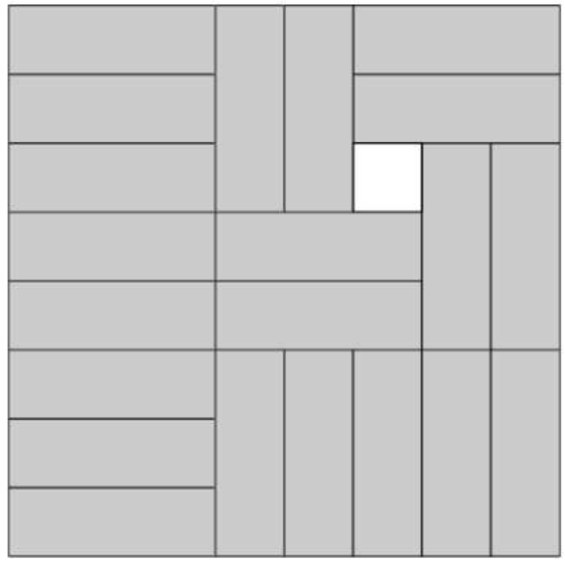


Figure 9: One possibility for fitting 21 gingerbreads into one layer.

We show that there are exactly four possibilities to fit 21 gingerbreads into the box. To do this, we color the base of the box with three different colors as shown in Figure 10.

We observe that one gingerbread—regardless of how we place it in the box (vertically or horizontally)—always covers exactly one pink, one blue, and one yellow square. In total, we count 21 blue, 21 yellow, and 22 pink squares. Thus, fitting 21 gingerbreads into one layer inside the box leaves an empty pink square. Since the packing problem is both mirror-symmetric and rotationally symmetric, the only candidates for the empty pink square are those that remain pink under reflection or rotation (by 90°, 180°, or 270°). For example, assume that the bottommost left, pink square remains blank, then this is equivalent to the uppermost left, *yellow* square remain blank – but we already eliminated this possibility. Hence, there are only exactly **four squares** that might remain empty in the most efficient packing (see Figure 11). In Figure 9, we see that it is indeed possible to pack 21 gingerbreads into the box in such a way that one of the four pink squares remains empty.

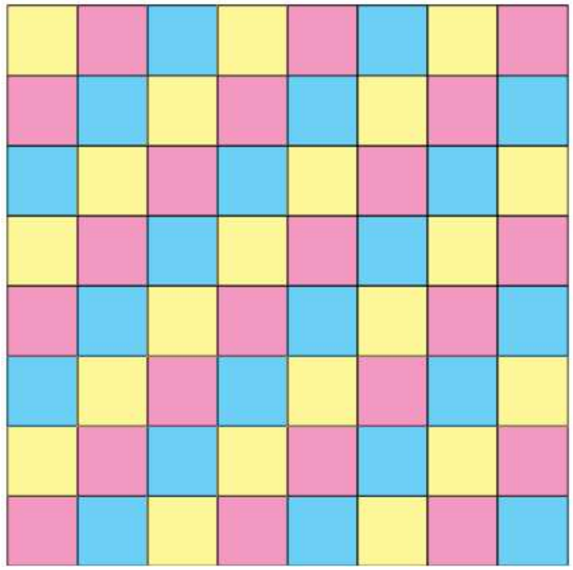


Figure 10: Coloring of the base of the box in three different colors.

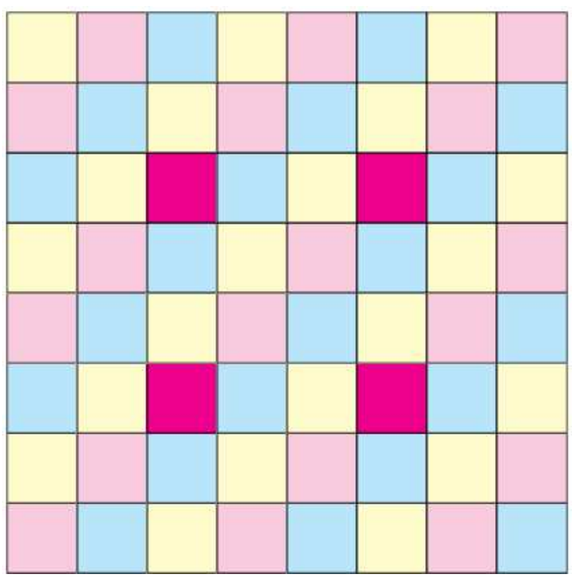


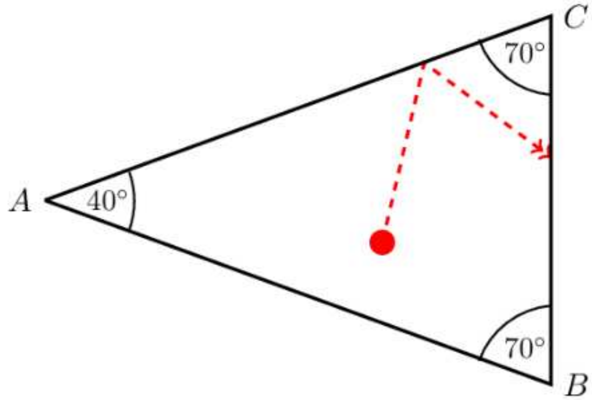
Figure 11: The dark pink squares are the only possible positions for the empty region.

2020 Biliards Table

Author: Hennie ter Morsche(TU Eindhoven)
Project: 4TU.AMI

Challenge

There is a large triangular billiards table standing in the pixies' lounge. The angle at corner A is $40°$, whereas the two angles at B and C each are $70°$. If the ball hits one of the rails AB or AC, it is perfectly reflected so that the angle of reflection is equal to the angle of incidence. However, if the ball hits the sticky rail BC or if it hits one of the three corners A, B, or C, it gets stuck and stops moving.



Conveniently, Ruprecht plays with a point-shaped ball that initially is located somewhere in the interior of the triangle and that moves only along straight lines. Ruprecht wants to make a single shot that scores as many rail contacts as possible before the ball gets stuck at some rail or point.

What is the largest possible number of such rail contacts?

Artwork: Julia Nurit Schönnagel

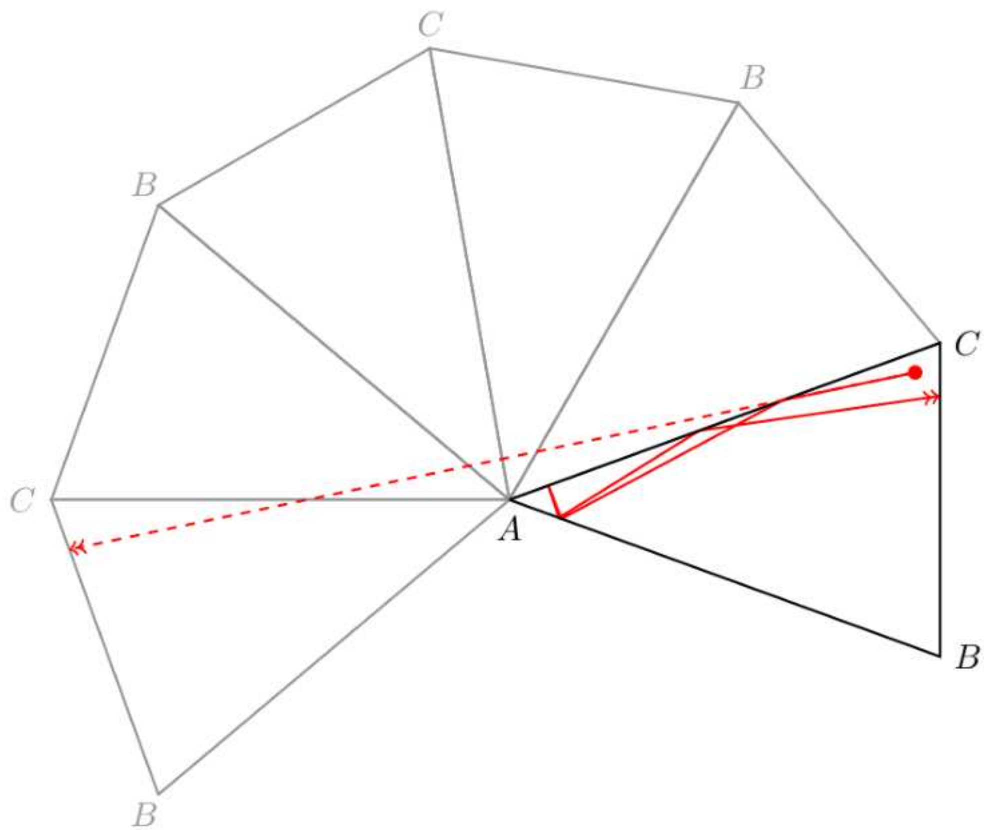
Possible answers:

1. The largest possible number of rail contacts is 4.
2. The largest possible number of rail contacts is 5.
3. The largest possible number of rail contacts is 6.
4. The largest possible number of rail contacts is 7.
5. The largest possible number of rail contacts is 8.
6. The largest possible number of rail contacts is 12.
7. The largest possible number of rail contacts is 16.
8. The largest possible number of rail contacts is 24.
9. Ruprecht may score infinitely many rail contacts.
10. The largest possible number of rail contacts depends on the length of BC.

Solution

The correct answer is: 2.
Since the third rail BC is absorbent, the ball can only run back and forth between the two rails AB and AC. Since this configuration is symmetric, we may assume that the ball touches AC first, then AB, then again AC, and so on.

The following figure shows the billiards table (at the right), together with five symmetric copies:



Since the angle of reflection is equal to the angle of incidence for each reflection at both of the rails AB and AC, the path of the ball corresponds to a straight line running from the starting point to the point of absorption. The dashed line shows such a possible path, which allows for exactly five rail contacts. (The solid line represents the corresponding movement of the ball on our billiards table.)

The rail AC of the original triangle and the rail AB of the fifth copy enclose an angle of $5 \times 40° = 200°$. Since this angle is larger than $180°$, one cannot find a straight line that runs through more than five copies of the triangle ABC. Hence, six (or more) rail contacts are not possible.

2021 Magic Ribbons

Authors: Myfanwy Evans (Uni Potsdam), Frank Lutz (TU Berlin)
Project: Thematic Einstein Semester 2021: Geometric and Topological Structure of Materials

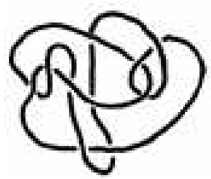
Challenge

Far, far up in the north is the home of the elves. When winter is near, the elves help Santa Claire to wrap presents with their magic ribbons. Spheric sounds ring upon touching the strings.

In the past year, the elves faced a special challenge when Santa Claire asked them to wrap a chocolate torus. They cut open one of their ribbons, wound it around the chocolate torus, and used magic to glue the ribbon back together. One of the elves realized that the ribbon on the torus became knotted, but soon forgot about it.

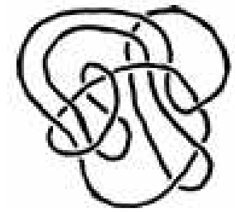
Every year, the magic ribbons are returned to the elves and stored to be reused for the next season. The special ribbon ended up in a bag with nine other ribbons. When the elves took out the ribbons in their preparations for this year's festivities, the ribbons appeared somewhat "entangled", and it took them quite some time to disentangle nine of them. Only then did they remember that one of the ten is special.

Which one is the special ribbon that was wrapped around the chocolate torus last year?

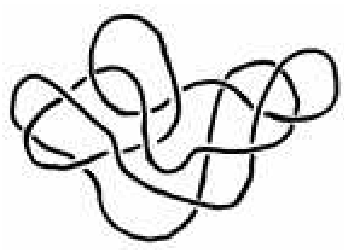


Ribbon no. 1.

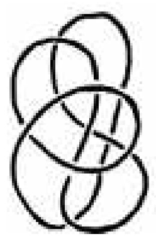
Ribbon no. 2.



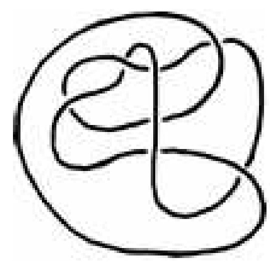
Ribbon no. 3.



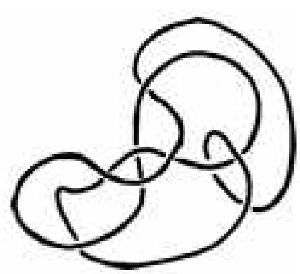
Ribbon no. 4.



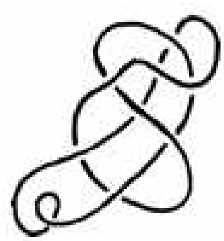
Ribbon no. 5.



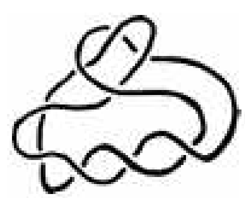
Ribbon no. 6.



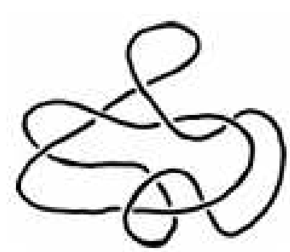
Ribbon no. 7.



Ribbon no. 8.



Ribbon no. 9.



Ribbon no. 10.

Artwork: Friederike Hofmann

Possible answers:

1. Ribbon no. 1.
2. Ribbon no. 2.
3. Ribbon no. 3.
4. Ribbon no. 4.
5. Ribbon no. 5.
6. Ribbon no. 6.
7. Ribbon no. 7.
8. Ribbon no. 8.
9. Ribbon no. 9.
10. Ribbon no. 10.

Project reference:

Physical properties of materials are governed to a large extent by their microstructure. Some materials are highly ordered like crystals, some are polycrystalline like rocks or metals, others are cellular like soap or metallic foams, are disordered like amorphous solids, and some even are entangled like DNA.

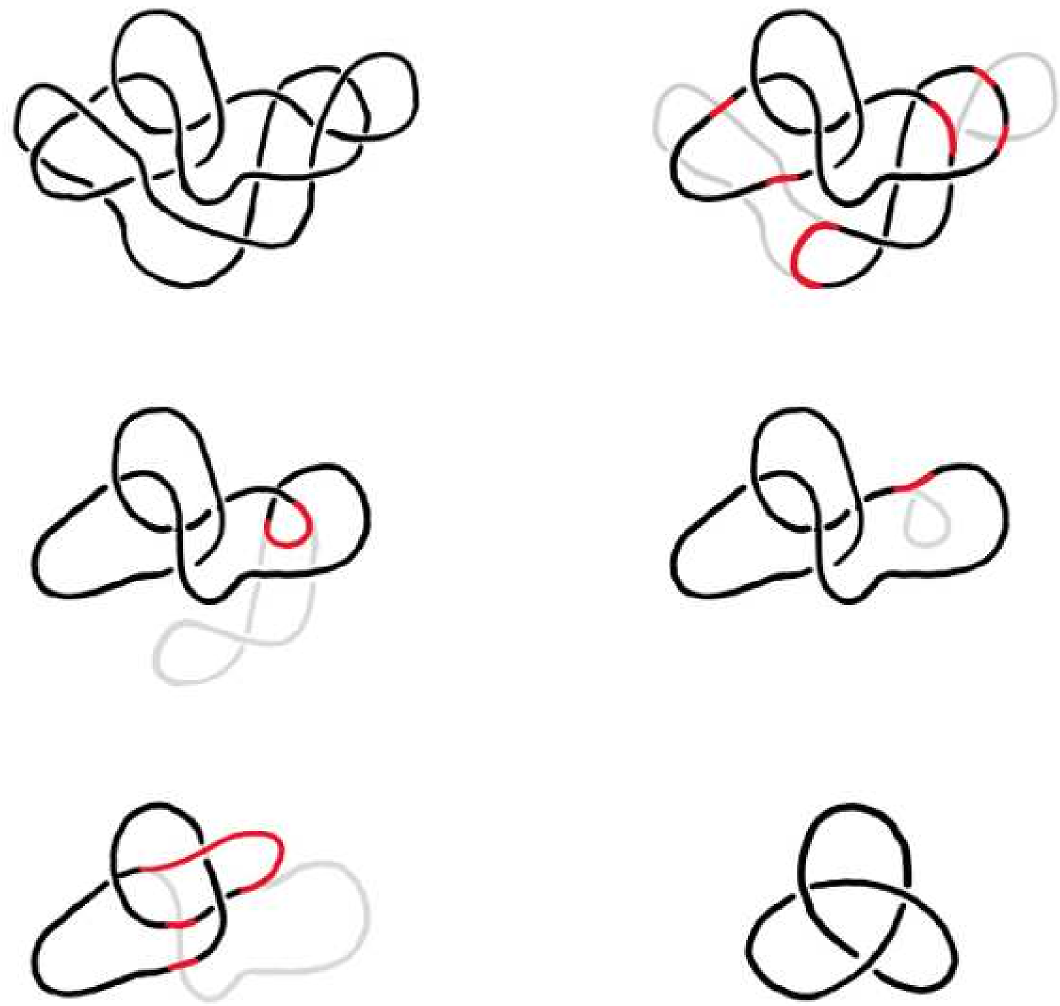
The Thematic Einstein Semester 2021 "Geometric and Topological Structure of Materials" was devoted to illuminate recent mathematical developments for a better understanding of materials by identifying or computing essential structural properties - eventually leading to improvements in production processes or to new designs of materials with controlled properties.

Solution

The correct answer is: 4.

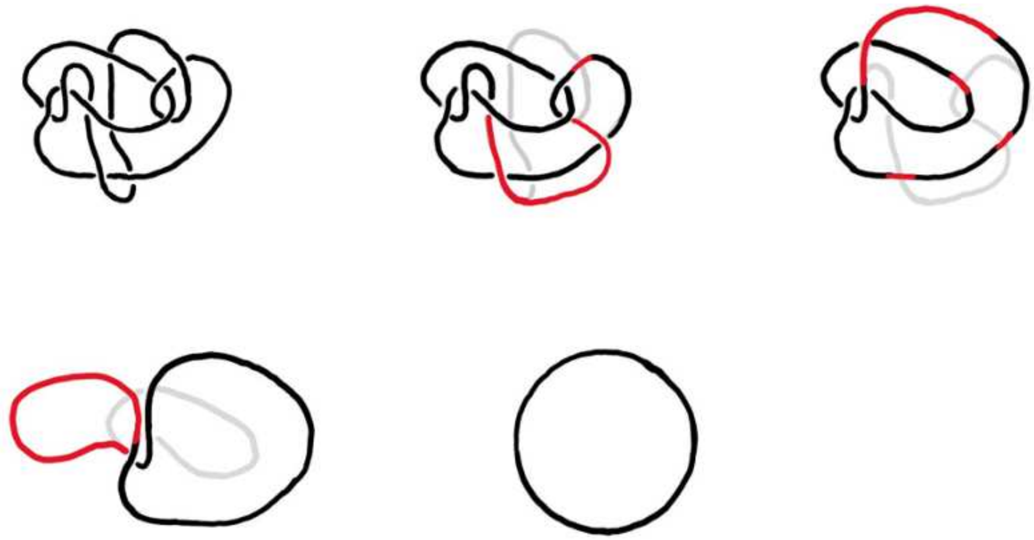
We will deform the fourth ribbon into the *trefoil knot*, which was wrapped around the chocolate torus. The trefoil knot is the simplest example of a non-trivial knot.

Ribbon no. 4:

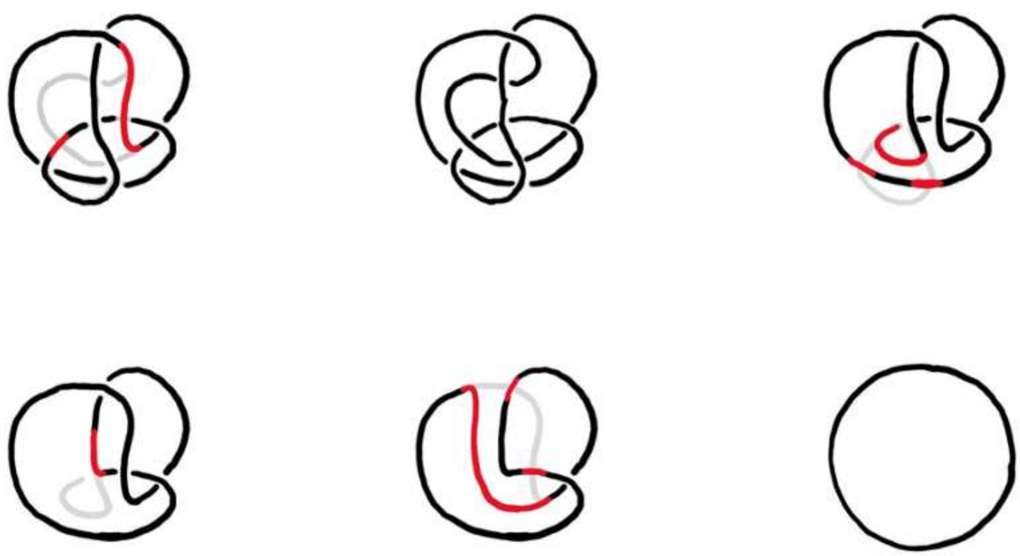


The other ribbons can indeed be transformed into the *unknot*, i.e., the trivial knot, which was wrapped around the other presents.

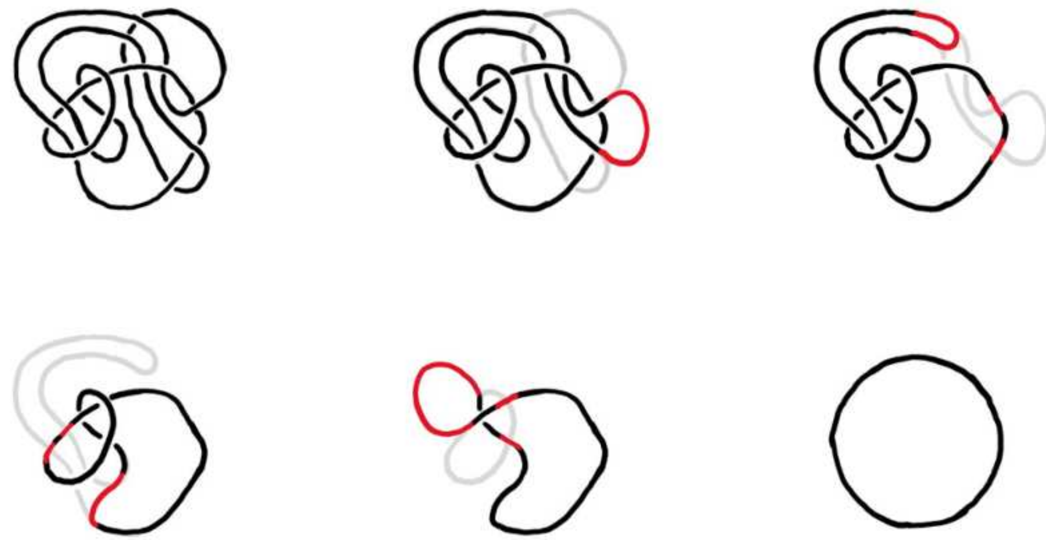
Ribbon no. 1:



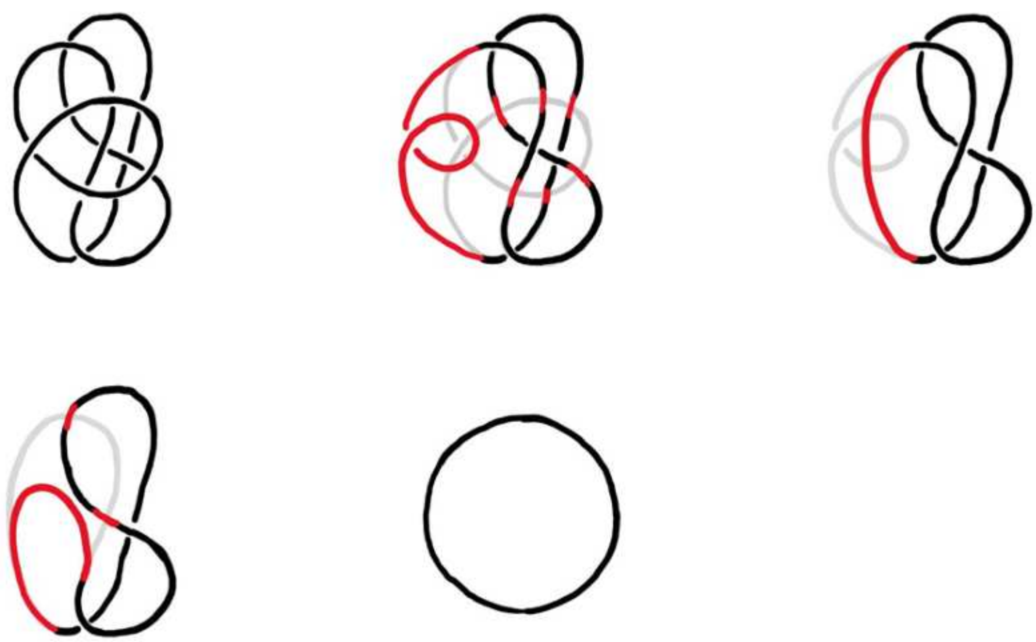
Ribbon no. 2:



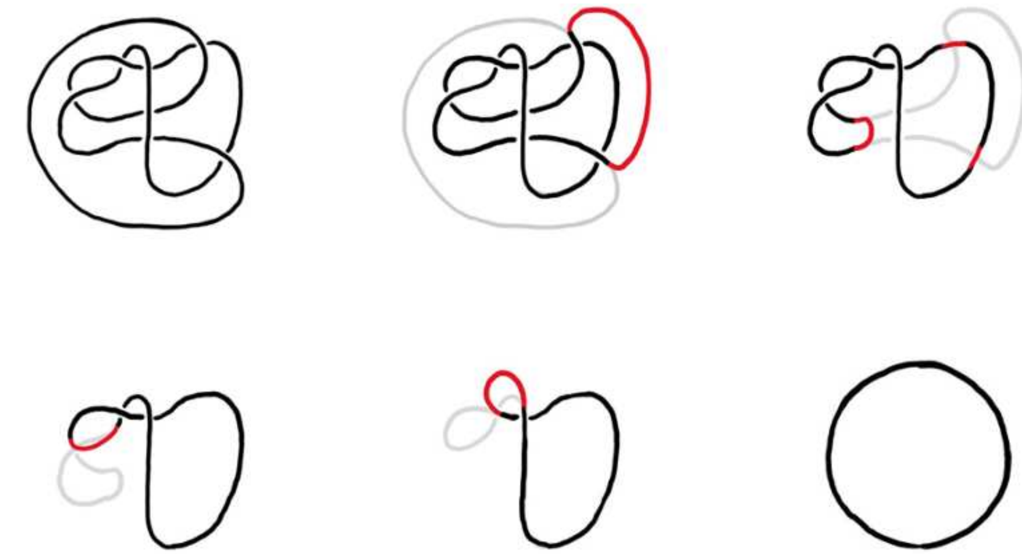
Ribbon no. 3:



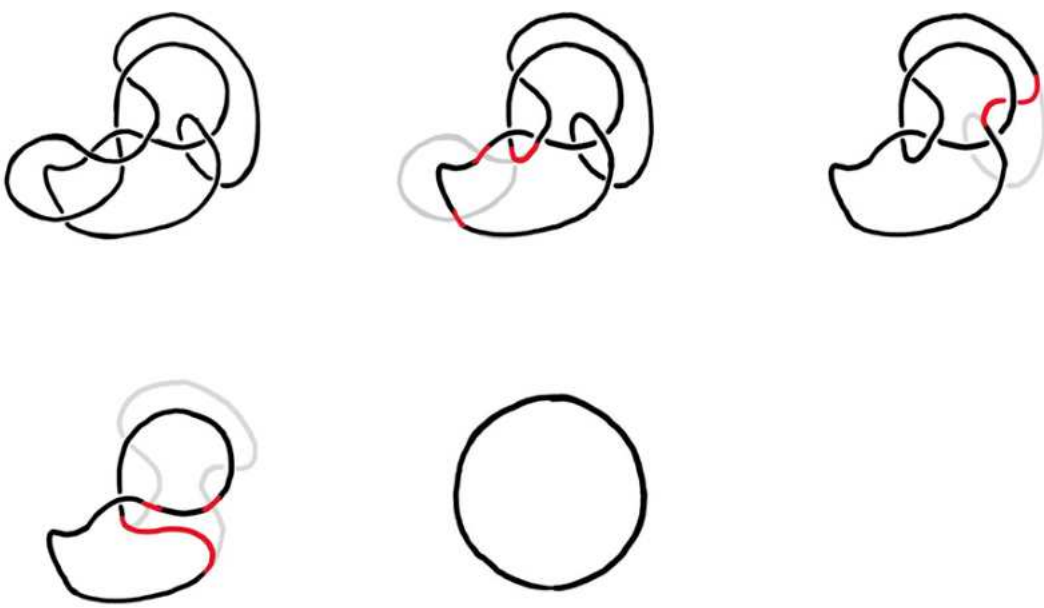
Ribbon no. 5:



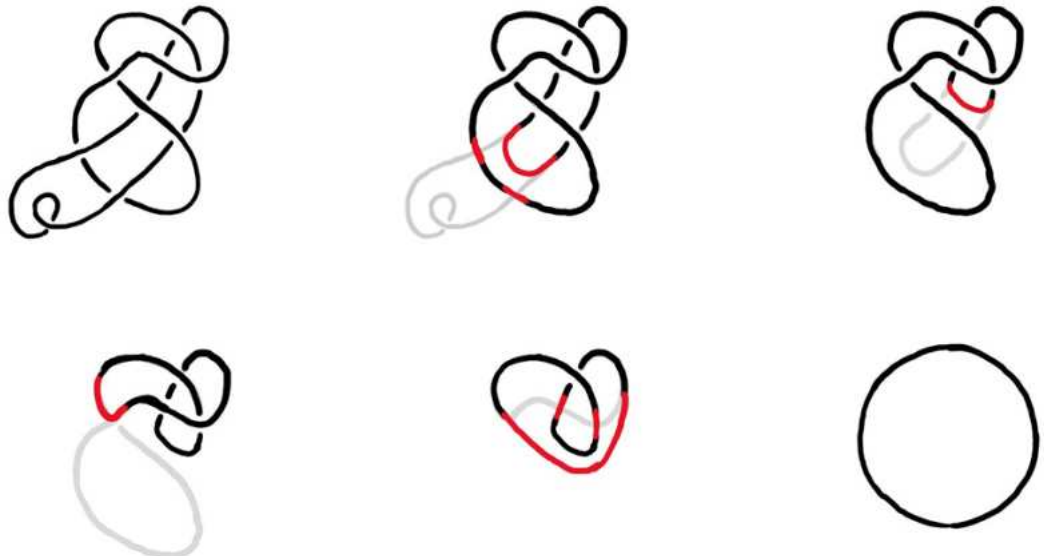
Ribbon no. 6:



Ribbon no. 7:

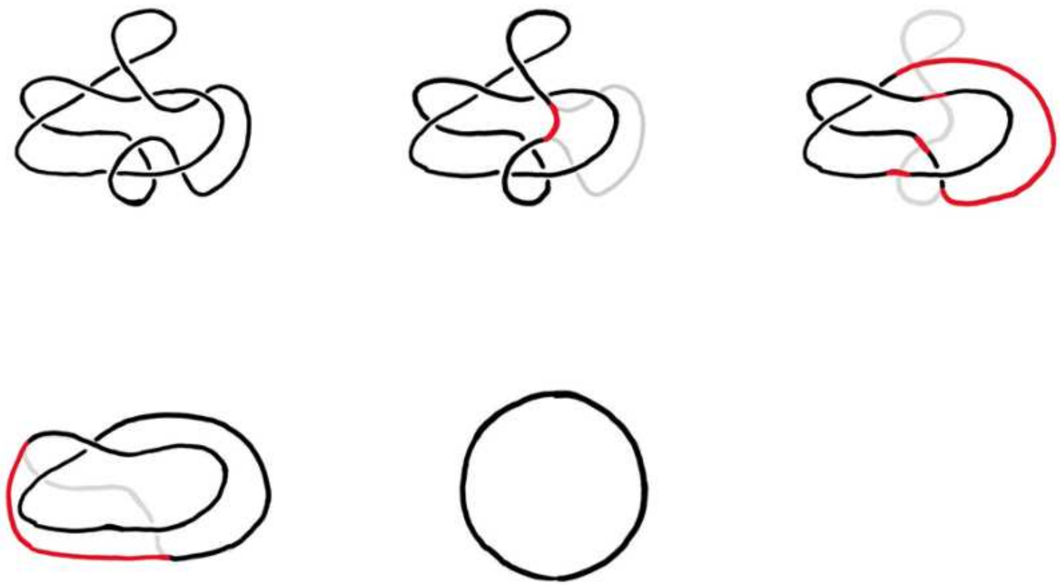


Ribbon no. 8:



Ribbon no. 9:

Ribbon no. 10:



The moves we performed on the ten ribbons are variants of the three *Reidemeister moves*, which transform a knot diagram into equivalent (or *isotopic*) knot diagrams.

2022 The Chocolate Game

Authors: Olaf Parczyk, Silas Rathke (FU Berlin)
Project: *Learning Extremal Structures in Combinatorics* (EF 1-12)

Artwork: Friederike Hofmann

Challenge

The supersmart elves Atto and Bilbo have a big bar of chocolate with n columns and m rows, where m and n are positive integers. We denote by (i, j) the piece in the i-th column and the j-th row of the chocolate. The piece (n, m) is filled with orange jelly and is therefore disgusting (this fact need not be proved).

Atto and Bilbo play the following game: *starting with Atto*, they take turns making moves. A move consists of choosing a piece (i, j) that is still available and eat up all the pieces (i', j') with $i' \leq i$ and $j' \leq j$. Whoever has to eat the piece (n, m) with the orange jelly loses.

Example: A possible move for $n = 7$ and $m = 4$ might look like this: If a player chooses the piece $(4, 2)$ in his turn, he must eat the pieces shaded in blue:

The piece $(7, 4)$ with the orange jelly is marked with a red X. In this example, a game, which Atto loses in the end, could look like this:

1. Atto (blue) starts with the piece $(4, 2)$.

2. Then, Bilbo (orange) chooses the piece $(3, 4)$.

3. Now, Atto chooses $(6, 4)$.
4. After that, Bilbo chooses $(7, 3)$.
5. Finally, Atto must eat the piece $(7, 4)$ with the orange jelly in his last move.

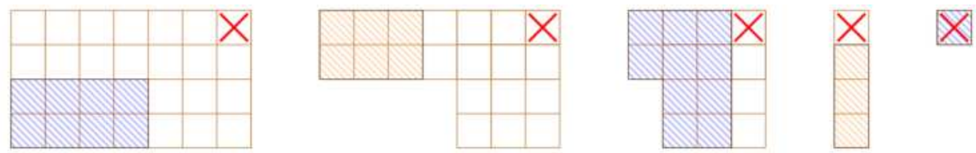


Figure 12: A possible game with a chocolate bar of size 7×4

Depending on m and n, there is *either* a strategy that can guarantee a win for Atto, *or* a strategy that can guarantee a win for Bilbo.

If we choose m and n independently and randomly from the set $\{1, 2, 3, \ldots, 10^6\}$, all numbers being chosen with equal probability, what is the probability p that Atto has a winning strategy?

Possible answers:

1. $p < 10\%$
2. $10\% \leq p < 20\%$
3. $20\% \leq p < 30\%$
4. $30\% \leq p < 40\%$
5. $40\% \leq p < 50\%$
6. $50\% \leq p < 60\%$
7. $60\% \leq p < 70\%$
8. $70\% \leq p < 80\%$
9. $80\% \leq p < 90\%$
10. $90\% \leq p \leq 100\%$

Project reference:

In the project EF 1-12 *Learning Extremal Structures in Combinatorics*, we use approaches from the field of artificial intelligence and machine learning to find new structures with certain properties. This also helps us to find new winning strategies for combinatorial games, quite similar to the one from this task.

Solution

The correct answer is: 10.

Interestingly, to the authors' knowledge, no one has yet been able to give an explicit winning strategy for all possible (m, n). However, we know that almost always Atto must have a winning strategy:

Claim: For $(m, n) = (1, 1)$, Bilbo has a winning strategy. In all other cases, Atto has one.

Proof:

1. If $(m, n) = (1, 1)$, Atto is forced in his first move to eat the piece with the disgusting orange jelly.

2. Let now $(m, n) \neq (1, 1)$. We want to prove that Atto has a winning strategy in this case. We will prove the claim by contradiction, i.e., we assume that Bilbo has a winning strategy. That is, no matter for which moves Atto decides, Bilbo always has an answer move that will allow him to win.

   In particular, Bilbo has an answer if Atto chooses the square $(1, 1)$ in his first move. Let (i, j) be such a response move by Bilbo to Atto's move $(1, 1)$ that will allow him to win. But this also means that Atto could have chosen (i, j) in his first move and would have won.

   This is a contradiction yielding that there can be no winning strategy for Bilbo for $(m, n) \neq (1, 1)$.

Thus, the probability we are looking for is:

$$p = \frac{10^6 \cdot 10^6 - 1}{10^{12}} = \frac{999,999,999,999}{1,000,000,000,000} = 0.9999999999.$$

2023 Revealing Sounds

Author: Svenja M. Griesbach & Max Klimm
Project: Information design for Bayesian networks (MATH+ AA3-9)

Challenge

For this riddle, you slip into the role of the Grinch, who constantly gets in the way of Santa Claus and his elves. To keep you from pranks on Christmas Eve this year, the elves have made you an offer: They have been baking cookies all day, and you are supposed to guess which type was baked today. If you guess the correct type, you will receive a huge jar of fresh cookies as a gift. However, if you guess wrong, you must, in turn, promise not to play pranks on Christmas Eve. Although you are often mischievous, when it comes to such offers, you can always rely on the honesty of the elves and will therefore keep the promise yourself.

As there are only three different types of cookies (Vanilla Crescents, Nut Triangles, and Chocolate Cookies) that the elves regularly and with equal probability bake, the elves are convinced that you will guess wrong in two out of three cases, giving them good chances for a relaxed Christmas. However, you have observed the behavior of the elves very well over the past few years and have collected some information about their behavior. For example, you noticed that when baking Vanilla Crescents, the elves always listen to *Driving Home For Christmas*. On the other hand, when baking Nut Triangles, they equally likely listen to either *All I Want For Christmas Is You* or *Last Christmas*. It's different when they bake Chocolate Cookies. Although they also only listen to either *All I Want For Christmas Is You* or *Last Christmas* on a loop, in two out of three cases, *All I Want For Christmas Is You* is playing.

Which statement about the three Christmas songs and cookie types is correct?

Possible answers:

1. The song *Driving Home For Christmas* is played most frequently.
2. If you know which song was played today, you can increase your average probability of winning to more than 70%.
3. There is no song for which you can be entirely certain which cookies were baked today.
4. If *All I Want For Christmas Is You* was played, you have the highest chances of winning if you bet on Nut Triangles.
5. Nut Triangles are baked more frequently than Chocolate Cookies and Vanilla Crescents.
6. If *Last Christmas* was played, you cannot rule out any type of cookie with certainty.
7. The probability that Chocolate Cookies were baked and the elves heard *All I Want For Christmas Is You* is $\frac{2}{3}$.
8. If the song *Driving Home For Christmas* was played, each type of cookie is equally likely.
9. The probability that you heard *Last Christmas* today is 20%.
10. *All I Want For Christmas Is You* is played most frequently, and *Driving Home For Christmas* is played least frequently.

Project reference:

In the Math+ project AA3-9, the *Information design for Bayesian networks* is being investigated to determine to what extent traffic equilibria can be improved through the transmission of information. It is assumed that, based on the provided information, traffic participants can draw conclusions about the actual traffic. This occurs according to a similar principle to how the Grinch in this task draws conclusions about the baked cookie type by listening to the music.

Solution

The correct answer is: 2.

To solve the puzzle, we must carefully examine the information provided by each song. Let's abbreviate the songs as S_1, S_2 and S_3 for *Driving Home For Christmas*, *All I Want For Christmas*, and *Last Christmas* respectively, and the cookie types as C_1, C_2 and C_3 for Vanilla Crescents, Nut Triangles, and Chocolate Cookies. Since all cookies are baked with equal probability, the following holds:

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}.$$

Answer 5 is therefore incorrect. Additionally, we can deduce the following *conditional* probabilities[5] from the text:

$$\begin{aligned} &P(S_1|C_1) = 1 && P(S_2|C_1) = 0 && P(S_3|C_1) = 0 \\ &P(S_1|C_2) = 0 && P(S_2|C_2) = \frac{1}{2} && P(S_3|C_2) = \frac{1}{2} \\ &P(S_1|C_3) = 0 && P(S_2|C_3) = \frac{2}{3} && P(S_3|C_3) = \frac{1}{3}. \end{aligned}$$

From this, using the law of total probability, we can derive initially how likely it is to hear each individual song. This states that the probability of an event is equal to the sum of the probabilities of all paths leading to this event:

$$\begin{aligned} P(S_1) &= P(S_1|C_1) \cdot P(C_1) + P(S_1|C_2) \cdot P(C_2) + P(S_1|C_3) \cdot P(C_3) \\ &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3} \\ P(S_2) &= P(S_2|C_1) \cdot P(C_1) + P(S_2|C_2) \cdot P(C_2) + P(S_2|C_3) \cdot P(C_3) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{7}{18} \\ P(S_3) &= P(S_3|C_1) \cdot P(C_1) + P(S_3|C_2) \cdot P(C_2) + P(S_3|C_3) \cdot P(C_3) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{18}. \end{aligned}$$

Answers 1, 9, and 10 can thus be ruled out. Bayes' theorem for conditional probabilities

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)},$$

gives us the remaining conditional probabilities:

$$\begin{aligned} &P(C_1|S_1) = 1 && P(C_2|S_1) = 0 && P(C_3|S_1) = 0 \\ &P(C_1|S_2) = 0 && P(C_2|S_2) = \frac{3}{7} && P(C_3|S_2) = \frac{4}{7} \\ &P(C_1|S_3) = 0 && P(C_2|S_3) = \frac{3}{5} && P(C_3|S_3) = \frac{2}{5}. \end{aligned}$$

[5] Here by conditional probability $P(A|B)$ is meant the probability of an event A occurring, given that event B is already known to have occurred. It is given by the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

This allows us to identify answers 3, 4, 6, and 8 as incorrect. Depending on the song we heard today, we now know which cookie type is most likely. Thus, we maximize our chances of winning as follows:

- If we hear song S_1 (Driving Home For Christmas), we bet on cookie type C_1 (Vanilla Crescents).
- If we hear song S_2 (All I Want For Christmas), we bet on cookie type C_3 (Chocolate Cookies).
- If we hear song S_3 (Last Christmas), we bet on cookie type C_2 (Nut Triangles).

Now, to calculate our winning probability, we need to consider the probabilities of each individual song. Let R represent the event where we guess correctly. We calculate the winning probability with the law of total probability:

$$\begin{aligned} P(R) &= P(S_1) \cdot P(C_1|S_1) + P(S_2) \cdot P(C_3|S_2) + P(S_3) \cdot P(C_2|S_3) \\ &= \frac{1}{3} \cdot 1 + \frac{7}{18} \cdot \frac{4}{7} + \frac{5}{18} \cdot \frac{3}{5} \\ &= \frac{13}{18} \approx 72.2\%. \end{aligned}$$

The correct solution is, therefore, answer 2. To finally refute answer 7, we only need to calculate:

$$P(S_2 \cap C_3) = P(C_3) \cdot P(S_2|C_3) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$