MATH+ KALENDER

Challenges & Solutions 2023

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Contents

1 Water Damage in the Technical Department

Author: Olaf Parczyk, Silas Rathke (FU Berlin)
Project: EF 1-12

Illustration: Friederike Hofmann

Challenge

In the technical department, ten elves work in ten offices on various technical problems that may arise in the world's largest gift production. However, due to climate change and the associated rise in sea levels, three offices have suffered water damage and are now unusable. So, we have ten elves but only seven offices. What should be done now?
The solution is found quickly: only seven elves come to the technical department each day, and the other three work from home.
Security elf Willi has the important task of equipping ten elves with keys for offices. Each key fits only one specific office lock and is given to an elf. An elf can even receive multiple keys. Importantly, offices can also be opened by multiple keys.

Willi has a clear rule for key distribution: after the keys are distributed, it should not matter which seven elves come to the technical department. It must always be possible to assign the seven elves to the seven functional offices, such that each elf has access to an office with the corresponding key. Exchanging or borrowing keys is strictly prohibited.
Of course, Willi could give each elf a key for each office, meaning he would need to make a total of seventy keys. However, Willi is considering whether it is possible with fewer keys. He is looking for the smallest number, k, of keys he must distribute in total to fulfill the previous rule. Which statement about k is correct?

Possible Answers:

1. $k < 20$
2. $20 \leq k < 25$
3. $25 \leq k < 30$
4. $30 \leq k < 35$
5. $35 \leq k < 40$
6. $40 \leq k < 45$
7. $45 \leq k < 50$
8. $50 \leq k < 55$
9. $55 \leq k < 60$
10. $60 \leq k$

Project Reference:

In the project "Learning Extremal Structures in Combinatorics", we use approaches from the field of artificial intelligence and machine learning to find new constructions for graphs with specific properties. The presented task can be well formulated as an extremal problem in graph theory, similar to the problems we study in this project.

Solution

The correct answer is: 3.
We show that $k = 28$ is the correct answer. To do this, we need to prove two things:

1. There is a distribution with 28 keys that satisfies the desired rule.

2. There is no distribution with 27 or fewer keys that satisfies the desired rule.

First, we provide a distribution with 28 keys. Willi creates one key for each office and distributes these seven keys to seven elves. These seven elves, whom we call Type A elves, do not receive any additional keys. The other three elves, called Type B elves, receive one key for each office. This totals $7 \cdot 1 + 3 \cdot 7 = 28$ keys. If the Type A elves, who have only one key, come to the office, they go to their respective office. If any of the other three Type B elves is present, they can easily go to any of the remaining offices, as they have a key to each office. Thus, this distribution with 28 keys indeed works.
Now, we show that there is no distribution with 27 keys. Suppose there is such a distribution. Then, on average, an office can be opened by $\frac{27}{7} \approx 3,857$ keys. In particular, there must be an office C that can be opened by at most 3 keys. Otherwise if every office had at least 4 keys, the average would be at least 4. However, if three elves with access to office C all stay at home, no one can open C. This shows that there can't be a distribution with 27 keys.

2 Gift Organisation

Author: Lotte Weedage (University of Twente)
Project: 4TU.AMI

Illustration: Friederike Hofmann

Challenge

The elves need your help! There is a large, perfectly square warehouse in Santa's village where they store all gifts. To be sure they do not miss any gift, they arrange their gifts as follows: the first gift is stored in the exact middle of the warehouse, and every next gift is stored 1 meter away in a counter clockwise spiraling motion around this gift. In Figure 1 we show an example: the first gift is in the middle, the second gift is stored one meter to the right of this gift, the third gift one meter above the second gift, et cetera. The system can also be seen as a two-dimensional grid: we call the location of the first gift $(0, 0)$, then the location of the second gift is $(1, 0)$.

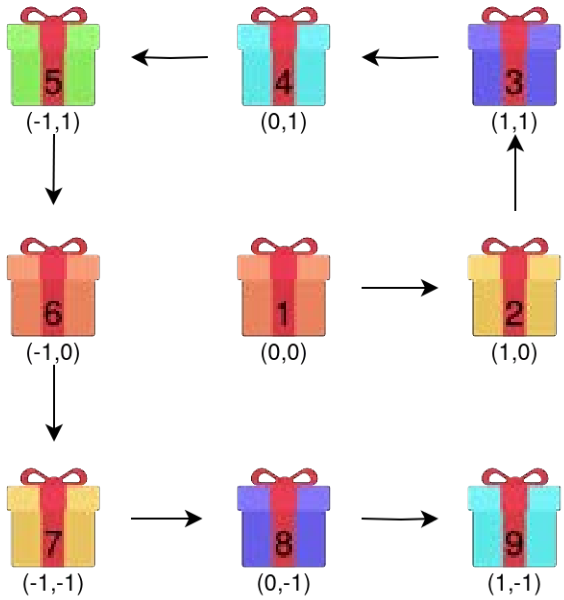


Figure 1: The organisation of gifts in the warehouse.

Moreover, there are 5 types of gifts: peach, yellow, purple, turquoise and green ones. The gifts need to be stored in this exact order. The first gift (in the center of the warehouse) is peach, the second gift is yellow, the third purple, et cetera. After the five colors, we start again with peach.

One day, when almost all gifts were already delivered and therefore not anymore in the warehouse, the cleaning elves had to clean the warehouse and did not know about this important ordering system of the other elves. They moved around all remaining gifts. Now, the gift elves are panicking since they do not know anymore which gifts need to be on location $(-49, 50)$ and $(-39, 49)$.

What is the color of the gifts at locations $(-49, 50)$ and $(-39, 49)$?

Possible Answers:

1. The gift at $(-49, 50)$ is peach and the gift at $(-39, 49)$ is turquoise.
2. The gift at $(-49, 50)$ is yellow and the gift at $(-39, 49)$ is turquoise.
3. The gift at $(-49, 50)$ is green and the gift at $(-39, 49)$ is green.
4. The gift at $(-49, 50)$ is green and the gift at $(-39, 49)$ is turquoise.
5. The gift at $(-49, 50)$ is purple and the gift at $(-39, 49)$ is peach.
6. The gift at $(-49, 50)$ is yellow and the gift at $(-39, 49)$ is purple.
7. The gift at $(-49, 50)$ is purple and the gift at $(-39, 49)$ is yellow.

8. The gift at $(-49, 50)$ is peach and the gift at $(-39, 49)$ is purple.
9. The gift at $(-49, 50)$ is turquoise and the gift at $(-39, 49)$ is green.
10. The gift at $(-49, 50)$ is purple and the gift at $(-39, 49)$ is turquoise.

Solution

The correct answer is: 3.

To solve this problem, we need to determine the number of steps required to reach $(-49, 50)$ from $(-39, 49)$.To do this, let's focus solely on the corners, specifically the locations where we change direction. In Figure 2, we illustrated the distances between the initial corners. For instance, going from corners $(0, 0)$ to $(1, 0)$ requires one step, and from corners $(1, 1)$ to $(-1, 1)$ takes two steps. Moving from $(-1, -1)$ to $(2, -1)$ entails three steps, as does the transition from $(2, -1)$ to $(2, 2)$ and the following two corners require four steps each. To

Figure 2: Spiral 'rolled out'.

travel from $(0, 0)$ to $(-1, 1)$ requires four steps. To travel from one upper/lower left/right corner to another upper/lower left/right corner, one must follow a squared path, akin to circumnavigating a 'city block'. Therefore, the number of steps required from $(0, 0)$ to one of the upper left corners is made of the distance from $(0, 0)$ to $(-1, 1)$ plus the number of steps to walk around the 'city block':

- From $(0, 0)$ to $(-2, 2) : 4 + 2 + 3 + 3 + 4 = 16$ steps.
- From $(0, 0)$ to $(-3, 3) : 4 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + 6 = 36$ steps.
- From $(0, 0)$ to $(-4, 4) : 36 + 6 + 7 + 7 + 8 = 64$ steps.

Notice that the number of steps required follows a pattern of squares: to reach $(-1, 1)$, it takes $2^2 = 4$ steps, while for $(-4, 4)$, it requires $8^2 = 64$ steps. This is unsurprising, as we're essentially calculating the areas of nested squares. Hence, for each corner $(-x, x)$ where $x = 1, 2, 3$, and so on, there are $(2x)^2$ steps needed to reach that corner.

To reach the location $(-49, 50)$, we first calculate the number of steps to the nearest corner, which is $(-50, 50)$. This will take us $(2 \cdot 50)^2 = 10{,}000$ steps. Since there are 5 different gift colors and 10,000 is divisible by 5, the gift at $(-50, 50)$ will be peach. Then, to reach $(-49, 50)$, we take one step back, which means the gift will be green, the last color in the sequence.

We can apply the same logic to the location $(-39, 49)$. To do so, we first move to the closest corner, in this case, $(-49, 49)$ which takes $(2 \cdot 49)^2 = 9{,}604$ steps. Therefore, we can expect a green gift at this location. Then, we move 10 steps back, indicating that the gift at location $(-39, 49)$ will also be green.

3 Revealing Sounds

Authors: Svenja M. Griesbach & Max Klimm
Project: Information design for Bayesian networks (MATH+ AA3-9)

Illustration: Julia Schönnagel

Challenge

For this riddle, you slip into the role of the Grinch, who constantly gets in the way of Santa Claus and his elves. To keep you from pranks on Christmas Eve this year, the elves have made you an offer: They have been baking cookies all day, and you are supposed to guess which type was baked today. If you guess the correct type, you will receive a huge jar of fresh cookies as a gift. However, if you guess wrong, you must, in turn, promise not to play pranks on Christmas Eve. Although you are often mischievous, when it comes to such offers, you can always rely on the honesty of the elves and will therefore keep the promise yourself.

As there are only three different types of cookies (Vanilla Crescents, Nut Triangles, and Chocolate Cookies) that the elves regularly and with equal probability bake, the elves are convinced that you will guess wrong in two out of three cases, giving them good chances for a relaxed Christmas. However, you have observed the behavior of the elves very well over the past few years and have collected some information about their behavior. For example, you noticed that when baking Vanilla Crescents, the elves always listen to *Driving Home For Christmas*. On the other hand, when baking Nut Triangles, they equally likely listen to either *All I Want For Christmas Is You* or *Last Christmas*. It's different when they bake Chocolate Cookies. Although they also only listen to either *All I Want For Christmas Is You* or *Last Christmas* on a loop, in two out of three cases, *All I Want For Christmas Is You* is playing.
Which statement about the three Christmas songs and cookie types is correct?

Possible Answers:

1. The song *Driving Home For Christmas* is played most frequently.
2. If you know which song was played today, you can increase your average probability of winning to more than 70%.
3. There is no song for which you can be entirely certain which cookies were baked today.
4. If *All I Want For Christmas Is You* was played, you have the highest chances of winning if you bet on Nut Triangles.
5. Nut Triangles are baked more frequently than Chocolate Cookies and Vanilla Crescents.
6. If *Last Christmas* was played, you cannot rule out any type of cookie with certainty.
7. The probability that Chocolate Cookies were baked and the elves heard *All I Want For Christmas Is You* is $\frac{2}{3}$.
8. If the song *Driving Home For Christmas* was played, each type of cookie is equally likely.
9. The probability that you heard *Last Christmas* today is 20%.
10. *All I Want For Christmas Is You* is played most frequently, and *Driving Home For Christmas* is played least frequently.

Project reference:

In the Math+ project AA3-9, the *Information design for Bayesian networks* is being investigated to determine to what extent traffic equilibria can be improved through the transmission of information. It is assumed that, based on the provided information, traffic participants can draw conclusions about the actual traffic. This occurs according to a similar principle to how the Grinch in this task draws conclusions about the baked cookie type by listening to the music.

Solution

The correct answer is: 2.

To solve the puzzle, we must carefully examine the information provided by each song. Let's abbreviate the songs as S_1, S_2 and S_3 for *Driving Home For Christmas*, *All I Want For Christmas*, and *Last Christmas* respectively, and the cookie types as C_1, C_2 and C_3 for Vanilla Crescents, Nut Triangles, and Chocolate Cookies. Since all cookies are baked with equal probability, the following holds:

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}.$$

Answer 5 is therefore incorrect. Additionally, we can deduce the following *conditional* probabilities[1] from the text:

$$\begin{aligned} &P(S_1|C_1) = 1 && P(S_2|C_1) = 0 && P(S_3|C_1) = 0 \\ &P(S_1|C_2) = 0 && P(S_2|C_2) = \frac{1}{2} && P(S_3|C_2) = \frac{1}{2} \\ &P(S_1|C_3) = 0 && P(S_2|C_3) = \frac{2}{3} && P(S_3|C_3) = \frac{1}{3}. \end{aligned}$$

From this, using the law of total probability, we can derive initially how likely it is to hear each individual song. This states that the probability of an event is equal to the sum of the probabilities of all paths leading to this event:

$$\begin{aligned} P(S_1) &= P(S_1|C_1) \cdot P(C_1) + P(S_1|C_2) \cdot P(C_2) + P(S_1|C_3) \cdot P(C_3) \\ &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3} \\ P(S_2) &= P(S_2|C_1) \cdot P(C_1) + P(S_2|C_2) \cdot P(C_2) + P(S_2|C_3) \cdot P(C_3) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{7}{18} \\ P(S_3) &= P(S_3|C_1) \cdot P(C_1) + P(S_3|C_2) \cdot P(C_2) + P(S_3|C_3) \cdot P(C_3) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{18}. \end{aligned}$$

Answers 1, 9, and 10 can thus be ruled out. Bayes' theorem for conditional probabilities

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)},$$

gives us the remaining conditional probabilities:

$$\begin{aligned} &P(C_1|S_1) = 1 && P(C_2|S_1) = 0 && P(C_3|S_1) = 0 \\ &P(C_1|S_2) = 0 && P(C_2|S_2) = \frac{3}{7} && P(C_3|S_2) = \frac{4}{7} \\ &P(C_1|S_3) = 0 && P(C_2|S_3) = \frac{3}{5} && P(C_3|S_3) = \frac{2}{5}. \end{aligned}$$

[1] Here by conditional probability $P(A|B)$ is meant the probability of an event A occurring, given that event B is already known to have occurred. It is given by the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

This allows us to identify answers 3, 4, 6, and 8 as incorrect. Depending on the song we heard today, we now know which cookie type is most likely. Thus, we maximize our chances of winning as follows:

- If we hear song S_1 (Driving Home For Christmas), we bet on cookie type C_1 (Vanilla Crescents).
- If we hear song S_2 (All I Want For Christmas), we bet on cookie type C_3 (Chocolate Cookies).
- If we hear song S_3 (Last Christmas), we bet on cookie type C_2 (Nut Triangles).

Now, to calculate our winning probability, we need to consider the probabilities of each individual song. Let R represent the event where we guess correctly. We calculate the winning probability with the law of total probability:

$$\begin{aligned} P(R) &= P(S_1) \cdot P(C_1|S_1) + P(S_2) \cdot P(C_3|S_2) + P(S_3) \cdot P(C_2|S_3) \\ &= \frac{1}{3} \cdot 1 + \frac{7}{18} \cdot \frac{4}{7} + \frac{5}{18} \cdot \frac{3}{5} \\ &= \frac{13}{18} \approx 72.2\%. \end{aligned}$$

The correct solution is, therefore, answer 2. To finally refute answer 7, we only need to calculate:

$$P(S_2 \cap C_3) = P(C_3) \cdot P(S_2|C_3) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

4 Christmas Chaos

Author: Marieke Heidema (Universität Groningen)

Illustration: Vira Raichenko

Challenge

Before Christmas, Santa's elves are always working hard on Christmas presents for children from all around the world: they read the wish-lists and letters, make presents, and wrap the gifts, before sorting them and putting them in Santa's sled.

In the days close to Christmas Eve, all the elves are working especially hard. This is also when there is a lot of chaos in Santa's workshop: you will find elves frantically running over all the bridges connecting the different parts of Santa's workshop, leaving behind a trail of ribbons and jumping over the rolls of wrapping paper that are covering the floor...

Last year, the elves made such a mess of the warehouse that Santa tripped and fell down, almost injuring himself, as he was inspecting the progress of the elves! This year, the elves want to make sure that Santa does not fall again. That is why the elves decided to clean up all the bridges connecting the 5 stations of the workshop:

(A) the letter reading room

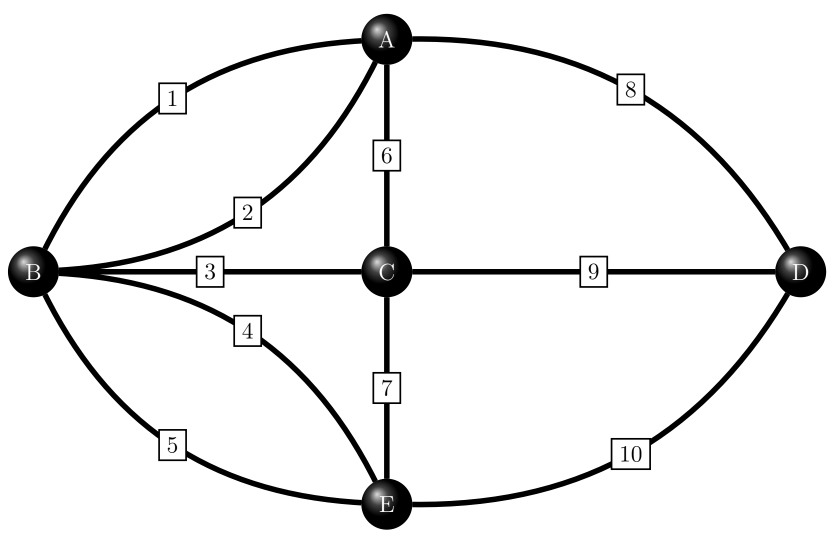
(B) the gift factory

(C) the gift-wrapping room

(D) the sorting room

(E) the warehouse with Santa's sled

Their goal is to ensure a smooth and safe passage for Santa as he oversees their festive preparations. The layout of Santa's workshop is as follows:



Here, the lines $1, 2, ..., 10$ each denote a bridge connecting the five different stations A,B,...,E of the workshop.

Santa will make his round through the entire workshop and the elves need to know what path they have to follow to clean all the bridges efficiently before Santa comes. The two important questions are:

- Is it feasible to tidy up the workshop while traversing each bridge only once?
- If this is possible, what are the start and end points within the workshop for cleaning process?

Possible Answers:

1. It is not possible to clean up all the bridges without crossing bridge number 7 more than once.
2. It is not possible to clean up all the bridges without crossing bridge number 8 more than once.
3. It is not possible to clean up all the bridges without crossing bridge number 9 more than once.
4. It is possible if the elves start at A and end at B.
5. It is possible if the elves start at A and end at C.
6. It is possible if the elves start at A and end at D.
7. It is possible if the elves start at A and end at E.
8. It is possible if the elves start at B and end at C.
9. It is possible if the elves start at B and end at D.
10. It is possible if the elves start at C and end at D.

Solution

The correct answer is: 9.
This challenge is an example of the "Seven Bridges of Königsberg" problem, a famous problem in mathematics. This problem is named after the city of Königsberg in Prussia, which had a river running through it, creating two large islands that were connected to each other, and to the two mainland, by seven bridges. The problem was to figure out if it is possible to walk through the mainland and the islands of the city, via the bridges, while crossing each of those bridges exactly once.

In the 18th century, Euler laid the foundations of graph theory by showing that this problem has no solution. Our variation on the problem, however, does have a solution! As Euler observed while tackling the "Seven Bridges of Königsberg" problem, it is only possible to walk over all the bridges in Königsberg once (and just once), if either

- The mathematical degree of each mainland/island, which is the number of bridges connected to it, is even for all mainlands/islands, or
- There are exactly two mainlands/islands with an odd degree. In particular, if this is the case, then the ones having an odd degree must be the start and end points of the route.

It is important to note that this is only true if it is possible to visit each station from every other station, or in mathematical terms, if we have a so-called *connected* graph. If we look at the map of Santa's workshop, we see the following:

(A) the letter reading room has 4 bridges connected to it,

(B) the gift factory has 5 bridges connected to it,

(C) the gift-wrapping room has 4 bridges connected to it,

(D) the sorting room has 3 bridges connected to it,

(E) the warehouse with Santa's sled has 4 bridges connected to it.

So, the letter reading room, the gift-wrapping room, and the warehouse with Santa's sled all have an even number of bridges connecting them to other stations, whereas the gift factory and the sorting room have an odd number of bridges connected to them. Hence, by Euler's observation, it is possible to walk over all the bridges exactly once and, in particular, the cleaning elves have to start at the gift factory and end at the sorting room (or vice versa). One possible route would be to cross the bridges in this order: 1-8-10-5-2-6-3-4-7-9.

5 Shipping Presents

Author: EWM-NL

Illustration: Ivana Martić

Challenge

Julia Robinson was one of the best female mathematicians of the 20th century, and is an old friend of Santa. She worked on the so-called Hilbert's 10th problem, but also worked on game theory. Now, Santa needs her help with a funny game that his elves have created to have some fun while working on an assembly line.
Santa is packaging and sending presents, with n assembly lines. The assembly process is an $n \times n$ square grid, and all n packages are on the diagonal of this square, see Figure 1. To be able to send them away, these packages need to be transported to the lowest row of the square. To do so, Santa's elves can move the presents downward. However, they can only do this by moving *two* packages at the same time downward by *one step*. Packages at

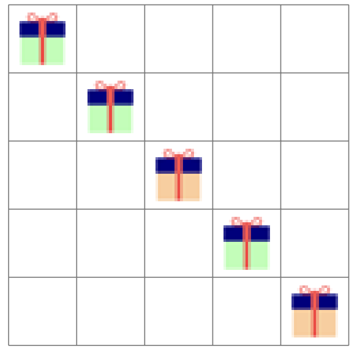


Figure 1: The assembly process for $n = 5$.

the bottom row cannot be moved any further, as otherwise they would fall off the assembly line.
Santa would like to end up with the lowest row full of all the n presents (so that all packages can be sent). However, sometimes his elves seem to get stuck and not be able to send all presents. When is it possible for Santa to get all his presents to the bottom of the assembly square with some algorithm at all and not be stuck?

Possible Answers:

1. For every possible value of n
2. For n even
3. For n odd
4. For all $n \geq 24$
5. For all $n \geq 12$
6. For all n where either $n - 1$ or n are a multiple of 4
7. For all n where either $n + 1$ or n are a multiple of 4
8. For all n where either $n - 1$ or n are a multiple of 3
9. For all n where either $n + 1$ or n are a multiple of 3
10. For no value of n

Project Reference:

EWM-NL is the Dutch association for women in mathematics.

Solution

The correct answer is: 6.

At the beginning, there are $(n-1)+(n-2)+\cdots+1 = n(n-1)/2$ empty spaces below the presents. With each move, this decreases by 2. Thus, $n(n-1)/2$ should be even. Therefore $n(n-1)/2 = 2k$ for some k, or equivalently $n(n-1) = 4k$. As either n or $n-1$ is odd, this means that the other one should be a multiple of 4.

This conditions is also sufficient. Indeed, the elves can get started on the leftmost two columns, and move the pair of presents down as much as possible. After this, the present in column 2 is in the most downward row, while the present in column 1 is still one place too high. The elves repeat this for all pairs of columns after the first two (columns 3 and 4, then 5 and 6, and so on). After this, there are two cases:

- n was divisible by 4: then half of the columns ($n/2$) still have a present that is one place too high. These presents can be moved down in pairs, as $n/2$ is even when n is divisible by 4.
- $n-1$ was divisible by 4: then half of the columns, rounded down ($(n-1)/2$) still have a present that is one place too high (as the last column that was not part of any pair had its present already at the lowest column). Now $n-1 = 4k$ for some k, so that $(n-1)/2 = 2k$, and an even number of presents with one empty space below it is left.

6 Conflict in the Elf Office

Authors: Olaf Parczyk, Silas Rathke (FU Berlin)
Project: EF 1-12

Illustration: Vira Raichenko

Challenge

Politics is not easy! Not even at the North Pole. There, ten elected bureaucracy elves decide on the laws in the eternal ice.
But now, a huge dispute has arisen about what should be spent on next year: Some want to use the money to make better gifts for the children. Others prefer to invest it in a lighter sleigh to relieve the reindeer, and still, others want to equip the elf village with more modern heaters and so on.
The dispute has become so bitter that a total of 20 hostilities have arisen. A hostility always consists of exactly two bureaucracy elves who have clashed so much that they no longer speak to each other (if elve A has a fight with elve B and elve B has a fight with

Elve A, then it counts as one hostility). Some elves might not speak to all other elves, and there might also be some elves who are at peace with all other elves.
To smooth things over, Chief Elf Olav invites the ten bureaucracy elves to a working meeting at the Elf Office. There, solutions should be worked on in different rooms. Unfortunately, a single room is not enough for this because feuding bureaucracy elves cannot work in the same room. Additionally, Olav has lost track of who is feuding with whom - he only knows that there are exactly 20 hostilities. Therefore, Olav wants to distribute the rooms in such a way that they would be sufficient for all possible types of hostilities. Even if that were to mean that one room had to be super crowded while in the other room there would be only one elf.
What is the smallest number k of rooms that Olav needs, such that he can divide the ten bureaucracy elves into k rooms in any case with 20 hostilities, without feuding elves ending up in the same room?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

Project Reference:

In the project "Learning Extremal Structures in Combinatorics", we employ approaches from the field of artificial intelligence and machine learning to discover new constructions for graphs with specific properties.
The task at hand can be effectively formulated as an extremal problem in graph theory, akin to the problems we investigate in this project

Solution

The correct answer is: 6.
To solve the task, we will show two things:

1. Five rooms are not always enough.
2. Seven rooms are one too many.

1\. First, we show that five rooms are not sufficient. To do this, it is enough to construct a scenario where this does not work. Let's assume we have six elves, each fighting with every other elf. In total, this results in $5+4+3+2+1 = \binom{6}{2} = 15$ hostilities. Since each elf is in conflict with every other, no two of the six elves can be placed in the same room, requiring six separate rooms, even if the other four elves are peaceful.
Therefore, Olav needs at least six rooms in this case.
2\. We now show that six rooms are sufficient. For this purpose, Olav first assigns each bureaucracy elf to his own room, so in total 10 rooms. We now demonstrate that this allocation is not optimal. In an optimal allocation, there must be enmity pairwise between all rooms. Otherwise, one could combine the two rooms and find a better allocation. Since $\binom{10}{2} > 20$, there are two rooms where not a single enmity exists between them. Thus, Olav can safely pack the elves in these two rooms together into one room. Now there are still nine rooms left. We show that we can further combine the rooms. Two rooms A and B cannot be merged if there is at least one elf from room A and one elf from room B who are enemies. In this case, we say that two rooms are at odds with each other. It is important to note that in an optimal case the number of rooms at odds is less than or equal to the number of hostilities of elves between these rooms, because for two rooms to be at odds it is enough to have one fighting pair of elves. In the next step, we check if we can combine another room. Now consider the number of possible rooms at odds, which is $\binom{9}{2} > 20$. This implies that there would be more than 20 hostilities among the elves. Therefore, nine cannot be the optimal number of rooms. Olav can continue this process until there are only six rooms left. Only then, due to $\binom{6}{2} < 20 < \binom{7}{2}$, it may happen that two rooms can't be merged.

7 The Triangle of Al-Bermuda

Author: Matthew Maat (Universiteit Twente)
Project: Combining algorithms for parity games and linear programming

Illustration: Vira Raichenko

Challenge

During their long journey through the desert, the wise men have fallen asleep while riding their camels. Upon waking up they are surprised, as they cannot see the star anymore. In fact, the whole sky is covered in a big cloud. This must be the infamous cloud of Al-Bermuda. A legend says that it is shaped like an isosceles right triangle, with its longest side being 100 miles long. It is not possible to orient as long as they are under the cloud. But once they manage to get out from under the cloud, they will notice immediately because they will see the star again. Unfortunately, they have hardly any water left, so the sooner they can escape the cloud the better. There's only enough for 100 miles. Caspar bets

Melchior that he can guarantee they will escape the cloud with less than 100 miles of walking if they follow his lead, so that hopefully they won't become thirsty.
Caspar starts to think about his strategy. If he just walks in a straight line for 100 miles, it is possible that they are now in the lower left corner of the triangle, and that they happen to walk exactly to the right (see line segment AB in Figure 1). Then they don't escape within 100 miles. Instead, if he walks in a circle with a circumference of 100 miles, it is possible that they started in point C (see Figure 1) and don't leave the triangle during the 100 miles of walking. After some more thinking, he comes up with 4 more strategies (see also Figure 1):

(a) Walk straight for 50 miles, then take a turn of size α, and walk straight for 50 more miles.

(b) Walk straight for x miles, take a $90°$ turn to the right, then walk straight for $100-2x$ miles, then another right, and then another x miles straight.

(c) The same as (b), but now at the second turn we turn left instead of right.

(d) Walk straight for z miles, and then turn right and walk straight for another $100-z$ miles.

Assuming Caspar picks the right values for parameters α, x, y and z if they exist, which of these strategies can Caspar use to guarantee himself to win the bet?

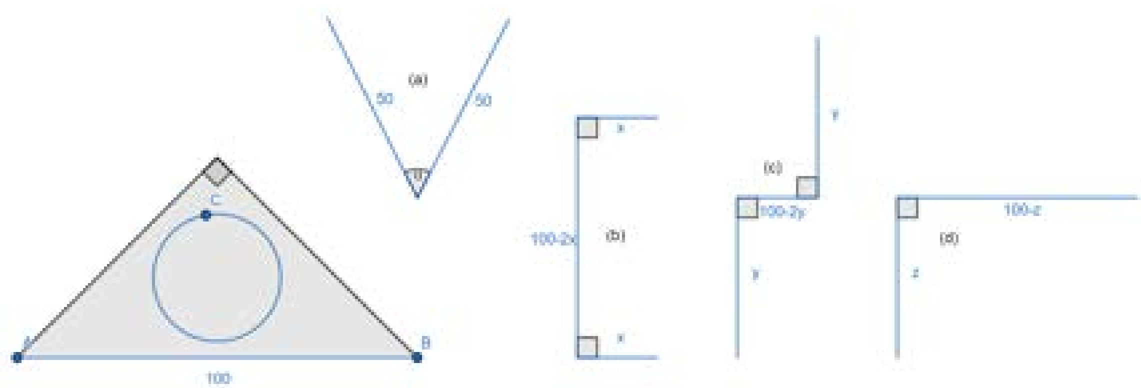


Figure 1: Left: the triangle of Al-Bermuda. Right: Caspar's ideas for the routes to take.

Possible Answers:

1. Caspar cannot guarantee the win
2. Only one of the four strategies
3. Only with (a) and (b)
4. Only with (a) and (c)
5. Only with (a) and (d)
6. Only with (b) and (c)
7. Only with (b) and (d)
8. Only with (c) and (d)
9. Three of the four strategies
10. Caspar can guarantee to win with all four strategies

Solution

The correct answer is: 2.
For convenience, we use a coordinate system, (see e.g. Figure 2) where the three corners of the triangle are $A(0,0)$, $B(100,0)$ and $C(50,50)$. The boundaries of the triangle are given by $y=0$, $y=x$ and $y=100-x$. Now we look at the strategies one by one.

(a) The route can be placed in the triangle such that the start and end are at the bottom and the x-coordinate of the turn is 50. Then it is clear that the route always stays within the triangle.

(b) The route can be in the triangle such that the start and end are at the bottom, and the turning points are exactly at the left and right edges at coordinates (x,x) and $(100-x,x)$ (where x is now the parameter for the route).

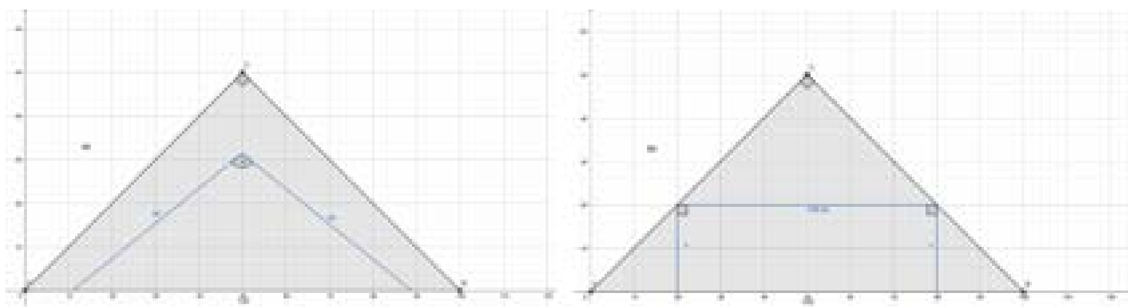


Figure 2: Type (a) and (b) routes inside the triangle.

(c) If we pick our parameter y equal to for example 10, then the route does not fit inside the triangle. The worst case would be if the trip is such that you start at the bottom of the triangle, make the first turn exactly at the left side, and the second turn at the bottom. To see why this is the worst case, consider the following observations:

- The route contains a straight section of 80 miles, while the short sides of the triangle, using Pythagoras, are only $\frac{100}{\sqrt{2}} \approx 71$ miles long. Thus, the only way the route would fit in the triangle would be if the 80 mile section is close to the bottom of the triangle and almost horizontal.
- If we would try to rotate the claimed worst case route clockwise a bit, then the left side of the route moves up, so we would have to shift the route to the right if we want to keep the start inside the triangle, resulting in leaving the triangle even earlier (provided we show, that the claimed worst case leaves the triangle).
- If we would try to rotate the claimed worst case route anticlockwise, the endpoint of the route would move almost straight up, while the left stays about the same, so we would also leave the triangle earlier.

Now we can focus on the worst case. Let h be the vertical segment from the bottom of the triangle to the first turning point, let P be the end point of the route and let d be the segment between the first turning point and P (compare figure 3). The length of d is obtained using Pythagoras, it is $\frac{6500}{\sqrt{6500}}$. Using the similarity of the triangle involving h, the first segment of the route and a section of the bottom of the Al-Bermuda triangle and the triangle involving d, the 80 mile and the last segment of the route, one can calculate h via

$$\frac{h}{10} = \frac{80}{d},$$

i.e. $h = \frac{800}{\sqrt{6500}}$. Hence, the coordinates of the first turning point are (h, h), and because d is horizontal, the coordinates of the end point P are $(\frac{7300}{\sqrt{6500}}, \frac{800}{\sqrt{6500}})$. This is about 0.47 miles above the line $y = 100 - x$, so they would escape the triangle in time.

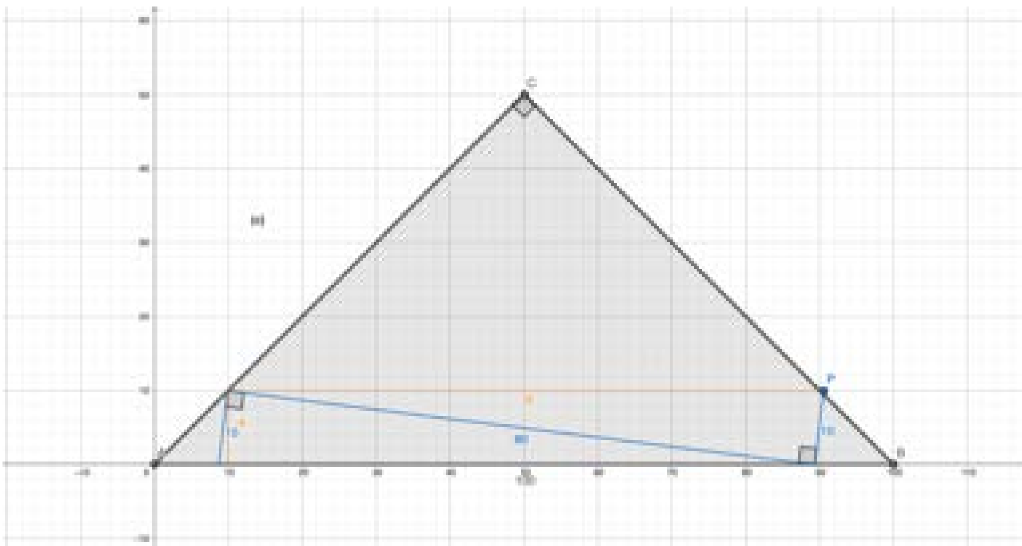


Figure 3: Worst case route for strategy (c) still leaving the triangle.

(d) Here we can distinguish two cases: $z < 50$ and $z \geq 50$. In the first case, we can choose the endpoint of the route to be A and the second line segment to lie on the bottom side of the triangle. The resulting route then stays inside the triangle. In the second case, we choose B to be the start of the route and the first line segment to lie on the bottom of the triangle. The resulting route then stays inside the triangle.

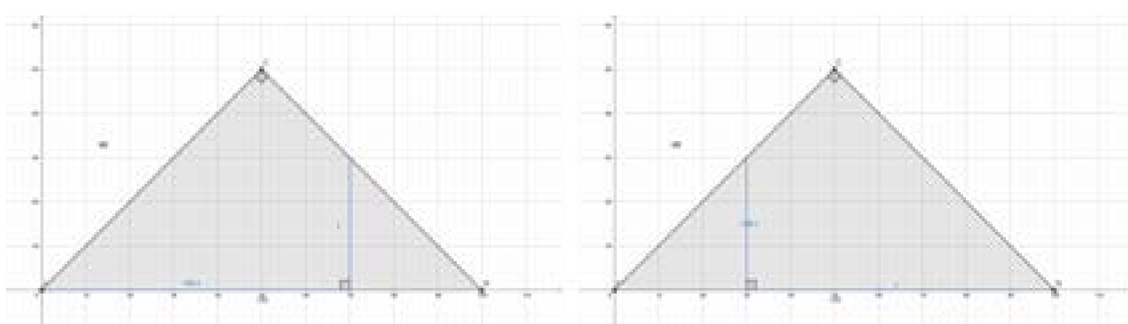


Figure 4: left: $z < 50$, right: $z \geq 50$. In both cases the route stays inside the triangle.

In conclusion, Caspar can only guarantee to win with strategy (c).

8 Cookies, Cards and Christmas Magic

Author: Margarita Kostre
Project: MATH+

Illustration: Friederike Hoffmann

Challenge

The elves Gwendelyn and Fredi work for Santa Claus. During their breaks, they like to play cards for cookies once a day. Since they both like to bake, every day they bring different cookies to work. Depending on which cookies they bring on the day, they either are more willing to give away their own cookies if they lose, or eager to get other cookies if they win. So if Gwendelyn wins on the following days, she receives the following number of cookies:

- On days 1 through 6: 20 cookies,
- On days 7 through 12: 40 cookies,

- On days 13 through 18: 30 cookies,
- An days 19 through 24: 50 cookies.

And if Gwendelyn loses on the following days, she must give away the following number of cookies:

- On days 1 through 6: 20 cookies,
- On days 7 through 12: 10 cookies,
- On days 13 through 18: 5 cookies,
- On days 19 through 24: 25 cookies.

Gwendelyn also has different working hours which affect her performance in the card game. Gwendelyn wins with a probability of:

- 50 percent on odd-numbered days that are divisible by three,
- 40 percent on even days,
- 20 percent on odd-numbered days that are not divisible by three.

On which day does Gwendelyn expect to win the most cookies?

Fredi gets so many cookies from Santa Claus as a reward after the long Advent season and he had so much fun with Gwendelyn that he has the idea to continue playing after Christmas, but this time with a coin. The rule is relatively simple: a fair coin is tossed by Gwendelyn until "heads" falls for the first time. This ends the game and Gwendelyn goes home with the cookies she has won. The number of cookies Gwendelyn wins depends on the total number of coin tosses. If she wins on the first toss, Gwendelyn receives a cookie. With two tosses (i.e. one "tails", one "heads") she receives two cookies, with three tosses 4 cookies, with four tosses 8 cookies and with each additional toss the amount of cookies doubles. Fredi doesn't win anything from the game, but as he doesn't want to eat too many cookies, he agrees. Gwendelyn is thrilled and wonders how many cookies she expects to win this time.

A) On which days during Advent does Gwendelyn expect to win the most cookies?
B) How many additional cookies does she expect to receive from Fredi after Advent?

Hint: In this task, the expected value is sought in the mathematical sense. The expected value describes the weighted average, whereby the weighting is based on the probabilities of the various possible values. The expected value is a measure of the "mean" or "average" value that would be expected in a large number of experiments or random events. The expected value does not have to be a value actually observed in the experiment.

Possible Answers:

1. 22 and 0
2. 9 and 152
3. 9 and 2048
4. 9 and infinity
5. 15 and 152
6. 15 and 2048
7. 15 and infinity
8. 21 and 152
9. 21 and 2048
10. 22 and infinity

The correct answer is: 4.

Solution:

The expected profit $E(t)$ of cookies on day t is calculated as the number of cookies to be won $G(t)$ multiplied by the probability of winning $p(t)$ minus the number of cookies to be lost $V(t)$ multiplied by the probability of losing:

$$E(t) = G(t) \cdot p(t) - V(t) \cdot (1 - p(t))$$

For example, the number of cookies expected on the seventh day is

$$E(7) = 40 \cdot 0.2 - 10 \cdot 0.8 = 0.$$

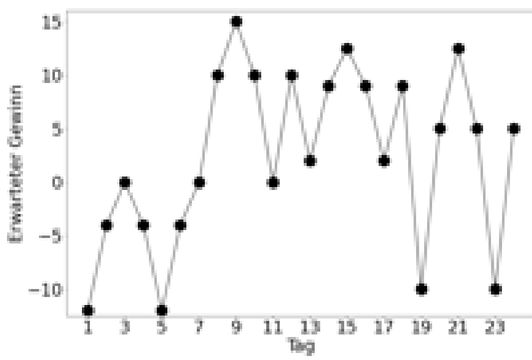


Figure 1: The expected profit on days 1 to 24 in cookies.

We calculate the expected profit in cookies for each day, see Figure ??. We observe that Gwendelyn expects the highest profit on the ninth day, namely 15 cookies.

For the second part, we first see that the probability of throwing "heads" on the k-th toss is $\left(\frac{1}{2}\right)^k$. We also observe that the profit with exactly k tosses is 2^{k-1} cookies. If we were to stop the game after n tosses at the latest, where $n \in \mathbb{N}$, the expected number of cookies that Gwendelyn receives after Advent would be

$$\left(\frac{1}{2}\right)^1 \cdot 2^{1-1} + \left(\frac{1}{2}\right)^2 \cdot 2^{2-1} + \left(\frac{1}{2}\right)^3 \cdot 2^{3-1} + \ldots + \left(\frac{1}{2}\right)^n \cdot 2^{n-1} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = \frac{n}{2}.$$

However, since we do not stop the game after a fixed number n, the expected number of cookies must be infinite.
This is called the *Saint-Petersburg-Paradox*. It states that, on average, an infinitely high win is expected in this game, although winning a large sum is very unlikely.

9 Secret Presents

Author: Felix und Nathanael Höfling
Project: EF4-10

Illustration: Christoph Graczyk

Challenge

Santa Claus, like every year, is very grateful for the great work of his elves and would like to give each of them a gift. However, he cannot write the names on the gifts, as the surprise

would be spoiled.
One day, Dwarf Alwin comes to visit, and Santa Claus explains the problem to him. "Dear Alwin, what should I do?" The clever Alwin immediately has an idea: You could write a secret code on each gift for every elf. It is important that no code is repeated. Also, similar names should get completely different codes so that they cannot guess the names. I will create a list of codes for you."
For each name, Dwarf Alwin calculates a four-digit secret code in the following way: First, he translates the letters of the name according to their position in the alphabet into numbers. For example, an E becomes 5, an M becomes 13, and Z gets 26, and so on. Then, he fills in a table with two rows, as shown in the table below. In the first column, he writes a 1 at the top and a 0 at the bottom. Then, in the first row, he adds the value of the next letter to the last written number. Before adding the result to the row, he makes sure that the number is less than 47. If the result is greater than or equal to 47, he subtracts 47 from the result. If the result is less than 10, a leading zero is added, for example, 7 becomes 07. In the second row, he follows the same process, except that he always adds the corresponding (above) entry from the first row to the last written number. Finally, he takes the last number from both rows and writes them one after the other, with the number from the second row first—this is the secret code of the name. Using this method, for example, Luise gets the code 3120, as shown in the picture.

LUISE $\longrightarrow$ 12 - 21 - 9 - 19 - 5					
1	13	34	43	15	**20**
0	13	00	43	11	**31**

A while later, Santa Claus happily holds the list of secret codes in his hand and gets to work. In the elf workshop, he orders a beautiful gift for each of his elves, but not under their names; instead, he uses the secret code. On the day before Christmas Eve, he prepares his sleigh.
At the North Pole, it is cold, and a terrible snowstorm is raging. As Santa turns to his reindeer, his coat opens slightly, the wind catches the list of secret codes, and blows it away. Santa is shocked.
Back in the elf workshop, he picks up the first gift. It has the inscription "2122". But which elf should he give it to? Can you help him? For whom is the gift?

Possible Answers:

1. Lotte
2. Flavia
3. Luiz
4. Elisabeth
5. Nathanael
6. Matteo

7. Evelina
8. Mathilda
9. Lutz
10. Fridolin

Project Reference:

Agent-based models are frequently employed to model cooperative behavior. These models depict tens of thousands of independently acting entities (such as people, animals, cars, mobile phones, etc.) that interact with each other. This interaction gives rise to large-scale structures describable with only a few parameters, such as the formation of groups or clusters. The extensive information in the original model can thus be condensed into a few key metrics. Conversely, it is challenging to deduce the positions of individual agents from these metrics. The mathematical formulation and computation of such relationships are the subject of current research.
Similarly, in the task, we consistently reduce long and short names to the same length. The inverse mapping, i.e., calculating the name based on the code, is nearly impossible. Unlike in agent systems, the codes have the additional property that similar names are assigned distinctly different codes. Thus, the code also serves as a checksum, enabling the detection of letter transpositions. There are many other applications for such codes, including secure password verification on a computer without the need to store the actual password.
Regarding the mathematical construction of the codes: the calculation is done based on the Adler32 hash function.

Solution

The right answer is: 7. Evelina

The secret codes are constructed in such a way that it is practically impossible to directly calculate the encrypted name from a code. However, since there are only ten possible answers, we can calculate the code for each possible answer from the name. For the correct answer, we will find the code "2122".
Evelina has the letter values 5, 22, 5, 12, 9, 14, 1. We enter the following numbers into the small table for code calculation:

1	06	28	33	45	07	21	**22**
0	06	34	20	18	25	46	**21**

The code “21|22" is obtained from the last column.

The secret codes for the other names are as follows: Lotte: 4226, Flavia: 0305, Luiz: 1822, Elisabeth: 2735, Nathanael: 4130, Matteo: 1928, Mathilda: 3122, Lutz: 4033 and Fridolin: 0841. By the way, Alwin has the code 1813.
To streamline the calculation, one can also write a small computer program, for example, in Python:

```python
def get_code(name):
    values = [ord(x) - ord('A') + 1 for x in name.upper()]

    a = [1,]
    b = [0,]
    for x in values:
        a.append((a[-1] + x) % 47)
        b.append((b[-1] + a[-1]) % 47)
    code = a[-1] + 100 * b[-1]

    print("Name: {0:s} -- Code: {1:04d}".format(name, code))
    print("Buchstabenwerte: ", *values)
    print("Zeile 1:", format_row(a))
    print("Zeile 2:", format_row(b))

    return code

def format_row(a):
    return " | ".join(["{0:02d}".format(x) for x in a])

get_code('Evelina')
```

10 Space Bending Warehouse Traversal

Authors: Ravi Snellenberg, Martin Skrodzki (TU Delft)
Project: *Holonomy*, Computer Graphics and Visualization

Ilustration: Ivana Martić

Challenge

The elves of the esoteric research department have found a way to increase the amount of storage in the current storage warehouse. This was badly needed due to the ever-increasing amount of presents that need to be stored each year. The way the elves accomplished this is by connecting the rooms in the current warehouse in a clever way using portals. Usually, one could place 4 square rooms around a corner, but with the way rooms are connected now, an elf would need to walk through 5 square rooms in order to walk around the corner point where the rooms meet instead of the usual 4, see Figure 1. However, due to the difficulty of creating these space-warping portals, the elves are only able to create them in a limited, separate area that is surrounded by walls. This area is called the control room. So to traverse the warehouse, the elves enter the control room instead of the physical warehouse. And when an elf crosses into another square in the control room they are walking into a new warehouse room. Which room is decided by the new way of connecting the rooms. This control room is given as a grid of 3×3 rooms with portals on every wall that seperates two square rooms within the control room, see Figure 1. Corners do not have portals on them, so that it is not possible to walk diagonally inside the control room but only orthogonally. Even with this limitation, it is still possible to reach any of the possibly infinite rooms in the warehouse by using the layout of the rooms in a clever way.

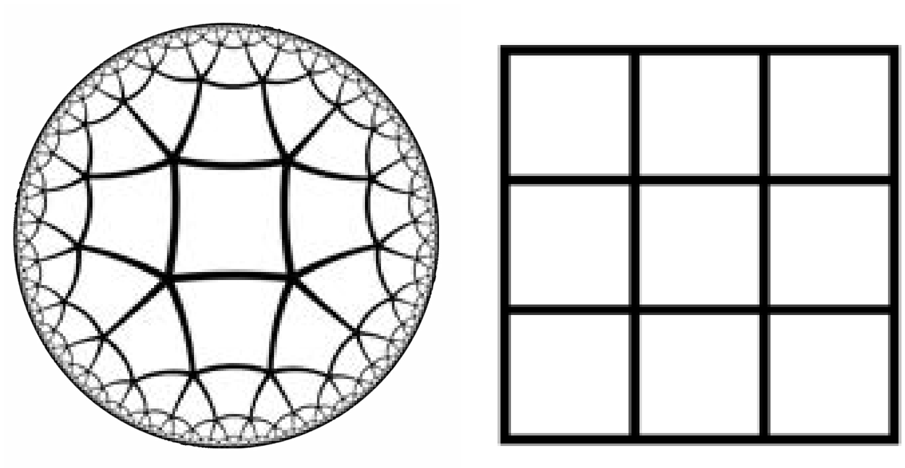


Figure 1: The warehouse room layout: The left figure is a map of the new warehouse layout. Each quadrilateral is actually a square room but they are distorted on the map due to the way we need to represent the extra fifth room at each corner. It uses the Poincaré disk model to display this special layout. The right figure is the control room.

Since the elves at the esoteric research department realised that this entire situation is really confusing, they created an infographic to hang on the front door of the control room, see figure 2.

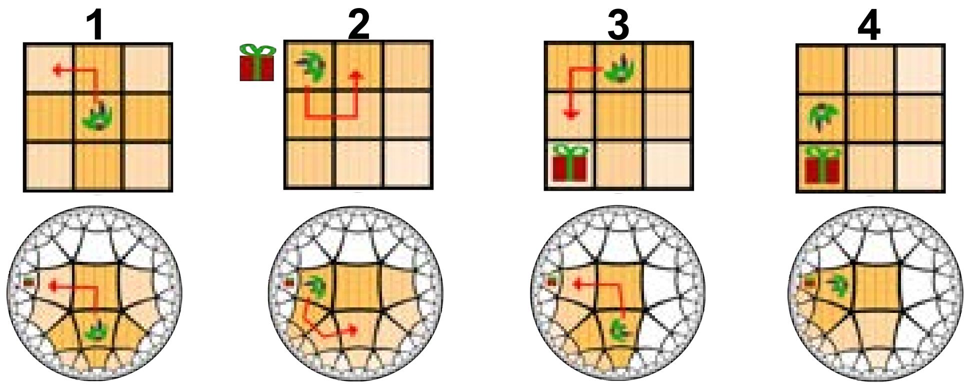


Figure 2: The infograpic. The lighter the floor gets, the further away is a room.

The infographic displays a hypothetical situation where an elf needs to reach a present that is not directly reachable within the starting control room layout. The top part of the infographic shows the control room the elf is confined to as they walk through the warehouse. The bottom part of the infographic shows a top view of the warehouse layout, which is linked to the control room view that is above it. The rooms that are colored in the infographic are the rooms that are reachable by walking a direct path (a shortest path) to a tile in the control room from the current position of the elf. Walking different shortest paths to the same tile in the control room does not always lead to the same room in the warehouse. For example, in the control room that is displayed in the first step of the infographic, the elf can reach each corner tile in two different ways. In the warehouse layout it can, for example, be seen that both ways of reaching the upper right corner tile in the control room (first up then right or first right then up) reach two different rooms in the warehouse, see figure 3.

The infographic demonstrates that by walking in the control room in a certain way, it is possible to reach the present, although it was not directly reachable from the starting position. It also demonstrates how the directly accessible area shifts as the elf walks through the warehouse.

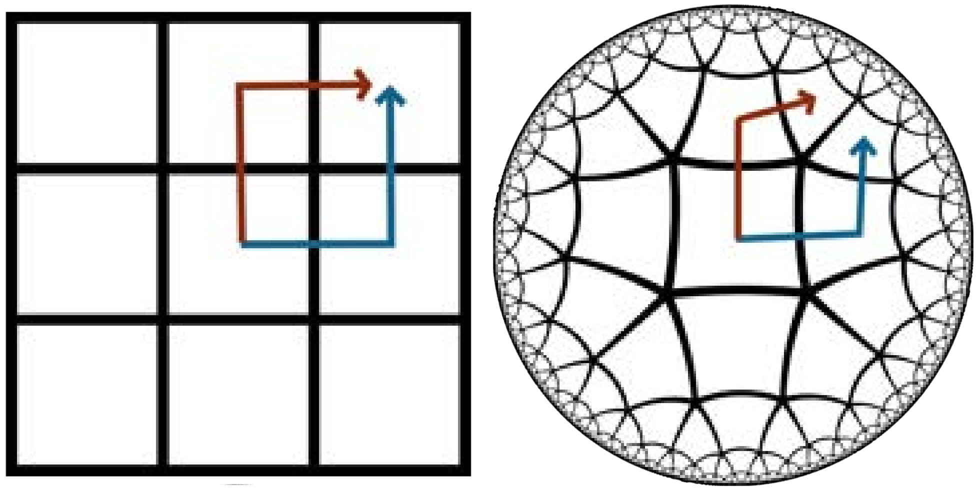


Figure 3: left: two paths to the upper right corner in the control room, right: the corresponding paths in the actual warehouse.

Now, the developmental elves of the esoteric research department wonder: What is the minimum amount of portals that need to be crossed to reach the present in the situation of figure 4 that could, without the control room size restriction, be reached by walking 3 steps forward? Assume that the elf starts in the center tile of the control room in this scenario.

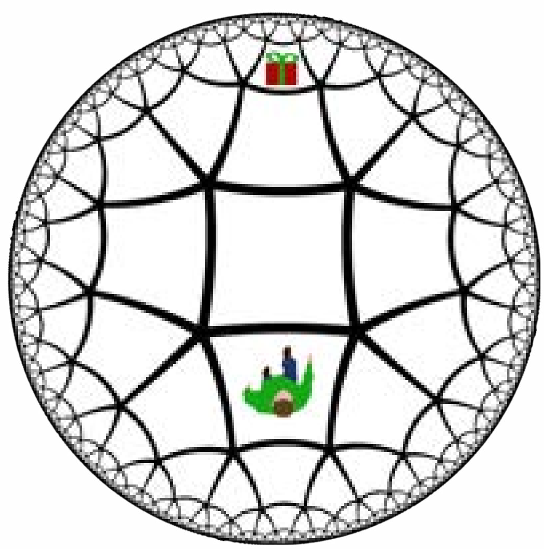


Figure 4: The question scenario.

Possible Answers:

1. 3
2. 5
3. 6
4. 7
5. 8
6. 9
7. 10
8. 12
9. 14
10. 42

Project Reference:

The scenario this problem describes is the environment of a Virtual Reality application called *Holonomy* (named after the mathematical property that makes this traversal possible). *Holonomy* is being developed to answer several different questions. For example, questions such as "Could such a demonstration help people understand the underlying mathematical properties more intuitively?", "How quickly do people adapt to navigation in a hyperbolic world?", "How could we help the player navigate such an environment?" And if we can figure out how to make navigation of these hyperbolic spaces easy, it could be used to create consistent worlds of any size that can be traversed while confined to a limited space. And calculating the shortest paths between two locations is an example of one of the challenges that we stumbled upon while creating *Holonomy*.

Solution

The correct answer is: 4.

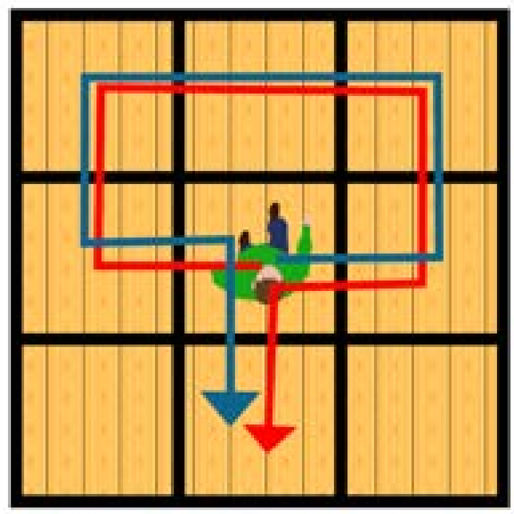


Figure 5: The optimal paths: both, the blue one and the red one, are valid.

We can see in Figure 5 that there is a path that only uses 7 portals. Indeed, one can easily check, that the corresponding path in Figure 4 leads to the present. Thus, the only thing left to show is to exclude the possibility of a shorter path.
If there were no limitations, then one could reach the present using 3 portals. This, however, is not possible with the limitations of the control room. To reach the present, the elf has to make use of a corner in some way. Going around a corner and then to the present, the elf has to traverse at least 6 portals. All paths using exactly 6 portals require the "big" room in the center of Figure 4 to be on that path. Thus, the elf has to walk around one of the corners of the big room. However, all these paths cannot be done inside the control room, excluding 6 traversals as an option. This means, that there cannot be a path that uses less than 7 portals.

11 Traveling Santa Problem

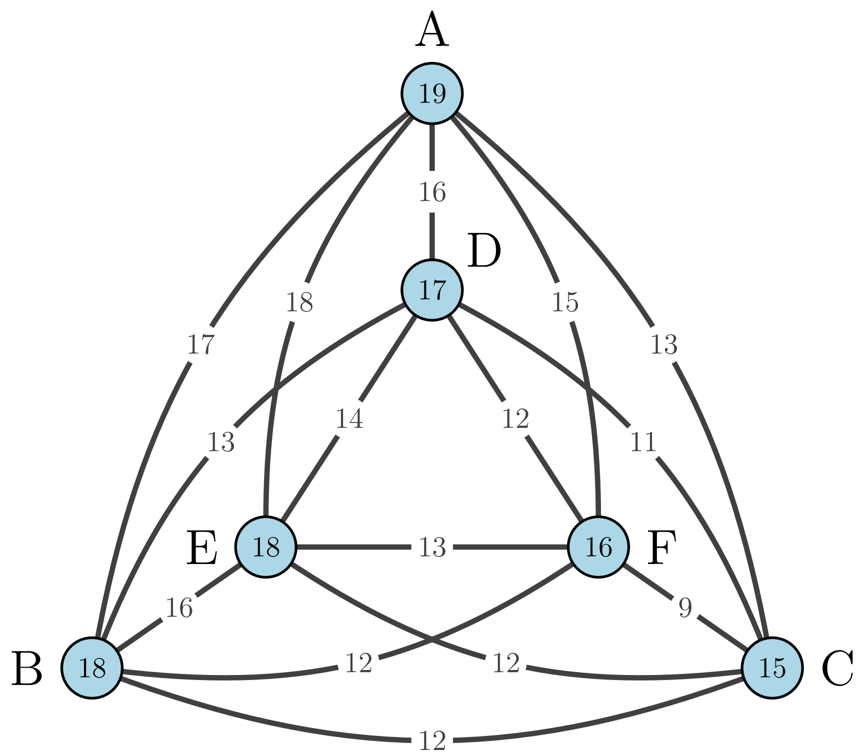
Author: Dion Gijswijt (TU Delft)
Project: 4TU.AMI

Illustration: Zyanya Santaurio

Challenge

Presently, Santa is on a mission to deliver gifts to children. His journey spans six cities, denoted as A to F. Starting from Santa's Workshop (SW), he must select one among A, B, C, D, E, or F to begin his visit.

The reindeers, tired from playing, can't travel long distances anymore. This necessitates Santa to chart the shortest route possible. He has a map, which indicates flying distances between cities along connecting lines, and the circles display the distances from SW to each city.



Santa has to choose now the shortest sequence for visiting the cities before returning to Santa's Workshop. For instance, if he takes the route SW-F-A-E-B-C-D-SW, the total journey length adds up to $16 + 15 + 18 + 16 + 12 + 11 + 17 = 105$. What is the length of the shortest tour that Santa can take?

Possible Answers:

1. 104
2. 103
3. 102
4. 101
5. 100
6. 99
7. 98
8. 97
9. 96
10. 95

Solution

The correct answer is: 7: 98

First, Let us notice that visiting any of the cities A, B, C, D, E, F twice, or the Santa's Workshop thrice, is always non-optimal. Let's assume we have a detour from city_1 to city_2 by walking a second time over city_3:

$$\ldots - \text{city}_1 - \text{city}_3 - \text{city}_2 - \ldots - \text{SW}$$

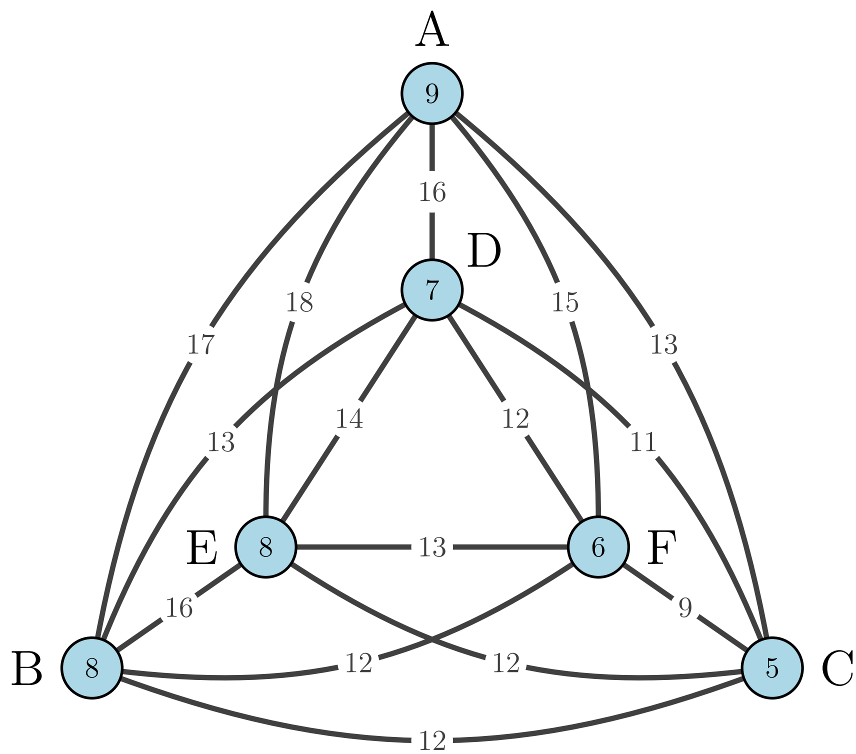
The direct tour from city_1 to city_2

$$\ldots - \text{city}_1 - \text{city}_2 - \ldots - \text{SW}$$

costs maximal 18, as this is the length of the longest distance. The detour costs at least 9+11=20, which is the sum of the two shortest distances. Thus, it is not worth to visit a city twice. To visit Santa's workshop thrice is also not optimal, as it never shortens the distance to any city.
Therefore, from now on, we will assume that cities A, B, C, D, E, F are visited exactly once and Santa's Workshop is visited only at the beginning and the end.

To simplify the problem, we successively change all distances from one city to the others (and Santa's Workshop) by the same amount. Changing the lengths in this way does not change which tours are optimal. This is because for a possibly optimal tour each city has to be visited exactly once (as explained above). Therefore one has one edge for visiting each city and one for leaving it (This is also true for Santa's Workshop). Changing the length by some amount d then reduces the cost of each possibly optimal tour by exactly $2d$. But this means that an optimal tour remains optimal, as no tour can become shorter than any other tour with this change.

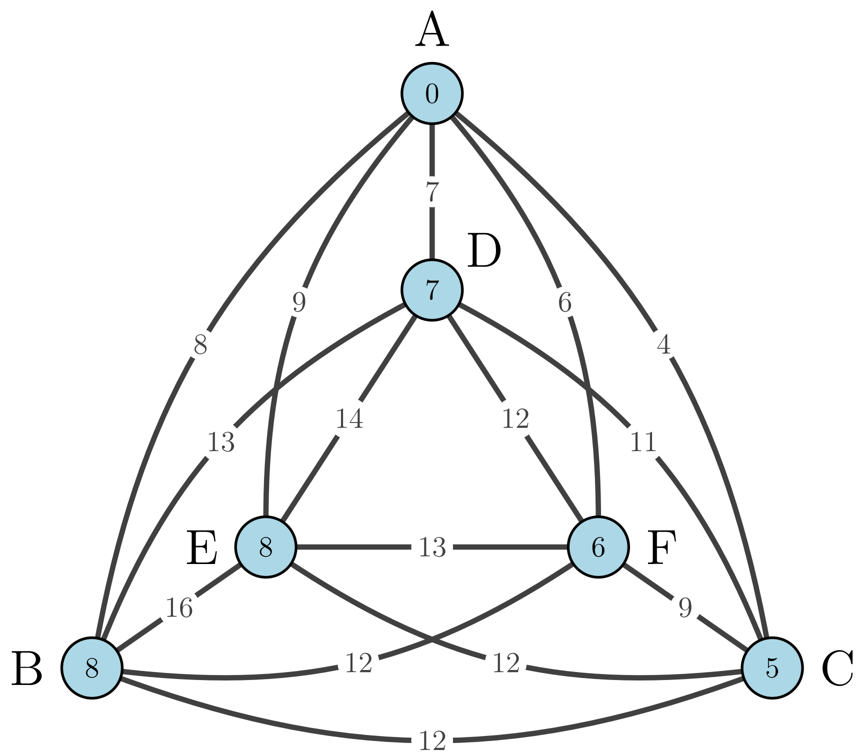
First, we decrease the distance from Santa's Workshop to every city by 10. This decreases the length of every tour by $2 \cdot 10 = 20$ since every tour goes from Santa's Workshop to some city and back from another city to Santa's Workshop. This gives us the following distances:



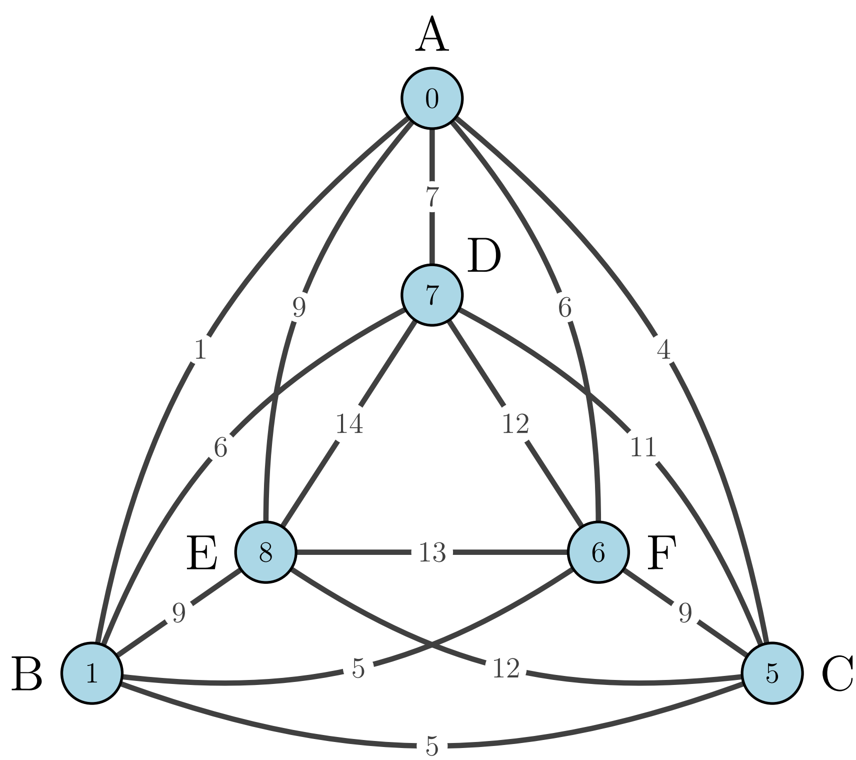
In the next step we decrease distances in the following way:

- the distances from city A we decrease by 9,
- the distances from city B we decrease by 7,
- the distances from city C we decrease by 4,
- the distances from city D we decrease by 6,
- the distances from city E we decrease by 8,
- the distances from city F we decrease by 5.

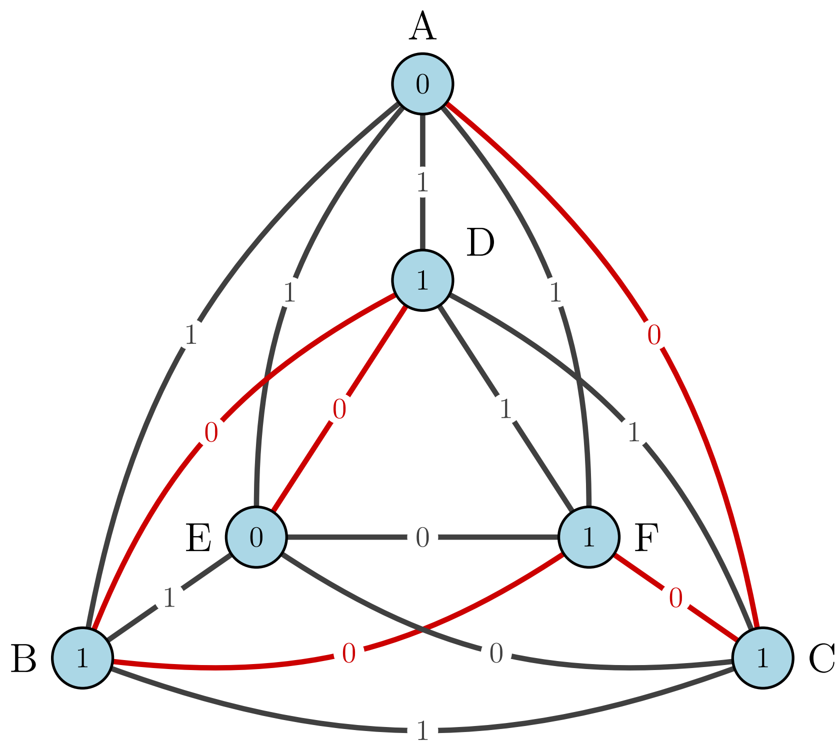
So for instance, decreasing the distances from A gives us the following result:



Then we decrease the distances from B by 7 and receive following graph:



We do the same procedures for the remaining nodes and receive the following graph:



We see that all distances are now 0 or 1 and the tour SW-A-C-F-B-D-E-SW has length 0 (in fact this tour and the reverse tour are the only tours of length 0), so this is an optimal tour. Because of the change in distances, every tour is $2(10+9+7+4+6+8+5) = 98$ longer in reality. So the shortest tour has length 98.

12 The Road to Bethlehem

Author: Matthew Maat (Universiteit Twente)
Project: Combining algorithms for parity games and linear programming

Illustration: Zyanya Santaurio

Challenge

After consulting with king Herod, the wise men are getting ready to travel to Bethlehem from Jerusalem. There is only one problem that they would like to solve before leaving: taxes. Because of the large amount of gold and spices they carry, they will have to pay taxes in every town that they enter. To make things worse, those smart tax collectors have pointed all road signs towards the places where you need to pay the most taxes. On the map (see Figure 1) you see the roads and towns between Jerusalem and Bethlehem represented by arrows and circles respectively, and how much tax the men need to pay in every town. In red, you see the directions of the signs in the towns.
The men decide to send Balthasar forward with the following assignment:

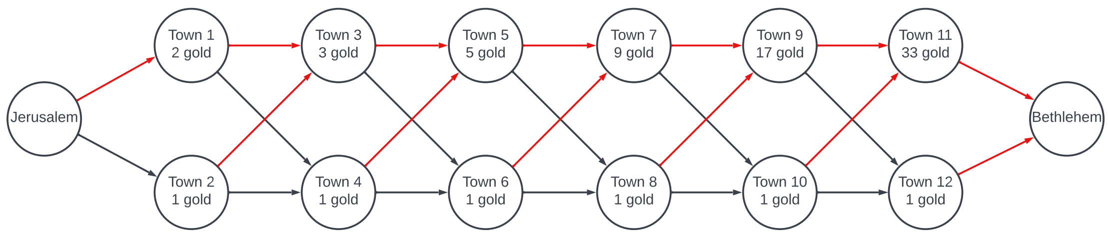


Figure 1: The roads (arrows) from Jerusalem to Bethlehem, with the amount of tax written in the circles. The caravan can only travel in the direction of the arrows.

- Starting from Jerusalem, walk to Town 1, then Town 2, 3, etc. Since Balthasar is on foot, he may ignore all the roads that exist.
- If Balthasar is in a town where he can change the sign such that the trip from that town to Betlehem becomes cheaper, then he changes the sign, runs back to Jerusalem, and starts the whole process over.
- If Balthasar changes the sign, he may only point the signs following the direction of the arrows of the roads. For example, in Town 7, the sign can only point towards Town 9 or Town 10.
- If Balthasar reaches Bethlehem, he will send a carrier pigeon to let the others know that they can come to Bethlehem.

On his first trip, Balthasar changes the sign in Jerusalem towards Town 2, since then the total tax on the trip from Jerusalem becomes $1 + 3 + 5 + 9 + 17 + 33$ instead of $2 + 3 + 5 + 9 + 17 + 33$ gold pieces. Likewise, on his second trip, Balthasar will switch the sign in Town 1 towards Town 4. On his third trip, he changes the sign in Jerusalem back to Town 1. This is, because the trip from there to Bethlehem becomes cheaper. Indeed, the new trip costs $2 + 1 + 5 + 9 + 17 + 33$ which is cheaper than the cost $1 + 3 + 5 + 9 + 17 + 33$ of the old trip.
How many times will Balthasar change a sign in total? (if the sign is changed in a town multiple times, we count it multiple times)

Possible Answers:

1. 55
2. 57
3. 63
4. 83
5. 114
6. 120

7. 126
8. 177
9. 240
10. 367

Project Reference:

The algorithm that Balthasar uses is called the *strategy improvement* algorithm. The algorithm uses a so-called *improvement rule* that decides in which order we make improvements (or in which order Balthasar visits the towns). In this case we use what is called Bland's rule. Although this is a one-player game (we only minimize the price), we can also use the algorithm when there are towns controlled by a maximizer or a random player. It is unknown whether there is an improvement rule that guarantees a small number of iterations. As you can see, even with a small number of towns, Bland's rule can take a large number of iterations.

Solution

The correct answer is: 8.
Note that Town $2n$ is always 2^{n-1} cheaper than Town $2n-1$. Instead of tracing every trip of Balthasar, we can prove three things that remain invariant for all trips of Balthasar:

1. Every time the sign is switched in Town $2n$ or $2n-1$, the cost of the trip from that town decreases by 2^n (counting Jerusalem as Town 0)
2. The cost of the trip from the towns where no switch is made are always unaffected.
3. The difference in cost between trips from Town $2n$ and $2n-1$ stays always 2^{n-1} (for $n > 0$ of course).

We start by considering the very end of Balthasar's trips, namely $n = 6$, and then work our way backwards to the beginning of the trips. The signs cannot be switched in Town 11 and 12, so statements 1 and 2 are true by default. The difference in cost between the two towns is indeed always $32 = 2^{6-1}$. Now we choose some $n < 6$ and pretend the three statements to be true for values greater than n. We will argue later, why this assumption is justified. By this assumption, the cost of the trip decreases by 2^{n+1} after a switch in Town $2n+1$ or $2n+2$; no other costs change; and the difference in costs of the trips from these towns remain 2^n. Then we want to prove our statement for towns $2n$ and $2n-1$. Suppose Balthasar changes the sign in Town $2n-1$. Then, since Town $2n$ is cheaper, the signs in the previous towns must already point towards Town $2n$ (otherwise he would have changed signs in an earlier town). Therefore no trips except from Town $2n-1$ change. Moreover, the difference in cost between the trips from towns $2n+1$ and $2n+2$ is 2^n by our assumption, so the cost of the trip from Town $2n-1$ decreases by 2^n. Also, if the trip from Town $2n-1$ was 2^{n-1} more expensive than from Town $2n$, then it is 2^{n-1} gold cheaper than the trip from Town $2n$ afterwards.
Suppose on the other hand that Balthasar changes the sign in Town $2n$. He must have made the switch to the best town from Town $2n-1$ already, so the trip from Town $2n-1$ is cheaper than from Town $2n$, and the previous towns already point to Town $2n-1$. Again, from the assumption, the difference in costs between Towns $2n+1$ and $2n+2$ is 2^n, so the switch makes the trip from Town $2n$ cheaper by 2^n gold. After the switch, the trip from Town $2n$ becomes 2^{n-1} cheaper than from Town $2n-1$ if it was 2^{n-1} more expensive before the switch.
So, if the assumption is true, then we have proven the three statements for our chosen value of n. We already know, that $n = 6$ satisfies the assumption and therefore the statements hold for $n = 5$. But then the assumption is true for $n = 5$, so we can use it on $n = 4$ by the same logic. This chain continuous until we reach $n = 0$.[1] Thus, the statements hold for all of Balthasar's trips.
So the total amount of signs switched in Town $2n$ or $2n-1$ is the difference in cost between the optimal route and the starting route from that town, divided by 2^n. That is 1 switch in Towns 9 and 10, 3 switches in Towns 7 and 8, and then 7, 15, 31 and 63 switches in respectively Towns 5,6, Towns 3,4, Towns 1,2, and Jerusalem. In total that is $63 + 2 \cdot (31 + 15 + 7 + 3 + 1) = 177$ switched signs.

[1] This type of argument is formally known as *induction*.

13 Santa's Bill

Author: Tobias Paul (HU Berlin)
Project: EF 4-7

Ilustration: Friederike Hofmann

Challenge

To foster creative freedom in gift-giving, Santa Claus has introduced a new system: each gift consists of individual parts and is intended to be assembled freely by the Christmas elves. At the beginning, each elf receives one component of the large gift. To connect the pieces appropriately, the elves scurry around in a chaotic manner. With n elves present, it takes $\frac{1}{n(n-1)}$ days for two elves with matching parts to find each other. At the same time only two elves can look for each other. Once two elves have found each other, they immediately assemble their parts. Right after that, one of the two elves collects his wages from Santa Claus and then disappears into a well-deserved evening. The other elf now keeps the larger piece and continues searching in the commotion for matching parts. After

that, $n-1$ elves are left and they repeat the analogous procedure where it takes $\frac{1}{(n-1)(n-2)}$ days for some two matching elves to find each other. This process continues until the last two (by now very large) pieces become the finished gift, and the last two elves are allowed to leave.
Of course, Santa Claus must ensure that the gifts are completed on time. Additionally, he needs to know how high his labor costs per gift can be for the budget. According to the collective agreement, a compensation of one cookie per day is agreed upon for each elf, with the amount paid out to the exact crumb. As the demands for the gifts continue to grow, requiring more and more individual parts, he wants to set up a universal framework for gift assembling. Namely, he would like to find numbers d and c such that

(a) any gift, regardless of the number of pieces it is made of, can be assembled in d days

(b) any gift, regardless of the number of pieces it is made of, costs at most c cookies.

At the moment, Santa is not sure if such values d and c exist, but if they do, Santa would like them to be as small as possible. Can you help Santa find the answer, i.e., find the minimal such d and c (if they exist)?

Possible Answers:

1. $d = 0.5$ and $c = 1$
2. $d = 0.5$ and $c = 2$
3. $d = 0.5$ and c cannot be found
4. $d = 1$ and $c = 1$
5. $d = 1$ and $c = 2$
6. $d = 1$ and c cannot be found
7. $d = 2$ and $c = 1$
8. $d = 2$ and $c = 2$
9. $d = 2$ and c cannot be found
10. both d and c cannot be found.

Project Reference:

The described process is called a *coalescent process*. In this process, two particles merge to form a larger one. In this example, it involves the Kingman coalescent with a slightly adjusted merging rate. In biology, the coalescent describes the genealogy of a subpopulation. The time until the last coalescence is then the time until the most recent common ancestor. In task part (a), we thus observe how long it takes for the most recent common ancestor to be reached for any arbitrarily large subpopulation. Part (b) then deals with the total sum of the so-called branch lengths.

Solution

The right answer is: 6.

The first part of the solution: $d = 1$.
Let us first show that a gift of size n can be assembled in 1 day, regardless of what n is. For the first two elves with compatible gift pieces to meet $1/(n(n-1))$ days pass. In the next step $1/((n-1)(n-2))$ days pass, and so on. In total, until the last two elves have met, the number of days that have passed is

$$\frac{1}{n(n-1)} + \cdots + \frac{1}{2 \cdot 1} = \left(\frac{1}{n-1} - \frac{1}{n}\right) + \cdots + \left(\frac{1}{1} - \frac{1}{2}\right) = 1 - \frac{1}{n}. \tag{1}$$

This proves that the minimal number of days d exists and satisfies $d \leq 1$. i.e., any gift can be assembled in 1 day.
Let us now show that the equality $d = 1$ holds. Assume the contrary, that d is strictly smaller than 1. This means that $d = 1 - \varepsilon$ days are needed for any gift to be assembled, where $\varepsilon > 0$ is some real number.
Suppose now we have a gift which has n pieces, where $n > 1/\varepsilon$ (i.e., a gift made out of very large number of pieces). According to the formula (1) it takes precisely $1 - 1/n$ days to finish such a gift. In particular, since d was a number of days in which any gift can be finished, this means that

$$1 - \frac{1}{n} \leq \; d \; = \; 1 - \varepsilon.$$

This is equivalent to $n \leq 1/\varepsilon$, which is a contradiction. Hence d cannot be smaller than 1.

The second part of the solution: c cannot be found.
Let us now show that a c such that any gift costs at most c cookies does not exist. Suppose to the contrary, that such c exists. We will now choose a number n such that the gift made out n pieces costs more than c cookies to make, which is a contradiction.
The number n we choose is $n = 2^m$, where m is a natural number such that $m > 2c$. Let us show that a gift made out of n pieces costs more than c cookies.
At the beginning of the process there are n elves running around. It takes $1/(n(n-1))$ days for some two of them to meet, so the elf that leaves right after that needs to be payed $1/(n(n-1))$ cookies. Then, there are $n-1$ elves left and it takes another $1/((n-1)(n-2))$ days for some two among them to meet, so the elf that leaves right after that needs to payed

$$\frac{1}{n(n-1)} + \frac{1}{(n-1)(n-2)} = \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) = \frac{1}{n-2} - \frac{1}{n}$$

cookies. In general, for any $1 \leq k \leq n-2$, the k'th elf that leaves gets paid

$$\begin{aligned}\frac{1}{n(n-1)} + \cdots + \frac{1}{(n-k+1)(n-k)} &= \left(\frac{1}{n-1} - \frac{1}{n}\right) + \cdots + \left(\frac{1}{n-k} - \frac{1}{n-k+1}\right) \\ &= \frac{1}{n-k} - \frac{1}{n}\end{aligned}$$

cookies. The last two elves leave at the same time and get payed

$$\frac{1}{n(n-1)} + \cdots + \frac{1}{2 \cdot 1} = \left(\frac{1}{n-1} - \frac{1}{n}\right) + \cdots + \left(\frac{1}{1} - \frac{1}{2}\right) = 1 - \frac{1}{n}$$

cookies each. The total amount of cookies that all the elves need to be payed is

$$\begin{aligned}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \cdots + \left(\frac{1}{2} - \frac{1}{n}\right) + 2\left(1 - \frac{1}{n}\right)\\ = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1}.\end{aligned} \tag{2}$$

We will now show that

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} > c$$

when the number of pieces $n = 2^m$ is as chosen at the beginning, which would mean that the n elves need to be be payed with more than c cookies. This would finish the proof. To this end, we can first notice that for each $r \geq 1$ the sum of reciprocal values of the 2^{r-1} following numbers $2^{r-1}, 2^{r-1}+1, \ldots, 2^r - 1$, denoted by S_r, can be estimated by

$$S_r = \frac{1}{2^{r-1}} + \frac{1}{2^{r-1}+1} + \cdots + \frac{1}{2^r - 1} \;\geq\; \frac{1}{2^r} + \frac{1}{2^r} + \cdots + \frac{1}{2^r} \;=\; 2^{r-1} \cdot \frac{1}{2^r} = \frac{1}{2}.$$

Therefore, since $n = 2^m$, we have that the total cookie cost (2) satisfies

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} = S_1 + S_2 + \cdots + S_m \geq \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{m}{2}.$$

Thus, in order to pay all the elves, Santa need at least $m/2 > c$ cookies for such a gift.

We conclude that there is no c such that every gift can be payed with up to c cookies. Let's hope that Santa Claus has a good oven.

14 Santa's Storage Struggle

Author: Wouter Fokkema
Project: 4TU.AMI

Illustration: Vira Raichenko

Challenge

Santa has too little space to store all his presents! Luckily, he found an old basement that still has some room. The basement is divided into cells. A top view of the basement is shown below. Santa wants to store as many presents in the basement as possible.

However, there are some rules:

- A present takes up exactly 1 cell.

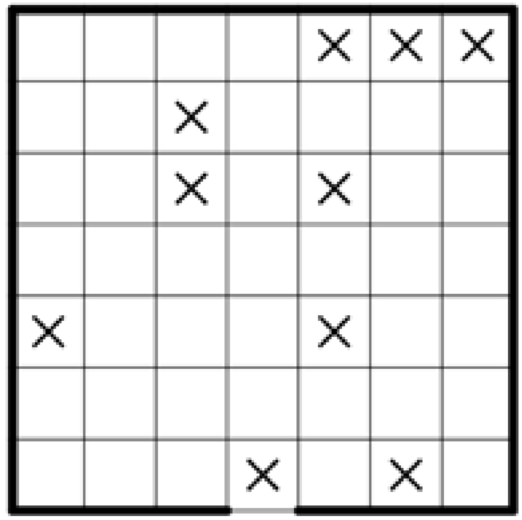


Figure 1: The layout of Santa's basement. The cross in the center of the bottom row marks the entrance. All other crosses mark places where the ceiling leaks.

- No presents are allowed at the entrance and in places where the ceiling leaks; these cells are indicated with a cross.
- 2 presents may not be in horizontally or vertically adjacent cells. (This is, because all presents look similar and two presents next to each other can easily be confused.)
- Starting from the entrance to the basement, it must be possible to visit every cell without a present by moving horizontally and vertically through the cells without presents. In other words, the cells that do not contain a present (including the cells with a cross) must form 1 area of cells that is connected orthogonally. (This is important, because the presents have to be inspected regularly. Thus it would be additional work, if one needs to move them around.)

Santa needs your help determining the maximum number of presents that can fit in the basement. What is that maximum number?

Possible Answers:

1. Eleven presents
2. Twelve presents
3. Thirteen presents
4. Fourteen presents
5. Fifteen presents
6. Sixteen presents

7. Seventeen presents
8. Eighteen presents
9. Nineteen presents
10. Twenty presents

Solution

The correct answer is: 6.

To find the solution, we count the number of horizontal and vertical connections between cells. To fit as many presents as possible in the basement, we need to use these connections as efficiently as possible. There are 84 connections in total. A present in the corner is adjacent to 2 connections, a present next to a border is adjacent to 3 connections, and a present in another cell is adjacent to 4 connections, see Figure 2. All connections that are not adjacent to presents are used to connect the cells without presents. Let W be the number of cells without presents. Then at least $W - 1$ connections are needed to connect these cells into 1 area.

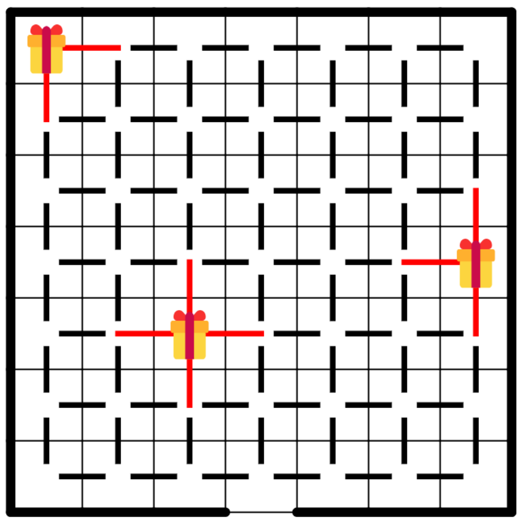


Figure 2: The different cell types and their connections.

By placing as many presents as possible in the corners and along the borders, there remain as many connections as possible to connect the remaining cells into 1 area. At most 3 corners can contain a present (due to the cross in one corner), and after these presents are placed, at most 6 more presents can fit along the borders. In total, the presents in the corners and along the borders use $3 \cdot 2 + 6 \cdot 3 = 6 + 18 = 24$ connections. Now there are 40 of the 49 cells left, and 60 of the 84 connections. Each additional present we place now takes up 4 connections and 1 cell. With this we can create the table below.

Presents	Remaining connections	Cells without present
9	60	40
10	56	39
11	52	38
12	48	37
13	44	36
14	40	35
15	36	34
16	32	33
17	28	32

We see that 16 presents is the largest number of presents such that there are enough connections left to possibly connect all the cells without a present. To reach this number, there must therefore be 3 presents in the corners and 6 presents along the borders. We must also use as few connections as possible to connect the remaining cells. This means that these connections should not form a cycle anywhere, because we could then remove 1 connection such that the remaining cells would still be connected. That is why in a solution that contains 16 presents, no 2x2 square of cells can exist that does not contain a present. By applying these insights, we can figure out the solution step by step and find out that 16 presents is indeed possible, see Figure 3.

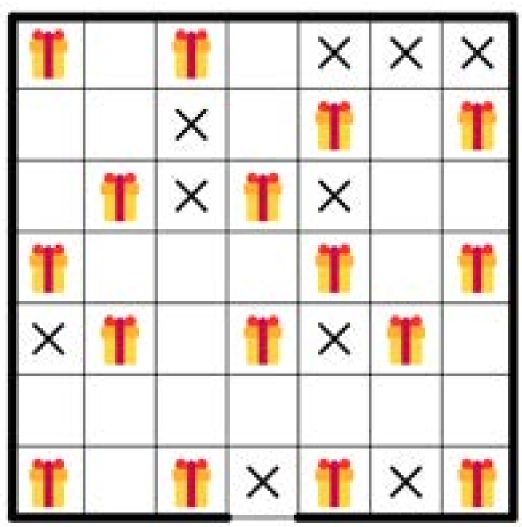


Figure 3: A way (indeed the unique one) to fit 16 presents into the basement.

15 Safe Cracking

Author: Owen Hearder (FU Berlin)
Project: BMS

Illustration: Zyanya Santaurio

Challenge

Aapo was excited and a little worried. Head elf Priita retired last Christmas and chose Aapo to be the new head elf. In order to begin preparing for next Christmas, Aapo has to read the instructions on how to be head elf. Since Priita had been head elf for so long, the instructions are locked in the archives.

The archives are locked in a very complex way. Priita told him the following: "There is a set of four digit numbers that are valid codes for the safe and all of these contain the digit 6. I described for you an algorithm, where applying steps 2 till 5 (a so-called iteration) yields a number. The valid codes are such numbers that, from a certain point onward in

the algorithm, this number appears and the result of further iterations from this point on do not change the number anymore."
Aapo looks at the paper she gave him. Indeed, the algorithm is described as follows:

1. Choose a number between 1 and 9999 with at least two different digits.
2. Add zeros to the front of the number so that you have four digits.
3. By rearranging the digits, build the highest and the lowest number.
4. Subtract the lowest number from the highest number.
5. Write this number down and with it follow the instructions from step 2 again.

"This will be all you need to unlock the safe.", Priita said with a grin on her face.
Aapo is panicking, since he thinks that it will take forever to find out a code! But Priita is optimistic and smiles at him.

Assuming Priita is right and the set of valid codes satifying described properties by Priita exists, how many valid codes are there?

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10.

Solution

The correct answer is: 1.

The idea of this challenge is that although we are given an infinite loop of instructions, at some point we converge to a specific number, i.e. applying Priita's instructions again to this number in the algorithm does not change it. This number is the code she chose for the safe.

We claim that 6174 is the only four-digit number, with not all digits the same and containing the digit 6, (indeed one can show that it is the only four-digit number besides 0000) that does not change under Priita's instructions. We call such a number a *fixpoint*.

First it is easy to see, that 6174 does not change after applying steps 2 till 5. Rearranging in step 3 yields 7641 and 1467, subtracting in step 4 gives $7641 - 1467 = 6174$

Next, we show that 6174 is the only fixpoint. Note that in the proof, we will use an overline when we refer to the digits of a number, and not the number itself.

Let $x_0 := \overline{abcd}$ be a four-digit number with digits a, b, c, d, such that applying Priita's instructions to x_0 results again in x_0. Let a_1, a_2, a_3, a_4 be a relabeling of a, b, c, d in ascending order. In the mathematical language this means

$$\{a_1, a_2, a_3, a_4\} = \{a, b, c, d\}$$
$$\text{and } 9 \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq 0.$$

Then, the highest number we can build using these four digits is $\overline{a_4a_3a_2a_1}$ and the lowest one is $\overline{a_1a_2a_3a_4}$. Subtracting the lower one $\overline{a_1a_2a_3a_4}$ from the higher one $\overline{a_4a_3a_2a_1}$ results in the following equation:

$$x_0 = 1000(a_4 - a_1) + 100(a_3 - a_2) + 10(a_2 - a_3) + (a_1 - a_4) \tag{3}$$

Let's assume $a_2 = a_3$, then

$$\begin{aligned} x_0 &= 1000(a_4 - a_1) + 100(a_3 - a_2) + 10(a_2 - a_3) + (a_1 - a_4) \\ &= 1000(a_4 - a_1) + (a_1 - a_4). \end{aligned}$$

Since we assume $x_0 \neq 0000$ and $a_4 > a_1$, it follows

$$x_0 = \overline{(a_4 - a_1 - 1)99(10 + a_1 - a_4)}. \tag{4}$$

Thus, x_0 must have two digits equal to 9, which have to be the last digits in the order, hence $a_4 = a_3 = 9$. Since we assume $a_2 = a_3$ it follows that $a_2 = a_3 = a_4 = 9$. Substituting $a_4 = 9$ in equation (4) results in $x_0 = \overline{(8 - a_1)99(a_1 + 1)}$. It follows that $9 = a_1 + 1$, because a_1 is the smallest digit and all other digits are 9. Therefore $a_1 = 8$. On the other hand $a_1 = 8 - a_1$, as $8 - a_1$ is smaller than 9 and thus cannot be a_2, a_3 or a_4. Therefore we get $a_1 = 4$, which is a contradiction. Therefore the assumption $a_2 = a_3$ is false and we conclude $a_2 < a_3$.

Next, let us look at the equality we get from (3) for every digit using $a_2 < a_3$:

$$\begin{aligned}
a &= a_4 - a_1\\
b &= a_3 - a_2 - 1\\
c &= a_2 - a_3 + 9\\
d &= a_1 - a_4 + 10
\end{aligned}$$

Addition of the first and fourth as well as of the second and the third of these equalities gives us $a + d = 10$ and $b + c = 8$.
Note that the number we obtain from following the instructions does only depend on the digits of the number we started with, and not on the number itself. Thus, for the moment we ignore the order of the digits and only consider all pairings of a, b, c, d with $a + d = 10$ and $b + c = 8$. In mathematical notation this means

$$\{a, d\} \in \Big\{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5, 5\}\Big\},$$

and similarly

$$\{b, c\} \in \Big\{\{0, 8\}, \{1, 7\}, \{2, 6\}, \{3, 5\}, \{4, 4\}\Big\}.$$

Using the condition that one of the digits is a 6, we have only nine options for $\overline{a_4a_3a_2a_1}$ and $\overline{a_1a_2a_3a_4}$:

1. 8640, 0468
2. 7641, 1467
3. 6642, 2466
4. 6543, 3456
5. 6444, 4446
6. 9621, 1269
7. 8622, 2268
8. 7632, 2367
9. 6552, 2556

By subtracting the higher from the lower number for all nine possible pairs, we receive nine candidates for a fixpoint. In the next step we do one iteration from the candidates and observe that only 6174 is a fixpoint of the iteration.

We further claim that by performing Priita's routine on any four-digit number with at least two different digits, we eventually will get the number 6174 after at most 7 steps. The proof of this claim is rather exceeding at this point, for which reason we want to refer students interested in it search for "Kaprekar's routine". Thus, assuming the correctness of Priita's claim, we know that whatever number Aapo chooses, he will obtain the number 6174 at some point and will not get any different number afterwards by further applying Priita's instructions.

16 The Sheep Hotel

Author: Matthew Maat (Universiteit Twente)
Project: Die Kombination von Algorithmen für Paritätsspiele und lineare Programmierung

Illustration: Zyanya Santaurio

Challenge

With the sound of the angels' choir still resounding in their memory, the shepherds Ananias, Benjamin, Caius and David head for Bethlehem. However, they first need to drop off their sheep at a sheep hotel, since they have large flocks of sheep of different types. All four of them go to different hotels. The standard rules for sheep hotels in the first century are as follows:

1. You are allowed to use as many rooms as you like.
2. It is not allowed to spread two different types of sheep over the exact same set of rooms. For example, if we divide all the black sheep over room number 1 and 2, we

may put all the white sheep in room 1, or you may divide the white sheep over rooms 1,2 and 3, but we may not divide the white sheep exactly over rooms 1 and 2. We are also not allowed to, for example, put all black and all white sheep in room 1.

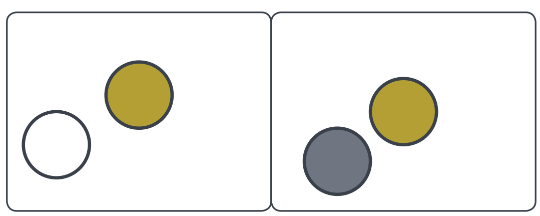
3. For every two types x and y of sheep, there is a type z (which may be x or y) such that the following holds for every room: Either there are no sheep of type x, y or z in the room, or there are sheep of type z in the room, and also sheep of type x and/or y.

4. The price is determined by the room with the most types of sheep. You pay as many gold pieces as there are types of sheep in that room.

For example, we can consider a shepherd with black (bl), white (w) and brown (br) sheep. We can fulfill the rules as follows (cf. figure 1):

- For rule 1: The shepherd uses two rooms.
- For rule 2: He puts all white sheep in room 1, all black sheep in room 2, and divides the brown sheep over rooms 1 and 2.
- For rule 3: If x is black sheep and y is brown sheep, we choose z to be brown sheep. (If we swap the roles of x and y we will always choose the same z. With this convention, the table in figure 1 is shortened accordingly.)

  If x is black or white sheep and y is brown sheep, then we choose z to also be brown sheep.
- For rule 4: In this case, the room with the most types of sheep contains two types of sheep. Therefore the shepherd pays two gold.



x, y	bl, w	w, br	bl, br
z	br	br	br

Figure 1: Cheapest distribution for a shepherd with 3 types of sheep. top: distribution inside the rooms. bottom: Table of corresponding mapping $x, y \mapsto z$.

Now it is given that Ananias has 4 types of sheep, Benjamin has 5, Caius has 6, and David has 7. Being experienced with the rules, each shepherd quickly finds the distribution of sheep that costs the least for them. It turns out that all of them only need to use 3 rooms each for their cheapest distribution. Ananias pays A gold, Benjamin pays B gold, Caius pays C gold, and David pays D gold. What is $A + B + C + D$?

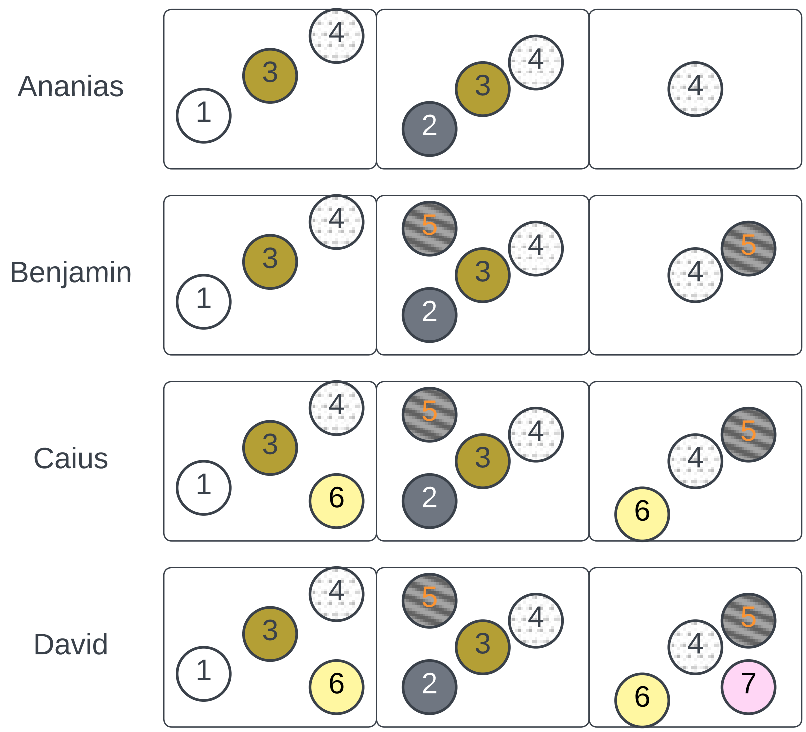
Possible Answers:

1. 13
2. 14
3. 15
4. 16
5. 17
6. 18
7. 19
8. 20
9. 21
10. 22

Solution

The correct answer is: 3.

We claim that the distribution in figure 2 is optimal regarding their total cost, which is $3 + 4 + 4 + 4 = 15$ gold. A mapping $x, y \mapsto z$ that shows that rule 3 is never violated with this distribution is presented in shortened form below.



x,y	1,2	1,3	1,4	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7
z	3	3	4	4	6	6	3	4	5	4	5

x,y	3,4	3,5	3,6	3,7	4,5	4,6	4,7	5,6	5,7	6,7
z	4	4	4	4	4	4	4	4	5	6

Figure 2: **Top:** Optimal distribution of sheep over 3 rooms, for each shepherd. **Bottom:** Table of the corresponding mapping $x, y \mapsto z$ in shortened form: If x, y maps to z, then also y, x maps to z and all pairs with $x = y$ map to x, i.e. $x, x \mapsto x$. Note that the map is valid for all shepherds, i.e. to obtain the mapping for Benjamin for example, ignore all entries with the digits 6 and 7 in it.

Before we continue with the proof, that this is a optimal distribution, we want to briefly describe how to come up with this solution. The crucial observation to be made is, that there is only one way (up to relabeling of the rooms and types of sheep) of distributing David's 7 types of sheep such that rule 2 is fulfilled. From this distribution, one can find distributions for the other shepherds by removing a suitable type of sheep each time.

Now we still need to show that they cannot distribute the sheep in a cheaper way. To make the argument easier, we say that sheep type x is a *subset* of type y if there are sheep of type y in every room that has sheep of type x. We say type z is the *union* of types x and y if z is exactly the type of sheep that is prescribed by rule 3 - so the sheep of type z are in exactly those rooms that also contain sheep of type x or y.
We start with Ananias and his 4 types of sheep. Suppose Ananias can divide his sheep while only paying 2 gold. Take two arbitrary types x and y with $x \neq y$. By rule 3, there is a z that is the union of x and y. Assume that $z \neq x$ (if $z = x$ we can swap x and y), then type x is a subset of type z. So the rooms that have sheep of type x have at least two types, as they already have x and z. Now we look at the other two types beside x and z. Call them r and s. If type r are in rooms that don't have sheep of type z, then the union of r and z must be different from r, z and cannot be x. But then the union must also have sheep at the rooms where the sheep of x are, so we have three types in the same room, which is a problem. So the sheep of type r can only be in rooms where type z is, and of course they cannot be where x is (since then there would be 3 types of sheep). Suppose the sheep of type r do not fill all the rooms that have type z but not type x. Then the union of r and x is not z. This is because there is at least one room with sheep of type z not featuring types x or r. Thus, we have three kinds of sheep in the rooms with x-sheep again. So the sheep of type r must be exactly in the rooms with z but without x. But the same can be said about s. But then types r and s disobey rule 2. So we conclude that it is not possible to pay 2 gold for Ananias.
Then to Benjamin. Note that in our proof for four types of sheep, we could always find three types x, y and z such that x is a subset of y and y is a subset of z, so this also holds for five types. Let r and s be the remaining two types of sheep (so r and s are not x, y or z). Suppose Benjamin can do it with 3 gold. Since there are already 3 types of sheep in the rooms with type x, there cannot be any more types in those rooms, so r and s must be in rooms without type x. But now what could the union of x and r be? It cannot be s, because then there is a room with both x and s. So this union must equal y or z. And the same holds for the union of x and s: it must be y or z. But that determines the rooms for types r and s: one of them goes into the rooms that have type z but not x, and the other goes into the rooms with y but not x. But now the rooms that have y and z but not x also contain both r and s: four types of sheep in total. We conclude that it is not possible with 3 gold pieces. Since dividing the sheep becomes harder with more types, it follows that Caius and David also cannot do it with 3 gold.

17 The Colorful Christmas Presents

Author: Lukas Protz
Project: MATH+

Illustration:Friederike Hofmann

Challenge

To relieve the elves from coloring gift boxes in the shape of a cube, the eager elf Eifi has purchased a machine. This machine is intended to take over the coloring process for the elves. As eager as Eifi is, he has already had all the lids for the gift boxes made by the machine. They are all colored with yellow stars on a dark blue background.

Unfortunately, Eifi forgot to ask Santa Claus beforehand if the elves are even allowed to use the machine. Santa is not thrilled that all the gift box lids look the same. In order to quickly find the right gift when distributing them on Christmas, he would prefer that all the gifts look different. He also finds colorful gifts much more beautiful than monochrome ones.

Eifi tries to reassure Santa by showing him that the machine can randomly color each of the remaining five sides of a gift. There are even nine different colors available for coloring.

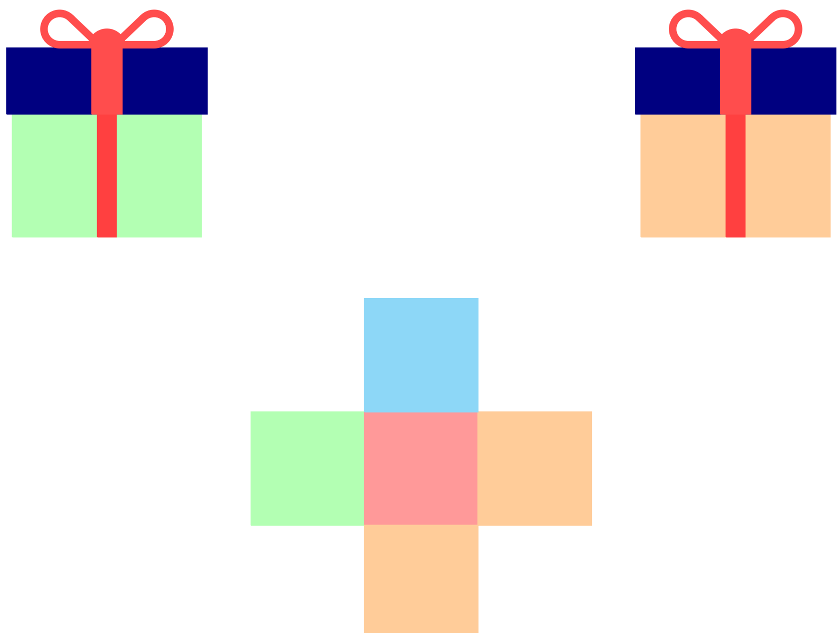
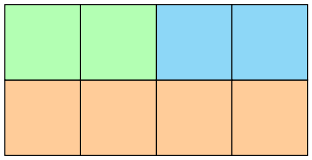


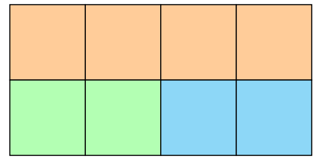
Figure 1: A present already wrapped. Top left: The present from one side, top right: The same present from the opposite side, bottom: The net of the present without the lid.

Santa Claus, however, is not convinced of that. It could still happen that a present is colored in only one or two colors. Therefore, he makes Eifi an offer. Initially, he must ascertain the variety of distinct color combinations achievable for the gift, excluding the lid, by employing the nine available colors. Two presents are considered differently colored if they differ not solely by reflections or rotations. Let a be the proportion of all colorings such that the number of sides which have the same color as some other side is at most two. The elves are allowed to use the machine if the proportion a is greater than 0.7. Otherwise, they must, as always, color all the presents themselves so that all sides always have different colors. What is the proportion a of presents with at most two identically colored sides?

Explaination of reflection and rotation

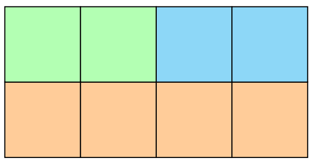
To explain the concept of reflection we are considering two 2×4 grids as shown below:

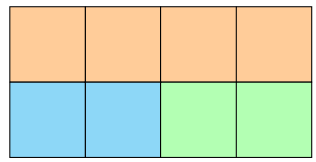




These two grids are not different, as they only differ by a horizontal reflection between the two rows.

To explain the concept of rotation we are again considering two 2×4 grids as shown below:





These two grids are not differnet, as they only differ by a rotation of π (or 180°).

Possible Answers:

1. $a \leq 0,1$
2. $0,1 < a \leq 0,2$
3. $0,2 < a \leq 0,3$
4. $0,3 < a \leq 0,4$
5. $0,4 < a \leq 0,5$
6. $0,5 < a \leq 0,6$
7. $0,6 < a \leq 0,7$
8. $0,7 < a \leq 0,8$
9. $0,8 < a \leq 0,9$
10. $0,9 < a$

Solution

The correct answer is: 7.

To determine the solution, we can systematically count all possible different colorings. To do this, we differentiate based on the number of colors used. We will model the cube by the net of its sides without the lid (as in the example). We note that two cubes differ by a rotation or reflection in space if and only if their nets differ by a rotation or reflection in the plane.

1. Only one color is used. In this case, there is only one way to color the box. Since there are nine different colors, we get nine presents of different colors.

2. Exactly two colors are used. Here, there are a total of five different coloring possibilities (excluding rotations and reflections). To describe the distribution of colors better, we use the notation $1-4$ when one side is colored in one color and the remaining four sides are colored in the other. Similarly, $2-3$ means that two sides are colored in one color and three in the other. The five possibilities are presented in Figure 2.

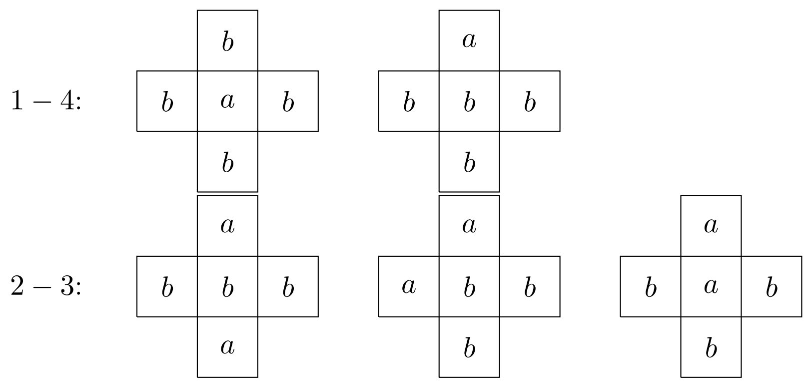


Figure 2: The five possibilties to color a present with two colors. Lowercase letters were used as placeholders for the colors.

   The number of ways to choose two colors is then $9 \cdot 8$, resulting in a total of $5 \cdot 9 \cdot 8$ possible presents for this case. The colors must be chosen in order so that we can decide which color is used on fewer sides of the present. If we were to choose the colors without order, we would lose the ability to determine the color distribution.

3. Exactly three colors are used. We use the notation from above. The options for this case are $1-1-3$ or $1-2-2$, see Figure 3.

   For all cases except the first and the last two, there is a reflection in each case (dashed gray), which swaps the names of the colors but not the distribution of the colors themselves. Therefore, for these possibilities, there are not $9 \cdot 8 \cdot 7$ colorings (ways to choose three colors in order from the nine colors), but only half as many. In total, there are $\left(\frac{4}{2}+3\right) \cdot 9 \cdot 8 \cdot 7$ colorings.

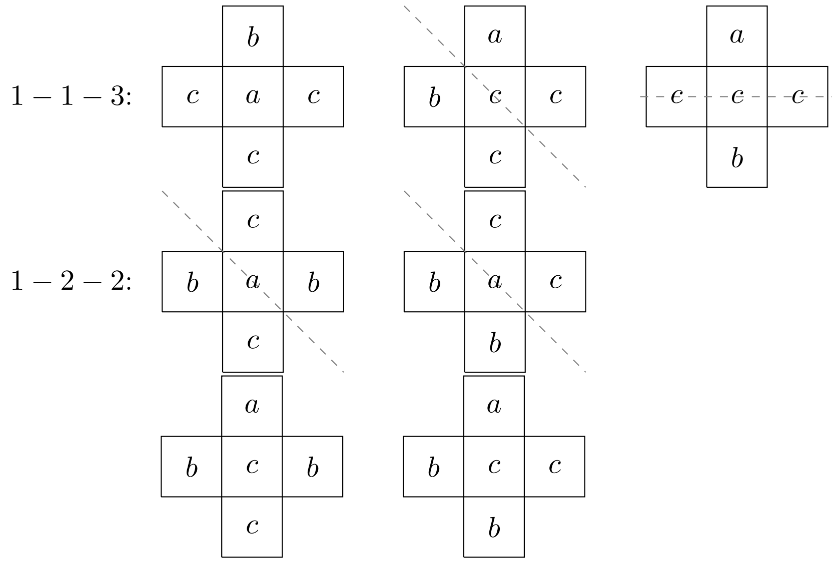


Figure 3: The seven possibilities to color a present with three colors.

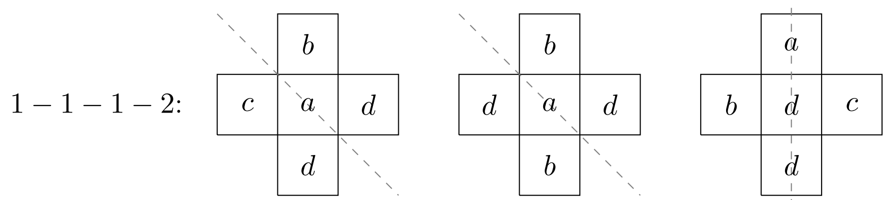


Figure 4: The three possibilities to color a present with 4 colors.

4. Exactly four colors are used. In this case, there is only the option $1-1-1-2$, giving us the cases shown in Figure 4.

   In each case this time, there is a reflection that swaps the color names but not the color distribution. Therefore, there are a total of $\frac{3}{2} \cdot 9 \cdot 8 \cdot 7 \cdot 6$ colorings.

5. Exactly five colors are used. This means each side is painted in a different color. In total, due to reflections and rotations, eight symmetric versions are obtained. Therefore, the number of colorings is $\frac{1}{8} \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.

Adding up all the coloring possibilities for the individual cases, we get 9315. The number of colorings where at most two sides have the same color as some other side is obtained by adding the possibilities from the last two cases. It amounts to 6426. Thus, the result is

$$a = 0.6899\ldots$$

So, unfortunately, the elves have to paint all the presents themselves, unless Eifi gets not only the lids but even all the gift wrappings in advance next year...

18 Phylogeny of Elves

Author: Andrei Comăneci (TU Berlin)
Project: Graduiertenkolleg „Facets of Complexity“ (GRK 2434)

Illustration:Zyanya Santaurio

Challenge

The young Christmas elves have recently learned in the Elf School about the intriguing concept of evolution and the classification of elves into a big family tree. These elves have always been known for their festive spirit and dedication to helping Santa Claus, but now they are eager to explore their own history. They were taught about how elves have evolved over time, adapting to different climates and environments.
For instance, Figure 1 displays the evolutionary history of four families of elves. Christmas and snow elves are the most closely related, sharing a common ancestor 2 million years ago. On the other hand, moon elves are the least closely related, as a common ancestor with any

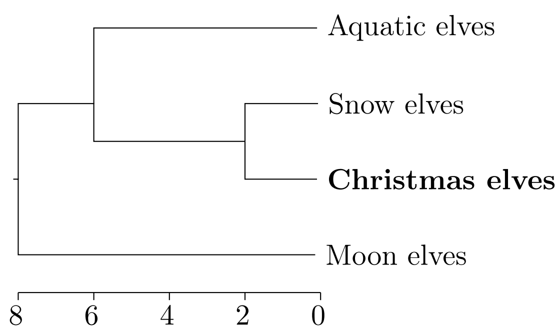


Figure 1: Phylogeny of elves. The x-axis shows the time in million years

other species dates back 8 million years. The times of the most recent common ancestors are collected in Table 1 and they measure the dissimilarity of the elf families.

	A	S	C	M
Aquatic elves (A)	0	6	6	8
Snow elves (S)	6	0	2	8
Christmas elves (C)	6	2	0	8
Moon elves (M)	8	8	8	0

Table 1: Dissimilarity matrix of elf families

Ten young elves wanted to learn more about their evolutionary history, so they searched for “The Great Book of Elvish Evolution” in the North Pole Archives. Unfortunately, this old book was severely damaged over the years, resulting in the loss of much of its information. However, the dissimilarity matrix was mostly intact. The only few missing entries are marked with W, X, Y, and Z in Table 2.

	A	B	C	D	E	F	G	H	I
A	0	8	8	8	8	8	8	8	8
B	8	0	5	**Y**	**Z**	7	7	7	**W**
C	8	5	0	1	7	7	7	7	7
D	8	**Y**	1	0	7	7	7	7	7
E	8	**Z**	7	7	0	1	4	4	4
F	8	7	7	7	1	0	4	4	4
G	8	7	7	7	4	4	0	**X**	2
H	8	7	7	7	4	4	**X**	0	1
I	8	**W**	7	7	4	4	2	1	0

Table 2: Dissimilarity matrix with missing entries

The little elves realized that not every value could be valid for the missing entries and it is possible to reconstruct the phylogeny from the matrix once they have the complete matrix. After considering possibilities, each of the ten elves proposed potential values for the missing entries. However, only one of the proposals is a valid option. Please tell us which of the ten elves was correct.

Remark: A graph is a pair $G = (V, E)$, where V denotes the set of vertices and E a set of edges, i.e., a set of (some) pairs of distinct vertices. A cycle in a graph is a sequence of distinct vertices $v_1, \ldots, v_n$ with $n \geq 3$ such that $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\} \in E$. A tree is a graph without cycles, and a rooted tree is a tree with a distinguished vertex, called the root.

Possible answers:

1. W = 7, X = 1, Y = 7, Z = 7
2. W = 8, X = 1, Y = 1, Z = 7
3. W = 7, X = 2, Y = 5, Z = 5
4. W = 7, X = 2, Y = 1, Z = 7
5. W = 8, X = 1, Y = 5, Z = 7
6. W = 8, X = 1, Y = 7, Z = 7
7. W = 7, X = 2, Y = 1, Z = 5
8. W = 7, X = 1, Y = 5, Z = 5
9. W = 7, X = 2, Y = 5, Z = 7
10. W = 8, X = 1, Y = 5, Z = 5

Project Reference:

Phylogenetic trees are a mean of representing the evolutionary history of the species under study (for more details, one could check, for example, the website: http://tolweb.org/tree/). The matrix representation is an alternative way to view the relationships between the species. In a project from TU Berlin, we focus on analyzing phylogenetic trees using tropical geometry. Missing data is common in practice, so filling in the missing entries is a popular preprocessing step.

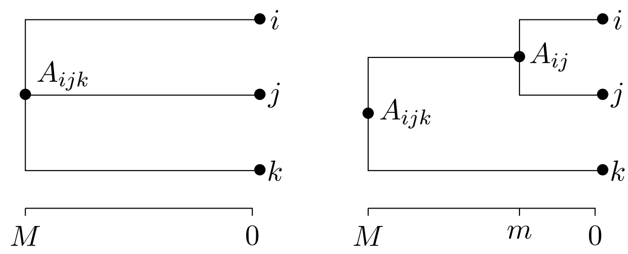


Figure 2: Phylogenies on three species

Solution

The correct answer is: 9.

Let us define a graph which has as the set of vertices the set of elf types (A, ..., I) and the set of pairwise most recent common ancestors (MRCA). An ancestor is connected by an edge to its direct descendant. This graph is a rooted tree, with its distinguished vertex the most recent common ancestor of all elf types.

Selecting three distinct species and looking at the evolutionary history restricted only to them, we obtain one of the two cases from Figure 2. We denoted the labels by i, j, and k. Let the MRCA of i and j be denoted by A_{ij} and the corresponding entry in the dissimilarity matrix by d_{ij}. Similarly, we define A_{ik}, A_{jk} as well as d_{ik}, d_{jk}. The MRCA of all of them is denoted by A_{ijk}. First we notice that two of the ancestors A_{ij}, A_{jk}, A_{ik} have to be equal, otherwise the graph contains the cycle $i, A_{ij}, j, A_{jk}, k, A_{ik}$. Without loss of generality, let $A_{ik} = A_{jk}$ (Otherwise we relabel i, j, k such that this assumption is true). For a similar reason, we get that $A_{ik} = A_{jk} = A_{ijk}$. Therefore, we obtain two situations depicted in Figure 2 by differentiating whether A_{ij} is equal to A_{ijk} or not.

1. If $A_{ij} = A_{ik} = A_{jk} = A_{ijk}$, we obtain the situation on the left. The most recent common ancestor (MRCA) for each pair of the selected species is the same. In other words, the values d_{ij}, d_{ik}, and d_{jk} of the dissimilarity matrix are all equal to M (as defined in Figure 2).

2. If $A_{ij} \neq A_{ik} = A_{jk} = A_{ijk}$, we obtain the situation on the right. In terms of the dissimilarity, this means that d_{ij} has a value smaller than d_{ik}, the latter being equal to d_{jk}.

In any case, we found that for every three distinct species i, j and k, the highest value among d_{ij}, d_{ik}, and d_{jk} occurs twice. A dissimilarity matrix with this property, holding for every triplet $\{i, j, k\}$, is called *ultrametric*. We use the ultrametric property to fill in the missing entries.

The entry W represents the distance to the MRCA of B and I. To apply the property above, we have to find another species S such that $d_{\mathrm{BS}} \neq d_{\mathrm{IS}}$. For example, considering $S = \mathrm{C}$, we have $d_{\mathrm{BC}} = 5 < 7 = d_{\mathrm{IC}}$. The property above says that the maximum of the values $d_{\mathrm{BI}} = W$, $d_{\mathrm{BC}} = 5$, and $d_{\mathrm{IC}} = 7$ is attained twice, which happens only when W = 7.

A similar procedure is performed to obtain the other values:

- looking at the triplet $\{G, H, I\}$, we obtain X = 2: the maximum of X, 1, and 2 must be attained twice;
- looking at the triplet $\{B, C, D\}$, we obtain Y = 5: the maximum of Y, 1, and 5 must be attained twice;
- looking at the triplet $\{B, E, F\}$, we obtain Z = 7: the maximum of Z, 1, and 7 must be attained twice.

The corresponding complete dissimilarity matrix with all entries is listed in Table 3.

	A	B	C	D	E	F	G	H	I
A	0	8	8	8	8	8	8	8	8
B	8	0	5	5	7	7	7	7	7
C	8	5	0	1	7	7	7	7	7
D	8	5	1	0	7	7	7	7	7
E	8	7	7	7	0	1	4	4	4
F	8	7	7	7	1	0	4	4	4
G	8	7	7	7	4	4	0	2	2
H	8	7	7	7	4	4	2	0	1
I	8	7	7	7	4	4	2	1	0

Table 3: Complete dissimilarity matrix

Finally, knowing the values of the dissimilarity matrix, we can reconstruct the tree. We do it step by step. Namely, we first add the most recent common ancestor (MRCA) for elf types with dissimilarity one. From the table we read that $d_{CD} = d_{EF} = d_{HI} = 1$. See Figure 3.

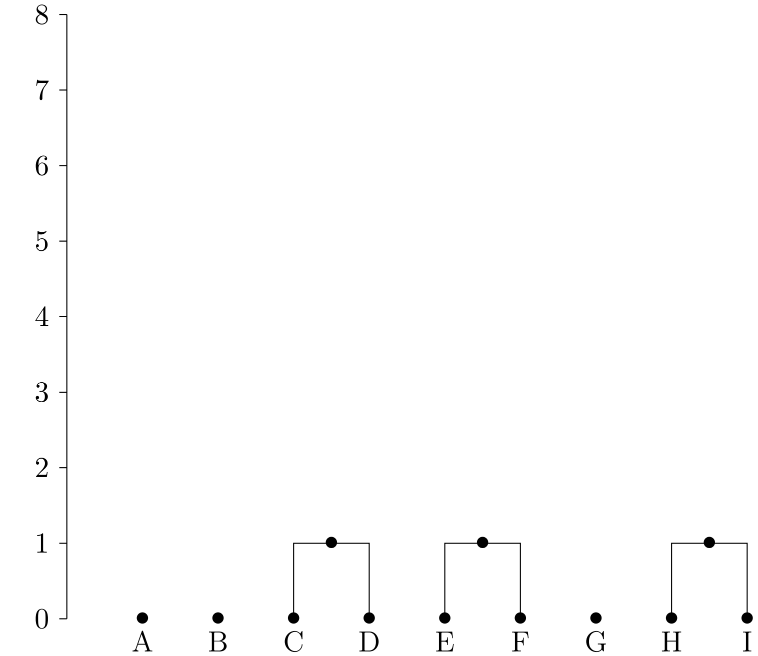


Figure 3: Part of the phylogenetic tree after entering dissimilarities equal to one

Next, we enter all dissimilarities equal to two. From the table we read that $d_{GH} = d_{GI} = 2$, so we place the MRCA of G,H and I at height two and connect it to the MRCA of H and I, which is at height one. See Figure 4.

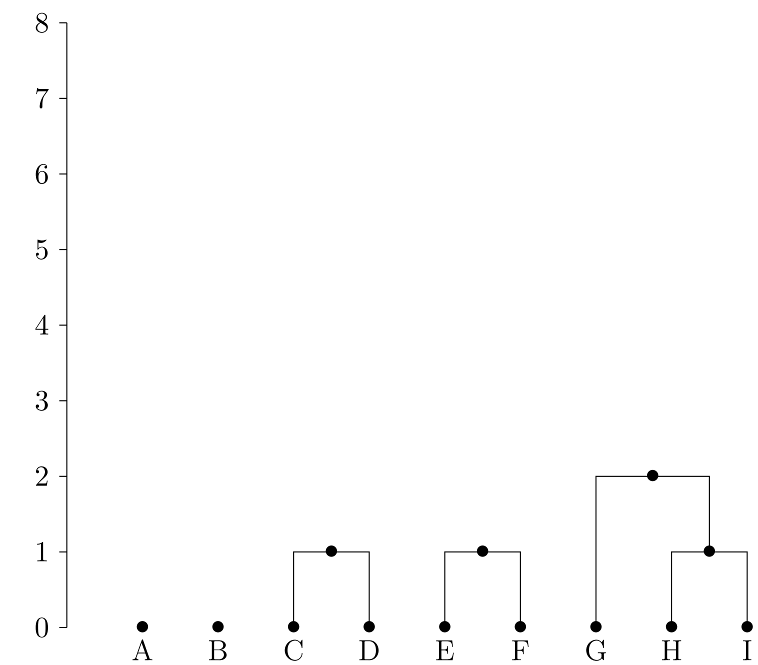


Figure 4: Part of the phylogenetic tree after entering dissimilarities equal at most two

By continuing this process further, we obtain the full tree depicted in Figure 5.

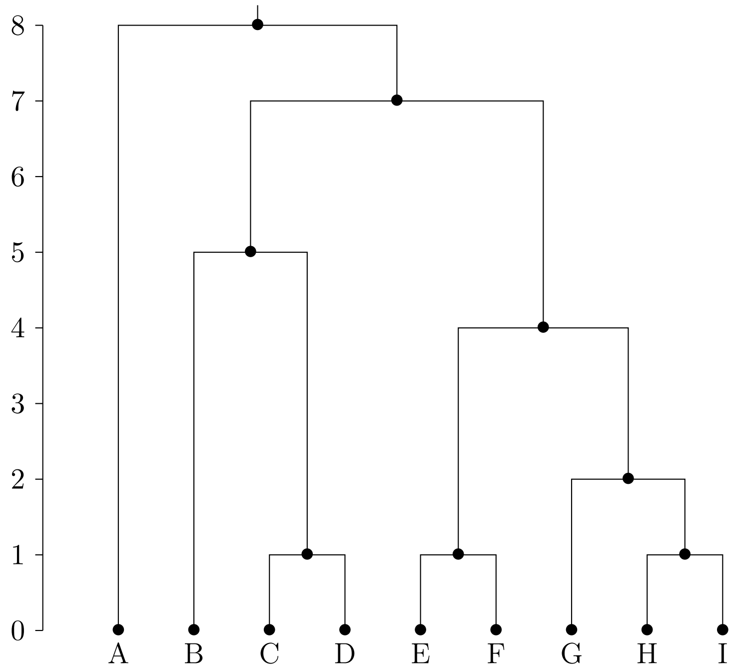


Figure 5: Phylogenetic tree corresponding to the solution

In general, the phylogenetic tree can be reconstructed from the matrix using any hierarchical clustering method, such as those described in the book [3].
One could also try to recover the tree from the incomplete matrix. A method similar to hierarchical clustering is described in [2].

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19 Triangle Trifle

Authors: Hajo Broersma, Jesse van Rhijn (University of Twente)
Project: 4TU.AMI

Illustration: Vira Raichenko

Challenge

During their downtime, some of Santa's elves like to play a game which they call "advanced tug-of-war". In this game, the elves are divided into two (possibly unequally sized) teams, distinguished by yellow and blue hats. All participating elves stand in a circle and tie ropes to the other elves, such that each two elves are connected by one rope. Each team tries to pull the other team over to their side by tugging on the ropes.

The ropes used for this game can have one of two colors, red or green. However, the colors impose no restrictions on the game and can be chosen arbitrarily. A spectating elf notices that sometimes, especially when many elves participate, it happens that three elves from one team among themselves are all connected by ropes of the same color. We call this a *perfect triangle*; see Figure 1 for an example.

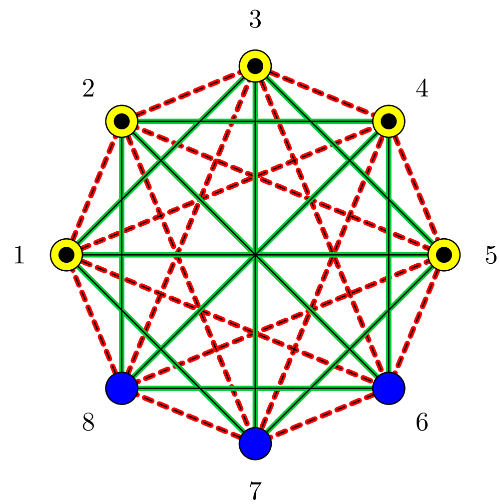

Figure 1: An example of an advanced tug-of-war game with eight elves. Green ropes are drawn as solid lines, red ropes are drawn as dashed lines. Blue dots represent elves with a blue hat, yellow dots (with a black central dot) represent elves with a yellow hat. Elves 1, 3 and 5 form a perfect triangle: they are pairwise connected by green ropes, and each of them wears a yellow hat.

What is the smallest number of elves that must participate in advanced tug-of-war in order to guarantee that there is a perfect triangle?

Possible Answers:

1. Six elves.
2. Seven elves.
3. Eight elves.
4. Nine elves.
5. Ten elves.
6. Eleven elves.
7. Twelve elves.
8. Thirteen elves.
9. Fourteen elves.
10. There is no such number (we can always assign colors to avoid a perfect triangle).

Solution

The correct answer is: 6.

To begin, we first consider a slightly simpler setting: if three elves are connected by ropes of the same color, we call them a *monochromatic triangle*. As defined in the challenge, a *perfect triangle* is then a *monochromatic triangle* in which all elves wear the same hat color. We start with a proof by contradiction showing that six elves suffice to guarantee that there exists a monochromatic triangle.

Hence, we assume there are six elves without any monochromatic triangle among them. Consider some elf e. This elf is connected to all other elves by five ropes in total. At least three of these ropes have the same color; say, green. Consider the three elves e_1, e_2 and e_3 on the other side of these ropes. If any two of these three elves are connected by a green rope, then they form a monochromatic (green) triangle together with e. We can then assume that these three neighbors of e are mutually connected by red ropes. But then e_1, e_2 and e_3 themselves form a monochromatic (red) triangle. Thus, six elves suffice to find a monochromatic triangle.

On the other hand, if there are only five elves, then it is easy to find some coloring of the ropes that avoids monochromatic triangles; see Figure 2. Hence, we conclude that six is the minimum number of elves needed to guarantee the existence of a monochromatic triangle.

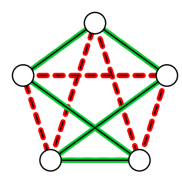

Figure 2: A coloring of the ropes between five elves that avoids monochromatic triangles.

Now we show that eleven elves suffice to guarantee the existence of a perfect triangle. Note that there must be at least six elves that wear the same hat color. From the previous result, we know that among these six elves there must be a monochromatic triangle. A monochromatic triangle among elves that wear the same hat color is exactly a perfect triangle, so indeed eleven elves suffice.

The final step is to show that with ten elves, we can assign hat and rope colors such that there is no perfect triangle. First, we divide the ten elves into two groups of five, giving all elves in one group blue hats and all elves in the other yellow hats. Then, we tie the elves within each group together according to Figure 2, and tie two elves of different groups together in any arbitrary way. Any perfect triangle would have to consist completely of elves from one of the two groups. Since the coloring in Figure 2 contains no monochromatic triangles, we see that there exists no perfect triangle with this assignment.

20 Santa's Digital Dilemma

Author: Christoph Graczyk (ZIB)
Project: IOL

Illustration: Christoph Graczyk

Challenge

Amidst the endless snow and twinkling stars of the North Pole, Santa Claus took a modern turn. To be more accessible to children globally, he set up a dedicated email inbox for their heartfelt Christmas wishes. However, crafting personalized responses to millions of emails seemed a stretch too far, even for Santa.

The elves, always eager to support Santa, convened an emergency council to solve the problem. The younger elves, like Elvin, advocated for the use of modern tech wonders, like Large Language Models. "Imagine," she exclaimed, "automating card-writing with the new computer we use for emails!" But Santa was hesitant, reluctant to lose the personal touch that defined his cards.

It was here that Elara, the oldest of the elves, suggested a middle ground. "Why not use something similar to a Bigram Model? It's the foundation of language generative tools. It can help select the introductory and concluding lines of the cards, based on our rich history of personalized cards. This way, Santa could still pen the main message. The model would only use Santa's own phrases as its vocabulary to generate the cards, instead of just the individual letters of the alphabet used in these models."

Elara's idea was simple. By analyzing past cards, they could predict how likely a phrase should appear given the previous phrases. This would retain the essence of Santa's messages while speeding up the process. To demonstrate, Elara delved into the archives, pulling out a random sample of 10,000 cards.

Each card Santa wrote typically had the same structure: an **opening line**, followed by his personal message, then a **closing wish** and a **goodbye phrase** before Santa's signature. Elara meticulously analyzed the samples from the archives. Following are some exemplary statistics for some specific phrases' frequency of usage

Opening lines:

- "As the winter wind whispers" - 220 occurrences.
- "Under the shimmering northern lights" - 180 occurrences.

Closing wishes: The choice of closing wish is depending only on the **given** opening lines:

- For cards that open with "As the winter wind whispers":
  - Beginnings:
    - "A snowflake echoing through the frosty air," - 70 occurrences.
    - "In every snowflake's unique journey," - 150 occurrences.
  - Endings:
    - "a melody of joy and hope." - 100 occurrences.
    - "is the story of a thousand stars." - 120 occurrences.
- For cards that open with "Under the shimmering northern lights":

- Beginnings:
  * "As the fireplace crackles softly," - 80 occurrences.
  * "With the serenity of a winter's night," - 100 occurrences.
- Endings:
  * "may your heart be merry and light." - 90 occurrences.
  * "let warmth and comfort in your heart dwell." - 90 occurrences.

Goodbye Phrases:

- "From the winter wonderland" - 245 occurrences.
- "Yours in festive cheer" - 155 occurrences.

However, in his excitement, Elvin overlooked a critical aspect of the model. Unlike the actual Bigram Model, which would learn the probability of the phrases depending on the previous phrase, Elvin's model was much simpler. His model selects phrases based on their overall frequency of usage with the additional criterion that the beginning of the closing wishes depend on the opening line. But, unlike Elara explained, his model treats the given endings of the closing wishes independently from the given beginnings of the closing wishes. This error could lead to the creation of unusual or even nonsensical combinations. For Santa the worst of them were the missing subject in the closing wish.

Question:
With the model based on the sample and considering Elvin's omission, what is the probability that a card created by Santa Claus this year,

- starts with one of the given opening sentences,
- contains a grammatically incorrect closing wish where the subject is missing,
- and ends with one of the given farewell sentences?

Hint: In the imperative sentence "... let warmth and security reside in your heart," it can be assumed that a subject (implicitly) is given.
The two incorrect sentences are:

- "As the fireplace crackles softly, is the story of a thousand stars." ("What is the story of a thousand stars?")
- "With the serenity of a winter's night, is the story of a thousand stars." ("What is the story of a thousand stars?")

Possible Answers:

1. 6.075%
2. 2.73375%
3. 0.0216%

4. 0.00972%
5. 0.243%
6. 0.436%
7. 0.78%
8. 10.45%
9. 3.34%
10. 0.89%

Project Reference:

The IOL research lab is dedicated to exploring the intersection of mathematical optimization and machine learning, with a focus on developing innovative techniques for learning and optimization. By integrating these two fields, the IOL group aims to create new approaches to solving complex problems that leverage the strengths of both optimization and machine learning.
The correct answer is: 3.

Solution

To calculate the probability of an error, we first have to construct all possiblly generated sentences.
First we generate all combinations for cards opening with **"As the winter wind whispers,":**

1. "A snowflake echoing through the frosty air, a melody of joy and hope."
2. "A snowflake echoing through the frosty air, is the story of a thousand stars."
3. "In every snowflake's unique journey, a melody of joy and hope."
4. "In every snowflake's unique journey, is the story of a thousand stars."

Next, we generate all combinations opening with **"Under the shimmering northern lights,":**

1. "As the fireplace crackles softly, may your heart be merry and light."
2. "As the fireplace crackles softly, let warmth and comfort in your heart dwell."
3. "With the serenity of a winter's night, may your heart be merry and light."
4. "With the serenity of a winter's night, let warmth and comfort in your heart dwell."

For the cross-combinations resulting from Elvin's oversight, we get 8 new combinations:

1. "A snowflake echoing through the frosty air, may your heart be merry and light."
2. "A snowflake echoing through the frosty air, let warmth and comfort in your heart dwell."
3. "In every snowflake's unique journey, may your heart be merry and light."
4. "In every snowflake's unique journey, let warmth and comfort in your heart dwell."
5. "As the fireplace crackles softly, a melody of joy and hope."
6. "As the fireplace crackles softly, is the story of a thousand stars." (Grammatically incorrect: missing subject)
7. "With the serenity of a winter's night, a melody of joy and hope."
8. "With the serenity of a winter's night, is the story of a thousand stars." (Grammatically incorrect: missing subject)

The ending **"...is the story of a thousand stars."**, combined with either of the beginnings from "Under the shimmering northern lights," results in an incorrect sentence with a missing subject. Thus we get the answer by simply computing the probability of these two combinations times the probability of either of the goodbye phrases.

It should be noted here that the sample space is different in each case. The opening phrase is taken from the sample data consisting of 10000 cards. Therefore the probability of picking a card with the beginning "Under the shimmering lights" is given by:

$$P(\text{"Under the shimmering lights"}) = \frac{180}{10000}.$$

Next, the probability of the ending "...is the story of a thousand stars.", given the error in Elvins programming, is the number of its occurences divided by the total number of endings for the given opening lines (which is 400), i.e.:

$$P(\text{"is the story of a thousand stars."}) = \frac{120}{400}.$$

Finally, we calculate the probability for either of the given goodbye phrases, which are sampled from the whole sample space consisting of 10000 cards, giving us:

$$P(\text{"From the winter wonderland"}) = \frac{245}{10000},$$

$$P(\text{"Yours in festive cheer"}) = \frac{155}{10000}.$$

Thus, the final answer in percent is given by

$$\frac{180}{10000} \cdot \frac{120}{400} \cdot \frac{245 + 155}{10000} \cdot 100 = 0.0216\%.$$

21 The Limping Ice Soccer Field Stability Tester

Authors: Olaf Parczyk, Silas Rathke (FU Berlin)
Project: EF 1-12

Illustration: Christoph Graczyk

Challenge

This year, the Women's Ice Soccer World Cup is held at the South Pole. However, ice-soccer is not played as often in "Down Under" as at the North Pole. That's why the ice-soccer

stadiums are a little outdated.
Martina, the stadium manager, experienced this first-hand: as she walked across the ice-soccer field, she collapsed immediately into the ice. After drying her feet and recovering from the shock, she decides to test the entire ice-soccer field to know whether the ice is also unstable at other spots.
To do this, she divides the field into a 4 $\times$ 9 grid as follows:

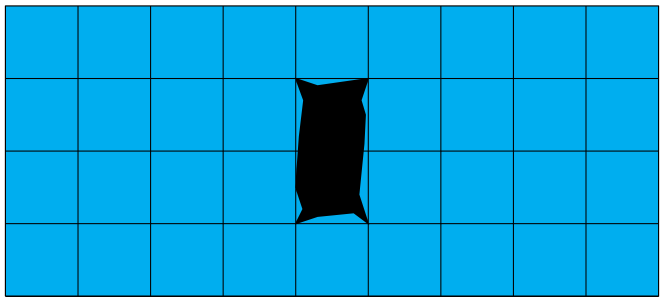

Figure 1: The 4×9 ice-soccer field with Martina's break-in point.

The two squares in the middle mark the point where she broke into the ice. She can no longer enter these two squares. However, she now wants to test all the other squares for stability.
To do this, she wants to start at any square of the soccer field and step on as many squares as possible, one after the other, without ever leaving the field. She is also not allowed to step on a square that she has already visited before. Only at the end she wants to return to the square where she started. But this is the only exception where a square can be entered twice.[1]
To make matters worse, she limps after having fallen into the ice. This means that she has to alternate between a long and a short step. With a long step, she has to walk to a square which is in chess a "knight move" away. As illustrated in Figure 2, after a long step, Martina ends up on a square that is two steps horizontally and one step vertically, or two steps vertically and one step horizontally away. [2]
With a short step, she must move to a horizontally or vertically adjacent square, as illustrated in Figure 3.
A possible sequence of steps could look as in Figure 4. In this sequence, she stepped on six squares.

[1] I don't know why she came up with these crazy rules. This story was passed from the South Pole to Germany and it may well be that some details have changed over time.

[2] At this point, the critical reader may ask whether the ice-soccer field is really so small that you can get from a square on the far left to a square on the far right with only four long steps. Yes, it is. And again, unfortunately, I don't know why that is the case. As I said, this story has been told so many times that the details may have changed...

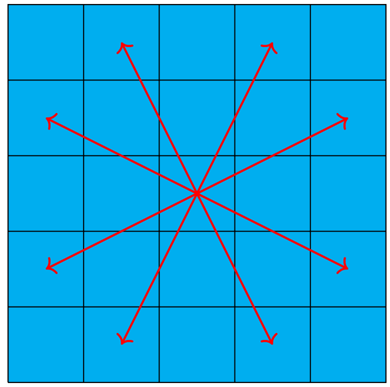

Figure 2: All possibilities for a long step starting in the central square.

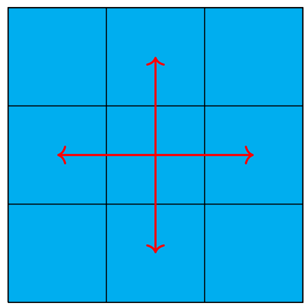

Figure 3: All possibilities for a short step starting in the central square.

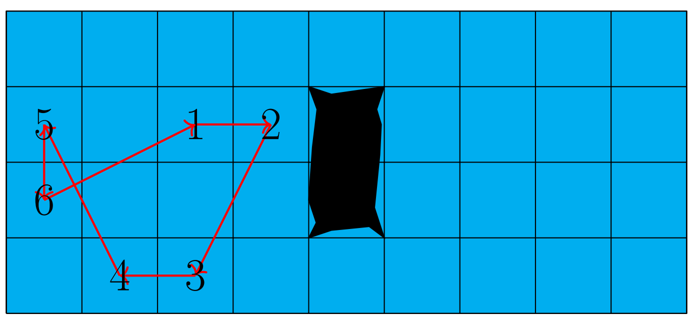

Figure 4: A possible path of Martina.

What is the largest number of squares she can visit in a one step-sequence?
All conditions summarized:

- The field has two holes that are not allowed to be stepped on.
- Martina chooses her starting field.
- She alternates between a long and a short step.
- She does not enter any square twice, except the starting square on which she ends her step-sequence.

Possible Answers:

1. 34
2. 33
3. 32
4. 31
5. 30
6. 29
7. 28
8. 27
9. 26
10. Less than 26

Project Reference:

In the project "Learning Extremal Structures in Combinatorics" we use approaches from artificial intelligence and machine learning to find new constructions for graphs with certain properties.
The presented challenge can be formulated as an extremal problem in graph theory, similar to the problems we study in this project.

Solution

The correct answer is: 3.

We first prove that Martina cannot step on 33 or more squares.
If we color the field like a chessboard, we notice that the colors of two squares, that are visited one after the other, are different for both types of steps. As we have to return to our first square at the end, this observation shows in particular that each valid sequence of steps has to visit an even number of squares. It is therefore impossible for Martina to step on exactly 33 squares.
We now have to exclude the possibility that Martina can step on all 34 squares. We show this by contradiction.

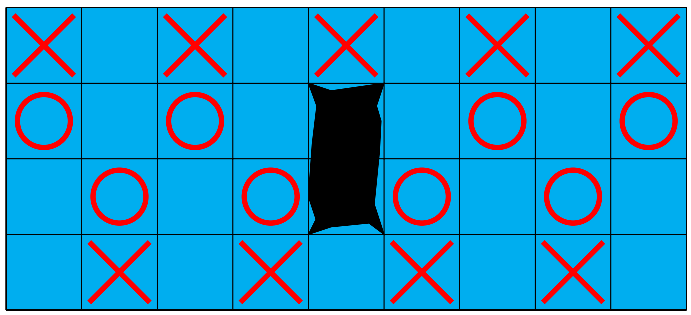

Figure 5: Martina cannot step on all X-squares.

Suppose there is a sequence of steps in which Martina visits each square exactly once. To this end, consider every second square in the first and fourth row, as marked by a red X in Figure 5. Since these squares all have the same color in the chessboard coloring, all X-squares are either reached by a long step or all X-squares are reached by a short step. In the first case, we can simply run the sequence of steps backwards: we still visit each square exactly once, but reach all X-squares with a short step now.
We can therefore assume that all X-squares are reached by a short step. However, this means that we leave an X-square by a long step. Note that like this we can only reach the squares that are marked by a circle. As there are more X-squares than circle-squares and each X-square needs its own circle-square as a successor, we can conclude that there cannot be a sequence of steps that visits each square exactly once.
Now it remains to show that there is a valid sequence that visits 32 squares. This is shown in Figure 6. [3]

[3] Before the critical and esteemed reader snorts and asks how one can come up with such a sequence of steps, here are a few hints: the proof above shows that there is at least one X-square that you cannot cover. The other X-squares must be assigned 1-to-1 to a circle-square that can be reached by a long step. Then, do the same procedure for the squares that do not yet have a circle or X. Here too, the squares in the first and fourth line must be assigned 1-to-1 to the squares in the second and third line. Once you have done this, you have already found all the long steps. Now you just have to connect these long steps with short steps to form a step-sequence. With a little trial and error, you will end up with a sequence of steps like the one shown here.

30	3	4	7	8			19	20
29	32	25	26		12	11	16	15
2	31	6	5		9	22	21	18
1	28	27	24	23	10	13	14	17

Figure 6: A possible step-sequence for Martina with 32 squares. Her path starts at 1 and is continued in ascending order.

22 Lights Out

Autor: Nikola Sadovek (FU Berlin)
Projekt: BMS und MATH+

Illustration: Vira Raichenko

Challenge

The elves have been very busy in their workshops making and packing all the gifts. Now it is time to deliver them from the workshops to Santa's house on a sleigh so that he can later distribute them to the children.

Since their work at the workshops is done, it is time for the elves to turn off the lights before closing. The lighting system in each workshop consists of $n \geq 7$ light bulbs aligned in a row, numbered from 1 to n from left to right. Each light bulb can be in one of two states: turned on or turned off.

Due to a technical error, when the switch on lightbulb k is pressed, in addition to changing the state of that bulb (turned off becomes turned on, and vice versa), the state of the three

light bulbs directly to the left of it (numbered by those indices $k-3, k-2, k-1$ which are greater than zero) and the three light bulbs directly to the right to it (numbered by those indices $k+1, k+2$ and $k+3$ which are at most n) are all changed.
For example, if the switch on light bulb $k = 2$ is pressed, only the states of the lamps $1, 2, 3, 4$ and 5 are changed. See Figure 1 for an illustration.

The elves would like to turn off all the light bulbs in every workshop using a sequence of such moves. For which values of $n \geq 7$ can the elves surely do that (i.e. for every initial state of the n light bulbs, there is such a sequence)?

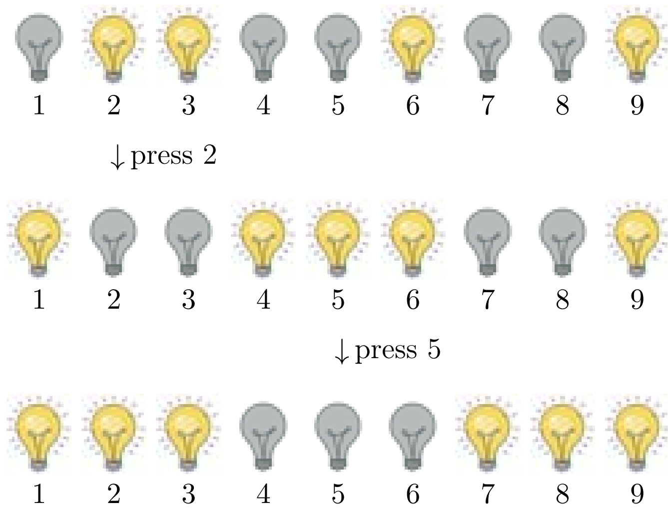

Figure 1: For $n = 9$ an initial state of n lamps in one workshop is depicted at the top row. The light bulbs $2, 3, 6$ and 9 are turned on (denoted by), while the rest are turned off (denoted by). The middle and the bottom row present the states of the lamps after pressing the switch on light bulbs 2 and 5.

Possible Answers:

1. Exactly for those values of $n \geq 7$ which have residue 2 or 3 when divided by 7.
2. Exactly for those values of $n \geq 7$ which have residue 4 or 6 when divided by 7.
3. Exactly for those values of $n \geq 7$ which have residue 0 or 2 when divided by 7.
4. Exactly for those values of $n \geq 7$ which have residue 2 or 4 when divided by 7.
5. Exactly for those values of $n \geq 7$ which have residue 1 or 4 when divided by 7.
6. Exactly for those values of $n \geq 7$ which have residue 3 or 5 when divided by 7.
7. Exactly for those values of $n \geq 7$ which have residue 1 or 3 when divided by 7.
8. Exactly for those values of $n \geq 7$ which have residue 0 or 1 when divided by 7.
9. Exactly for those values of $n \geq 7$ which have residue 0 or 3 when divided by 7.
10. Exactly for those values of $n \geq 7$ which have residue 5 or 6 when divided by 7.

Solution

The correct answer is: 8.

At the beginning, we reformulate the question. It is equivalent to asking for which values $n \geq 7$ the elves can switch off all the lights using a sequence of given moves when, at the beginning, only a single light bulb is turned on. This is indeed an equivalent question since, for any given constellation at the beginning, elves can just repeat the procedure for each light bulb which is turned on. Therefore, we will focus only on the case of a single turned-on light bulb, and we will call this process "turning off a single light bulb" for short.

Let us introduce some notation: if two numbers a and b have the same residue when divided by a number k, we will write $a \equiv_k b$. For example, a is divisible by k if and only if $a \equiv_k 0$. We split the proof in two parts.

1. If $n \equiv_7 0$ or $n \equiv_7 1$, then n is a solution. In this part we show that, if $n \equiv_7 0$ or 1, then the elves can "turn off any single light bulb".

Let us first show that if for some $n \geq 7$ they can "turn off any single light bulb" in a row of n lamps, then they can do so also in a row of $n + 7$ lamps.
Let $1 \leq k \leq n + 7$ be the bulb which is turned on. We can assume that $1 \leq k \leq n$ (otherwise we renumber the bulbs from right to left). By applying the hypothesis on lamps $1, \ldots, n$, there is a sequence of moves applied to those lamps such that after them they are all switched off.
During this procedure, some of the lamps $n + 1, n + 2$ and $n + 3$ could have changed their state. For each of them, that is turned on, the elves can apply the following procedure. If the lamp $n + i$ is turned on, for $1 \leq i \leq 3$, it can be "turned off" by changing the state of lamps $n + i + 3$ and $n + i + 4$. This does not affect the lamps $1, \ldots, n$.

Next, we will show that for $n = 7$ and $n = 8$ we can turn off any single light bulb. Together with the previous paragraph, this shows that we can "turn off a single light bulb" whenever n has residue 0 or 1 when divided by 7.

- Let $n = 7$ and $1 \leq k \leq 4$ be the index of the lamp which is turned on. If $k < 4$ it can be turned off by changing the state of the lamps $k + 3$ and $k + 4$. If $k = 4$, one can do so by changing the state of the lamps $1, 4$ and 7. See Figure 2 for an example.

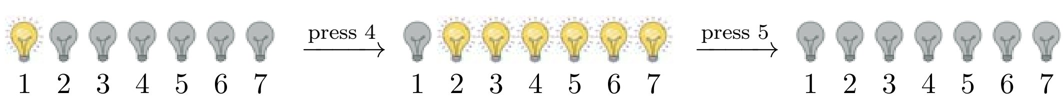

Figure 2: For $n = 7$ pressing the lamps 4 and 5 "turns off the first lamp".

- Let $n = 8$ and $1 \leq k \leq 4$ be the index of the lamp which is turned on. If $k > 1$ it can be turned off by changing the state of the lamps $k + 3$ and $k + 4$. If $k = 1$, one can do so by changing the state of the lamps $1, 5$ and 8. See Figure 3 for an example.

Figure 3: For $n = 8$ pressing the lamps 6 and 7 "turns off the third lamp".

2. If neither $n \equiv_7 0$ nor $n \equiv_7 1$, then n is not a solution. We will show that if $n \equiv_7 2, 3$ or 4, then the elves can never "turn off the first light bulb", and in the case when $n \equiv_7 5$ or 6, they can never "turn off the fourth light bulb".
For what follows, we make an observation. If the switch of some lamp is pressed twice, this has the same effect as not pressing the switch at all. Therefore, we can assume that the switch of the lamp k, for $1 \leq k \leq n$, is pressed a_k times, where $a_k \in \{0, 1\}$.
The proof method consists of looking first at lamp 1 and counting how many times its state is changed. Then, we look at lamp 2 and count also the state changes for it. By continuing further (up until lamp $n-3$), we will obtain valuable information on the numbers a_k. We will arrive at a contradiction by finally looking at some of the lamps $n-2, n-1$ and n, thus finishing the proof. We now expand this idea in details.

Let us now show that if 7 divides $n-2, n-3$ or $n-4$, we can never "turn off the first lamp".
Suppose that for some $n \geq 7$ we can "turn off the first lamp". The first lamp is effected whenever the elves press the switch on the lamps 1, 2, 3, and 4. Since the state of the first lamp needs to be changed an odd number of times, we get

$$a_1 + a_2 + a_3 + a_4 \equiv_2 1. \tag{5}$$

Looking at lamp 2, its state is switched whenever the elves press the switch of one of the lamps $1, 2, 3, 4$, or 5. Thus we get $a_1 + a_2 + a_3 + a_4 + a_5 \equiv_2 0$, thus, from (5) follows $a_5 = 1$. Similarly, looking at lamp 3 we get $a_6 = 0$, and looking at lamp 4 we get $a_7 = 0$.
Next, looking at lamp 5 we get

$$a_2 + \cdots + a_8 = a_2 + a_3 + a_4 + 1 + a_8 \equiv_2 0.$$

Hence, from (5) we get $a_8 = a_1$. Continuing this process for $k \leq n-7$ and looking at lamp $k+4$ we get $a_{k+7} = a_k$. Thus, the sequence of numbers a_k is periodic with period 7. See Figure 4 for illustration.

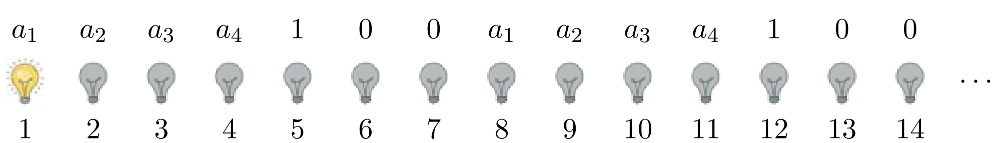

Figure 4: Periodicity with period 7 of the sequence a_k depicted above the lamps.

We now split the proof into cases.

- If $n \equiv_7 2$, by looking at lamp n, we get

$$0 \equiv_2 a_{n-3} + a_{n-2} + a_{n-1} + a_n \;=\; a_6 + a_7 + a_1 + a_2 = a_1 + a_2. \tag{6}$$

On the other hand, by looking at lamp $n-1$, we get

$$0 \equiv_2 a_{n-4} + \cdots + a_n \;=\; a_5 + a_6 + a_7 + a_1 + a_2 \;=\; 1 + a_1 + a_2,$$

which is a contradiction to (6).

- If $n \equiv_7 3$, by looking at the lamp n, we get

$$0 \equiv_2 a_{n-3} + a_{n-2} + a_{n-1} + a_n \;=\; a_7 + a_1 + a_2 + a_3 = a_1 + a_2 + a_3. \tag{7}$$

On the other hand, by looking at lamp $n-2$, we get

$$0 \equiv_2 a_{n-5} + \cdots + a_n \;=\; a_5 + a_6 + a_7 + a_1 + a_2 + a_3 \;=\; 1 + a_1 + a_2 + a_3,$$

which is a contradiction to (7).

- If $n \equiv_7 4$, by looking at the lamp n, we get

$$0 \equiv_2 a_{n-3} + a_{n-2} + a_{n-1} + a_n \;=\; a_1 + a_2 + a_3 + a_4,$$

which is a contradiction to (5).

In an analogous procedure as above, let us now show that if 7 divides $n-5$ or $n-6$, we can never "turn off the fourth lamp".
Suppose that for some $n \geq 7$ we can "turn off the fourth lamp". By looking at the first lamp we get

$$a_1 + a_2 + a_3 + a_4 \equiv_2 0. \tag{8}$$

Looking at the lamps 2 and 3 we get that

$$a_1 + \cdots + a_5 \equiv_2 0 \quad \text{and} \quad a_1 + \cdots + a_6 \equiv_2 0,$$

and thus, from (8) follows $a_5 = a_6 = 0$. By looking at lamp 4 we get

$$a_1 + \cdots + a_7 \;=\; a_1 + a_2 + a_3 + a_4 + a_7 \equiv_2 1,$$

and thus, from (8) follows $a_7 = 1$. Looking at lamp 5 we get

$$a_2 + \cdots + a_8 \;=\; a_2 + a_3 + a_4 + 1 + a_8 \equiv_2 0,$$

and thus, from (8) follows $a_8 = 1 - a_1$. Continuing this process analogously as before, we get that the sequence a_k is "almost periodic" with period 7.

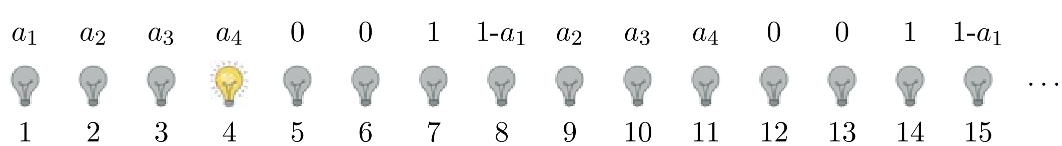

Figure 5: "Almost periodicity" with period 7 of the sequence a_k depicted above the lamps.

Namely for $1 \leq k \leq n-7$ we have

$$a_{k+7} = \begin{cases} a_k & \text{if } k \not\equiv_7 1 \\ 1 - a_1 & \text{if } k \equiv_7 1. \end{cases}$$

See Figure 5 for illustration. We finish the proof by looking at the two cases.

- If $n \equiv_7 5$, by looking at lamp $n-1$, we get

  $$0 \equiv_2 a_{n-4}+a_{n-3}+a_{n-2}+a_{n-1}+a_n \;=\; (1-a_1)+a_2+a_3+a_4+a_5 \equiv_2 1+a_1+a_2+a_3+a_4 \equiv_2 0,$$

  which is a contradiction to (8).

- If $n \equiv_7 6$ we obtain the same contradiction to (8) by looking at lamp $n-2$:

  $$\begin{aligned} 0 &\equiv_2 a_{n-5}+a_{n-4}+a_{n-3}+a_{n-2}+a_{n-1}+a_n \\ &= (1-a_1)+a_2+a_3+a_4+a_5+a_6 \equiv_2 1+a_1+a_2+a_3+a_4. \end{aligned}$$

24 Candy Presents

Author: Nikola Sadovek (FU Berlin)
Project: BMS and MATH+

Illustration: Julia Schönnagel

Challenge

Santa is currently in the process of designing gifts for kids. Each gift should be made of one or more of the $n > 0$ types of candy, denoted as $C_1, \ldots, C_n$. Some gifts can even contain all types of candys, even if some children might not like all types of candys. Santa is enthusiastic about creating unique gifts for each child and promoting the sharing of candies among them. To achieve this, Santa has asked for the help of the elves.
Specifically, the elves are tasked with designing the maximum number M of gifts, denoted as $P_1, \ldots, P_M$, which will then be distributed to M kids, with each child receiving exactly one gift. The presents are defined by which of the n candy types $C_1, \ldots, C_n$ they contain and they should satisfy the following two conditions.

(i) The uniqueness of the gifts is essential, and Santa requires that all presents be different. Two presents P_i and P_j are considered different if one of them contains a type of candy that the other one does not (although they are allowed to share some types of candy).

(ii) To encourage candy-sharing among the kids, Santa wishes that any combination of two presents should include all n candy types, $C_1, \ldots, C_n$. This way, if any two kids decide to combine their gifts, they will have access to all the candy types.

An example is illustrated in Figure 1 below.

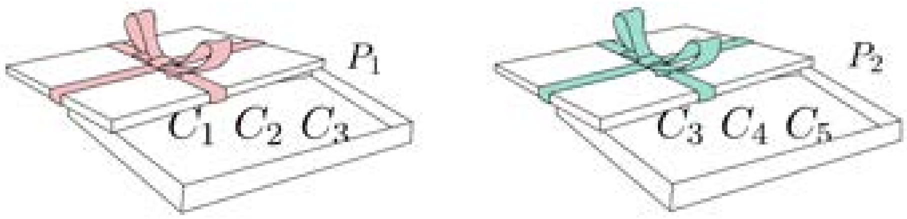

Figure 1: For $n = 5$ presents P_1 and P_2 satisfy conditions (i) and (ii).

What is the largest number M of presents $P_1, \ldots, P_M$ satisfying properties (i) and (ii) that the elves can design? Here we are considering two presents P_k and P_l as equal, if they contain the same candies, for example $P_k = \{C_1, C_2\}$ and $P_l = \{C_1, C_1, C_2\}$ are considered as the same.

Remark: An empty present, containing no candies, also counts as a present.

Possible Answers:

1. 3
2. 4
3. n
4. $n + 1$
5. $\frac{n(n-1)}{2}$
6. $\frac{n(n-1)}{2} + 1$
7. n^2
8. $n^2 + 1$
9. 2^{n-1}
10. $2^{n-1} + 1$

Solution

The correct answer is: 4. $n + 1$

To demonstrate that a configuration of $M = n + 1$ presents satisfying properties (i) and (ii) is possible, consider the following. For each $1 \leq i \leq n$ let the i'th present P_i consist of all candy types but the i'th candy type, C_i. Finally, let P_{n+1} consist of all n candy types. It can be verified that these presents are distinct, and any two of them combined contain all n candy types.
Especially, for $n = 1$ (i.e. there is only one candy type), Santa is packing one empty present and one with candy type 1, which makes $n + 1 = 2$ presents. For $n > 1$ the construction above does not need empty presents.
Now, let's show that a configuration of $n + 2$ presents satisfying properties (i) and (ii) is not possible. Suppose the contrary, and let $P_1, \ldots, P_{n+2}$ be such presents. Since they are distinct, at most one of them contains all the candy types. Equivalently, the remaining $n+1$ presents are such that each is missing at least one candy type. Without loss of generality, let us denote them as $P_1, \ldots, P_{n+1}$. Furthermore, for each k with $1 \leq k \leq n + 1$, let us define for each present P_k a set N_k that contains all numbers i with $1 \leq i \leq n$, such that the corresponding candy type C_i does not belong to P_k. In mathematical notation we write:

$$N_k \coloneqq \big\{ i \in \{1, \ldots, n\} \mid C_i \notin P_k \big\}.$$

In other words, N_k contains the indices with the information about the candy types not contained in this present. Per our assumption, all the sets $N_1, \ldots, N_{n+1}$ are non-empty. Since the presents have to be distinct to fulfill condition (i) , the sets $N_1, \ldots, N_{n+1}$ are also pairwise distinct. However, by the pigeonhole principl, as they are $n + 1$ subsets of the set $\{1, \ldots, n\}$ of n elements, the sets N_k cannot be pairwise disjoint, so there is a number $1 \leq s \leq n$ contained in two of them, $s \in N_a$ and $s \in N_b$, for some $1 \leq a < b \leq n + 1$. This contradicts condition (ii) because presents P_a and P_b do not contain the s'th candy type, C_s.

24 Interesting Working Conditions

Author: Lara Glessen
Project: MATH+ Adventskalender

Illustration: Friederike Hofmann

Challenge

Santa Claus is an interesting employer because the number of vacation days his nine elves get for the next year depends on their success in a new game he invents every year. This year he has devised the following game: He puts a hat, in either red or blue, on each of his elves' heads. He does not tell them how many red and blue hats he is giving out (so it is possible that he only distributes caps in one color). Each elf can see the hat colors of all the other elves, but not their own. The common goal of the elves is that as many of them as possible find out their own hat color, because Santa promises them all together as many days off as elves "guess" their own hat color correctly.

The rules of the game are as follows: As soon as the first elf has been put on their hat, the elves are no longer allowed to talk to each other. After Santa has distributed all the hats, he calls out one elf after the other (the order is unknown to the elves beforehand). After an elf has been called, they must immediately place themselves somewhere on the red line in the storage room. As soon as one stands on the red line, one may no longer move. When the last elf has lined up, the elves write on a piece of paper the color they believe their own hat is.
After he has explained all the rules and before he distributed the hats, he gives his elves some time to work out a strategy together on how they will position themselves on the red line. Fortunately, the elves are highly logically gifted and work out an optimal strategy. We call a strategy **A** optimal if there is no strategy in which more elves write down the correct hat color with one hundred percent certainty than with strategy **A**.
What is the largest number of elves that know certainly their hat colors with an optimal strategy?

Hint: We can assume that an elf can choose any position on the line for themselves. This means that they can squeeze between any two elves or stand next to an elf on the edge.

Possible Answers:

1. 0
2. 1
3. 2
4. 3
5. 4
6. 5
7. 6
8. 7
9. 8
10. 9

Solution

The correct answer is: 10.

We present a strategy—the elves can agree on beforehand—with which all of them end up knowing their hat color.
The first elf positions themselves anywhere on the line. For any elf k with $2 \leq k \leq 9,\ k \neq 8$ we introduce a strategy that basically tells their predecessor $k-1$ their own hat color:

1. If elf $k-1$ wears a red hat, elf k positions themselves left of the elf who is until now the furthest left.
2. If elf $k-1$ wears a blue hat, elf k positions themselves right of the elf who is until now the furthest right.

Since we also want the last elf to know their own hat color, we treat elf 8 differently:

1. If the elves 7 and 9 both wear a red hat, elf 8 positions themselves left of the elf who is until now the furthest left.
2. If elf 7 wears a red hat and elf 9 a blue one, elf 8 positions themselves right of the elf who is until now the furthest left.
3. If the elves 7 and 9 both wear a blue hat, elf 8 positions themselves right of the elf who is until now the furthest right.
4. If elf 7 wears a blue hat and elf 9 a red one, elf 8 positions themselves left of the elf who is until now the furthest right.

For an example scenario where the elves position themselves according to the above strategy, see Table 4.

Step	Elves
1	
2	
3	
4	
5	
6	
7	
8	
9	

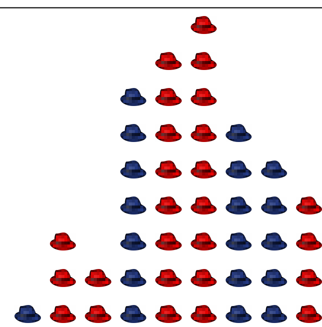

Table 4: The line-up assuming Santa chose 4 blue and 5 red hats for the elves; and called them in the order suggested by the table.

How do all nine elves know their hat color at the end? The elf k for $1 \leq k \leq 8$ can conclude their hat color by just observing whether their successor $k+1$ positions at the left or right end of the line. Elf 7 knows that if elf 8 has one of the two leftmost spots, then their own hat color must be red; and blue otherwise. Last, elf 9 observes whether elf 8 positions themselves at one of the ends of the line-up, then their hat color is the same as the hat color of elf 7. Otherwise, i.e. elf 8 squeezes between two elves, their hat color must be different to the one of elf 7. Thus, in the end all elves can know their hat color.

25 Bonus: Who lives in a Glass House...

Author: Felix Günther
Project: EF 2-1

Illustration: Ivana Martić

Challenge

Santa Claus wants to beautify his office and add a new glass roof to it. The hole in the ceiling, where the glass roof is to be placed, has the shape of a regular hexagon. Now Santa has six triangular glass plates in mind, which should be artfully assembled for the roof. It is important to him that the reflections of the roof in sunlight are uniformly beautiful and have no breaks. Therefore, the outer edges of the glass plates do not necessarily have to be parallel to the edges of the hexagonal hole; the plates can also be set at an angle. Due to the cold polar nights, Santa's ceiling is very thick.
To implement his plan, he hires the architecture firm MA&TH+, founded by the architecture elf Theresa and the mathematician elf Maryna. Maryna translates Santa's requirements

into mathematics: "So you want to arrange six triangular surfaces in the shape of a star: they have exactly one point in the middle in common, and at the edges, they fit together accordingly. For simplicity, let's call this figure a *cone*, even though it's not particularly round. The reflections of the cone in sunlight are determined solely by the position and arrangement of its side surfaces. Similarly, we can consider the *normal vector* for each of the six triangles: this is a vector (displacement arrow) of length 1 that is perpendicular to the respective surface and is outward pointing, i.e. pointing towards the sky. If we connect neighboring normal vectors on the surface of the sphere with radius 1, a spherical polygon is created, which we also call the *normal image*. The reflections of the cone in sunlight are uniformly beautiful if the normal image does not intersect itself."

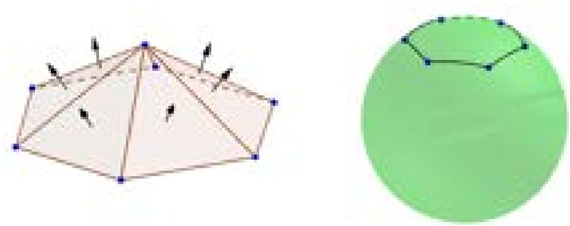

Figure 1: A pyramid as a glass roof and its normal image

Theresa suggests a regular pyramid as shown in Figure 1 as a glass roof. This construction has proven itself and precisely meets Santa's requirements – after all, the normal image is a regular spherical hexagon and, in particular, free of self-intersections. However, Santa finds this too boring. He envisions a more interesting shape, even though he cannot describe it very well. Therefore, he draws various spherical polygons that he would find exciting as normal images (see Figure 2). At first glance, however, Maryna and Theresa cannot say which of these correspond to a suitable cone.

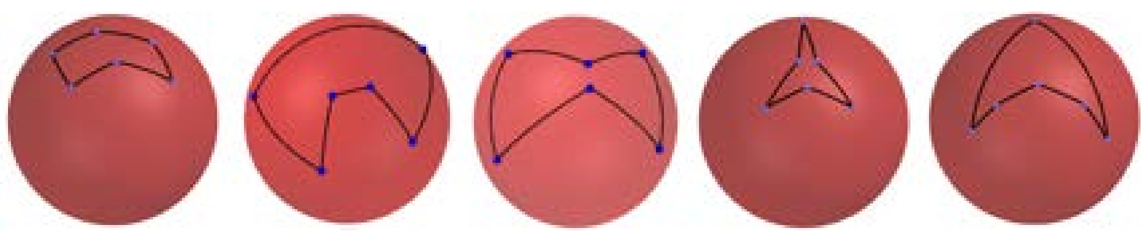

Figure 2: The spherical polygons of Santa Claus

Maryna and Theresa retreat to their office for consultation. They discuss the matter with their elf colleagues Aylin and Niveditha. Aylin remembers a recently proven theorem that could be helpful for Santa's problem. This theorem states that the oriented area F of the normal image is equal to 2π minus the sum of the interior angles of the triangles adjacent at the common vertex, where all angles are measured in radians. The orientation of the area is to be understood as follows: if we traverse the side surfaces of the cone counterclockwise, the area of its normal image is positive when we traverse the spherical polygon counterclockwise and negative when we traverse it clockwise. For example, the normal image of the regular

pyramid has a positive area.
Now Niveditha speaks up: "By the way, the (non oriented) area of a spherical polygon that does not intersect itself can be easily calculated from the angles of the polygon! For our polygon, we just have to add up the interior angles and subtract 4π from the result.
With this assistance, Maryna and Theresa enthusiastically start figuring out which spherical polygons drawn by Santa can be realized as normal images of a possible glass roof. After a while of pondering, they find out that they can relate the angles at the common vertex of the cone to the interior angles of the spherical polygon. Intrigued by their discovery, they go to their elf colleagues and explain what they have found: "If we number the side surfaces of the cone counterclockwise with f_1 to f_6, name the angles at the common vertex of the cone with α_1 to α_6 according to the names of the side surfaces, and also denote the interior angles of the spherical polygon with β_1 to β_6, then we can distinguish two cases: Either the area of the polygon is positively oriented or negatively oriented. In the first case, we can consider a triangle of the cone, e.g., f_3. The neighboring triangles f_2 and f_4 must then either be on the same side of f_3 or on different sides. If they are on the same side, then we have found that $\beta_3 = \pi - \alpha_3$ holds, and that $\beta_3 < \pi$, but if they are on different sides, then $\beta_3 = 2\pi - \alpha_3$ and $\beta_3 > \pi$ Of course, this also applies to all other triangles f_1, f_2, etc. If, on the other hand, the area of the spherical polygon is negatively oriented and the neighboring triangles of, for example, f_3 are on the same side, then $\beta_3 = \pi + \alpha_3$ holds, and $\beta_3 > \pi$ and if they are on different sides, then $\beta_3 = \alpha_3$ and $\beta_3 < \pi$."

	F positive	F negative
same side	$\beta = \pi - \alpha$, $\beta < \pi$	$\beta = \pi + \alpha$, $\beta > \pi$
different sides	$\beta = 2\pi - \alpha$, $\beta > \pi$	$\beta = \alpha$, $\beta < \pi$

Aylin and Niveditha examine Maryna and Theresa's calculations and agree with them: "Great! That means you just have to figure out which combinations of triangles with neighboring triangles on the same or different sides are possible."
Which spherical polygons drawn by Santa can be realized as normal images of a possible glass roof? Santa's ceiling is so thick that it does not impose any significant restrictions on the glass roof. (For your construction, you can assume that the ceiling is infinitely thick.)

Possible Answers:

1. Only the first.
2. Only the second.
3. Only the third.
4. Only the fourth.
5. Only the fifth.
6. Only the first, second, and third.
7. Only the first, second, and fifth.
8. Only the second and third.
9. All except the first.
10. All five.

Project Reference:

In the project EF 2-1 "Smooth Discrete Surfaces", we dealt with surfaces composed of flat polygons that behave like smooth surfaces. Our initial assumption was that the normal image around each vertex should be free of self-intersections. This *discretization* is motivated by the classical theory of smooth surfaces, where the normal image is also locally free of self-intersections, provided that the curvature at a vertex is not 0. Based on further analogies to smooth surfaces, we developed a theory of "smooth discrete surfaces." This theory is relevant for applications in architecture. For instance, architects want to design curved glass facades, but due to cost reasons, they rely on realizations using glass plates. It is quite expensive and complex to bend glass on a large scale. Therefore, they are interested in polyhedral surfaces that share many properties with smooth surfaces – among other things, the reflections in sunlight should be uniform. An algorithm based on our theory improves designs of glass surfaces in terms of their reflections.
The theorem mentioned in the task about the oriented area of the normal image also played a crucial role in our work. However, there was no general proof of this statement before – only a proof for special cases, such as the convex case from Figure 1, was known. We were eventually able to find an elementary proof, which was published in the *American Mathematical Monthly*: https://www.tandfonline.com/doi/full/10.1080/00029890.2023.2263299

Solution

The correct answer is: 8.

Thanks to Aylin, we know that for the oriented area F of the spherical polygon is

$$F = 2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6).$$

On the other hand, we know from Niveditha's hint that the area of the spherical polygon is

$$|F| = \beta_1 + \beta_2 + \ldots + \beta_6 - 4\pi.$$

Now let k be the number of triangles f_i for which the two neighboring faces are on the same side. Then for $6-k$ triangles, the two neighboring faces are on different sides.
If the spherical polygon is positively oriented, we obtain

$$\begin{aligned} 2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6) = F &= \beta_1 + \beta_2 + \ldots + \beta_6 - 4\pi \\ &= k\pi + (6-k)2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6) - 4\pi. \end{aligned}$$

The last equality holds for the following reason: Each angle β_i can be identified with $\pi - \alpha_i$ if the neighboring triangles of f_i lie on the same side and otherwise with $2\pi - \alpha_i$ if the neighboring triangles lie on different sides. Since we assume that there are k triangles whose neighboring triangles lie on the same side, we get k times the summand π and $6-k$ times the summand 2π, then we have to subtract the angle α_i, which gives us the equality

$$2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6) = k\pi + (6-k)2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6) - 4\pi.$$

If we add the angles α_i's on both sides and divide both sides by π, the result is $2 = k + (6-k)\cdot 2 - 4$, i.e. $k = 6$. This means that the neighboring faces of each triangle are on the same side. It follows that the interior angle β_i cannot have an angle greater than π (In this case, the normal image of the cone is a convex spherical polygon).
If the spherical polygon is negatively oriented, we obtain

$$\begin{aligned} 2\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_6) = F &= -(\beta_1 + \beta_2 + \ldots + \beta_6 - 4\pi) \\ &= -(k\pi + \alpha_1 + \alpha_2 + \ldots + \alpha_6 - 4\pi). \end{aligned}$$

The last equality is similar to the previous case. In particular, $2 = -k + 4$ holds, i.e. $k = 2$. This means that two triangles f_i have the neighboring faces on the same side, so the corresponding interior angle satisfy $\beta_i > \pi$. For the four remaining triangles f_j, the neighboring faces are on different sides, so the corresponding interior angle must satisfy $\beta_j < \pi$. In this case, the normal image of the cone therefore has four vertices with angles less than π and two vertices with angle greater than π.
This means that only the second and third spherical polygons of Santa Claus can be found as the normal image of a glass roof. Figure 3 shows that these can actually be realized. The fact that the glass roofs can also be inserted into the ceiling of Santa Claus is because the two normal images lie in an open hemisphere. If the sky is the north pole of the hemisphere, a glass roof could not be used unless its (outer) normal vector lies on or below the equator.

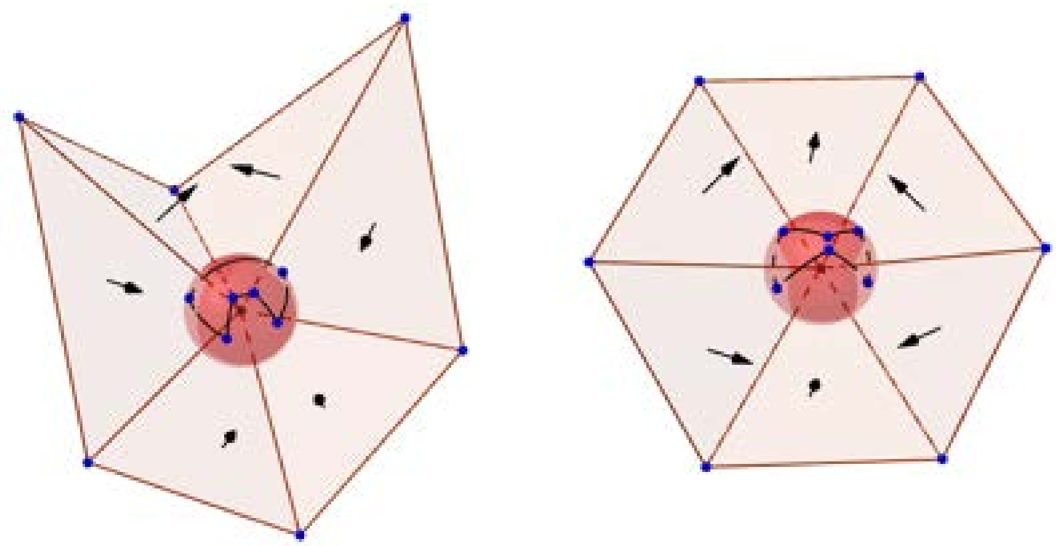

Figure 3: Glass roofs whose normal images correspond to the second and third spherical polygons of Santa Claus.