MATH+ KALENDER

Challenges & Solutions
2022

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1 Chocolate Bars

Author: N. Alexia Raharinirina (ZIB)

Project: *The Evolution of Ancient Egyptian – Quantitative and Non-Quantitative Mathematical Linguistics* (EF 5-4)

Artwork: Till Hausdorf

Challenge

To equip the elves Ra, Geb, and Bastet with enough food for their next journey to Snowtown, Santa Claus distributes chocolate bars into three bags: a red one, a green one, and a blue one. He decides to put the chocolate bars into the bags successively over several rounds, and starts by putting r bars into the red, g bars into the green, and b bars into the blue bag. Then, in each round, he puts twice as many chocolate bars into each bag as in the previous round.

After two rounds, the red bag contains 3 chocolate bars, the green one contains 6, and the blue one contains 9. In the last round, Santa puts 32 chocolate bars into the red bag, γ bars into the green one, and β bars into the blue one.

How many chocolate bars did Santa Claus put in all three bags in total?

Possible answers:

1. 189
2. 270
3. 378
4. 458
5. 570
6. 657
7. 745
8. 803
9. 926
10. 1034

Solution

The correct answer is: 3.

Let's denote the number of chocolate bars put in the red, green, and blue bag in the n^{th} round by

$$r(n), \quad g(n), \quad b(n),$$

respectively. Santa's rule for placing the chocolate bars into the bags is equivalent to the following equations

$$x(n) = 2 \cdot x(n-1), \quad \text{where } x \in \{r, g, b\} \quad \text{and } n \geq 2. \tag{1}$$

Recursively, we obtain explicit formulae for the number of chocolate bars that are put in each bag in the n^{th} round:

$$x(n) = 2 \cdot x(n-1) = 2 \cdot 2 \cdot x(n-2) = \ldots = 2^{n-1} \cdot x(1). \tag{2}$$

Hence, the total number of chocolate bars in each bag after the n^{th} round sums up to

$$\begin{aligned} X(n) &:= x(1) + x(2) + \ldots + x(n-1) + x(n) \\ &= x(1) + 2 \cdot x(1) + 4 \cdot x(1) + \ldots + 2^{n-2} \cdot x(1) + 2^{n-1} \cdot x(1) \\ &= \left(1 + 2 + 4 + \ldots + 2^{n-2} + 2^{n-1}\right) \cdot x(1) \\ &= (2^n - 1) \cdot x(1) \end{aligned} \tag{3}$$

After the second round, the total number of chocolate bars in each bag is given by

$$\begin{aligned} 3 &= R(2) = (2^2 - 1) \cdot r(1) = 3 \cdot r(1), \\ 6 &= G(2) = (2^2 - 1) \cdot g(1) = 3 \cdot g(1), \\ 9 &= B(2) = (2^2 - 1) \cdot b(1) = 3 \cdot b(1). \end{aligned}$$

Thus, we find that

$$r(1) = 1, \quad g(1) = 2, \quad b(1) = 3. \tag{4}$$

Furthermore, we know that in the last round, denoted by n_{last}, Santa put 32 chocolate bars into the red bag, i. e.

$$r(n_{last}) = 32 = 2^5 \cdot 1 = 2^{6-1} \cdot r(1).$$

Using equation (2) and (4), we get $n_{last} = 6$; which means that there were six rounds overall. By (3), the amount of chocolate bars in each bag after six rounds is

$$X(6) = (2^6 - 1) \cdot x(1) = (64 - 1) \cdot x(1) = 63 \cdot x(1).$$

Therefore, the total number of chocolate bars in all three bags is

$$R(6) + G(6) + B(6) = 63 \cdot r(1) + 63 \cdot g(1) + 63 \cdot b(1) = 63 \cdot (1 + 2 + 3) = 63 \cdot 6 = \mathbf{378}.$$

2 Arrowmatics

Authors: Hajo Broersma and Pim van 't Hof (Universiteit Twente)
Project: 4TU.AMI

Artwork: Till Hausdorf

Challenge

It is a public secret that all elves in Santa's crew are fond of archery, but that they dislike mathematics. In order to increase their motivation for mathematics, Santa Claus has invited the three elves Archy, Bowy, and Curvy for a combination of archery and mathematics.

Here is the general idea: Santa has installed a circular target. The elves get five arrows each. One after the other, the elves must use their bows to shoot all of their five arrows at the target. The target consists of a circular bull's eye and three concentric rings around it, as indicated in Figure 1.

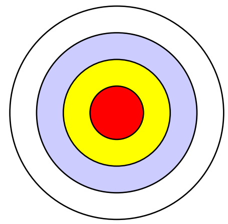


Figure 1: The circular target.

The bull's eye and every ring have each an associated fixed score value for each arrow that hits the associated target area. The elves know that these score values have been chosen from the infinite set of values

$$\{5, 10, 15, 20, 25, 30, \ldots\}$$

So each score value is a positive multiple of 5 and can be arbitrarily large. The exact score values are only known to Santa, but not to the elves. However, the elves do know the following: starting from the outer ring, the score values strictly increase when moving inside to the next ring, with the bull's eye having the highest score value of all four target areas.

After an elf has shot all five arrows at the target, Santa Claus will reveal the total score, without disclosing the scores of the individual arrows. Santa Claus challenges the elves to obtain as much information as possible on the score value of the bull's eye by carefully shooting at the target. For example, if one of the elves would be able to shoot all five arrows directly at the bull's eye, then from the total score revealed by Santa the elves could immediately deduce the exact individual score value of the bull's eye. Unfortunately, the elves are not that skilled, although all three of them manage to hit the bull's eye at least once.

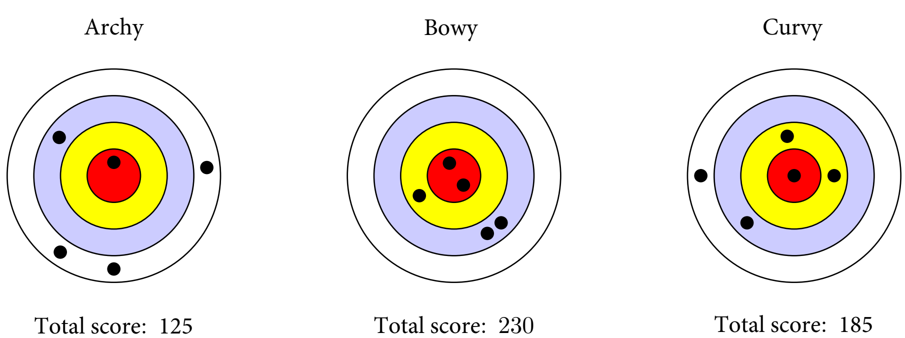


Figure 2: The spots where the arrows of Archy, Bowy, and Curvy have hit the target, and the corresponding total scores.

In Figure 2, the black dots indicate the spots where the arrows of each of the three elves have hit the target. The first target shows that Archy has hit the outer ring with three arrows, the next ring with one arrow, and the bull's eye with one arrow. Santa Claus reveals that Archy's total score is 125. From the second target, it becomes clear that Bowy is a better archer, with a total score of 230. Finally, after Curvy's five arrows have hit the target as shown on the third target, Santa reveals a total score of 185 for Curvy.

Unfortunately, with these three total scores it is not possible for the elves to determine the exact score value of the bull's eye. However, the above information limits the number of possible score values considerably.

Which of the following statements is correct?

Possible answers:

1. The number of possible score values of the bull's eye is limited to exactly 2.
2. The number of possible score values of the bull's eye is limited to exactly 3.
3. The number of possible score values of the bull's eye is limited to exactly 4.
4. The number of possible score values of the bull's eye is limited to exactly 5.
5. The number of possible score values of the bull's eye is limited to exactly 6.
6. The number of possible score values of the bull's eye is limited to exactly 7.
7. The number of possible score values of the bull's eye is limited to exactly 8.
8. The number of possible score values of the bull's eye is limited to exactly 9.
9. The number of possible score values of the bull's eye is limited to exactly 10.
10. The number of possible score values of the bull's eye is limited to exactly 11.

Solution

The correct answer is: 4.

From the assumption and the given total scores, it can be deduced in several different ways that the only five possible eligible score values of the bull's eye are: $60, 65, 70, 75, 80$. Here, we show just one specific approach.

We denote by x_1, x_2, x_3, x_4 the score values of the outer (first) ring, the second ring, the third ring, and the bull's eye, respectively. From the three subfigures and the revealed total scores, we obtain the following three equations:

$$\begin{aligned} 3x_1 + x_2 \quad\quad + x_4 &= 125 \\ 2x_2 + x_3 + 2x_4 &= 230 \\ x_1 + x_2 + 2x_3 + x_4 &= 185 \end{aligned}$$

Fiddling around with these equations, one can deduce the exact values of x_3 and x_1. One way to do this is by realising that taking 3 times the third equation and then subtracting the first and second equation, one obtains that

$$5x_3 = 555 - 125 - 230 = 200.$$

Hence, $x_3 = 40$. Substituting $x_3 = 40$ in the second equation yields:

$$2x_2 + 40 + 2x_4 = 230,$$

resulting in $x_2 + x_4 = 95$. Now, substituting $x_2 + x_4 = 95$ in the first equation yields:

$$3x_1 + 95 = 125,$$

resulting in $x_1 = 10$. Substituting $x_1 = 10$ and $x_3 = 40$ in all three equations, we get three times the same equation:

$$x_2 + x_4 = 95.$$

This indicates that we cannot determine the exact values of x_2 or x_4. The additional information we have not used yet is that all x_i are positive multiples of 5 and that $10 = x_1 < x_2 < x_3 = 40 < x_4$. This shows that x_2 can only attain the values $15, 20, 25, 30, 35$. Using $x_2 + x_4 = 95$ once more, the conclusion is that x_4 can only attain the **five values** $60, 65, 70, 75, 80$. One can easily check by substitution that all pairs $(15, 80)$, $(20, 75)$, $(25, 70)$, $(30, 65)$, $(35, 60)$ for (x_2, x_4) yield the correct total scores.

3 Chit-chat in the Workshop

Author: Tobias Paul (HU Berlin)

Project: *The Impact of Dormancy on the Evolutionary, Ecological and Pathogenic Properties of Microbial Populations* (EF 4-7)

Artwork: Friederike Hofmann

Challenge

This Christmas season, like all the years before, many presents have to be crafted by the Christmas elves. However, Santa is rather vexed, because it is hard to find qualified staff nowadays—this year there are only 12 elves to take care of the gift production. “If each of my little helpers manages to build one gift a day, we’ll still finish without much trouble,” Santa mumbles into his beard. So in the evening, he places crafting material at each workbench and instructs the twelve elves to finish their assigned gift the next day.

The next morning, the elves come to the workshop one after another. Elf Arvo arrives first and eagerly begins to work on his gift. Next, jolly elf Bjame walks in and sees Arvo already working. Since elves are gregarious chatterboxes, Bjame has a great desire to start a conversation with Arvo. But that would mean he would not be able to start and finish his own gift today—Arvo,

on the other hand, would not be distracted by the chit-chat and could continue working on his task. Being on the fence, Bjame is equally likely to decide to either go about his work or talk to Arvo.
When the third elf, Cortie, enters the workshop, he faces a similar choice: should he chat with Arvo, talk to Bjame, or do his assigned work? Regardless of Bjame's previous decision, Cortie chooses one of the three options with equal probability. (It is worth noting that elves are not at all squeamish and have no problem interrupting conversations or being interrupted).
And so it goes on and on until the twelfth and last elf, Lasse, arrives and has to decide whether he has enough discipline to concentrate on his gift or whether he would rather join one of the other eleven elves. The decision on what to do is again made with equal probability among all the other eleven elves and his gift, regardless of what the elves before him decided.

Elves who are already busy working will not be interrupted by conversations and are able to finish their gift as planned by Santa. On the other hand, no elf who decided to talk to another elf when arriving at the workshop will start working later.

Hence some elves are busy, while some forget the time chatting. In the evening, Santa Claus stops by to check on things and faces quite the chaos: not even close to as many gifts as planned got finished!

Will it still be possible to craft all the presents by Christmas Eve? To answer this question, Santa wants to calculate,

(a) how many gifts will be finished per day on average (rounded to whole gifts) and

(b) what is the probability that only one gift will be finished in one day.

Which answer is correct?

Possible answers:

1. (a) 3 and (b) $\frac{1}{24}$
2. (a) 3 and (b) $\frac{1}{12}$
3. (a) 3 and (b) $\frac{1}{6}$
4. (a) 4 and (b) $\frac{1}{24}$
5. (a) 4 and (b) $\frac{1}{12}$
6. (a) 4 and (b) $\frac{1}{6}$
7. (a) 5 and (b) $\frac{1}{24}$
8. (a) 5 and (b) $\frac{1}{12}$
9. (a) 5 and (b) $\frac{1}{6}$
10. (a) 6 and (b) $\frac{1}{24}$

Projekt reference:

In our EF 4-7 project, *The Impact of Dormancy on the Evolutionary, Ecological and Pathogenic Properties of Microbial Populations*, we are concerned, among other things, with biodiversity and genetic diversity. In this context, the gifts from the task correspond to genetic mutations in a classical model, the *Kingman coalescent*. Thus, the number of gifts in question is simply the number of (distinct) mutations in a sample of 12 individuals. This number can be determined explicitly. As is usual for mathematicians, we are interested in the asymptotic behavior of the number mutations for $n \to \infty$ in a more complex model for rare mutations, the *seed-bank coalescent*. Here n describes the size of the population under study.

Solution

The correct answer is: 2.

We number the elves Arvo, Bjame, Cortie, ... and Lasse consecutively from 1 to 12 and set $n_i = 1$ with $i \in \{1, \dots, 12\}$ if elf i chooses to work on his gift and $n_i = 0$ otherwise.

According to the task, the probability for elf i to process his assigned gift is

$$\mathbb{P}(n_i = 1) = \frac{1}{i}.$$

Thus, the elf i completes on average $\frac{1}{i}$ gifts per day. Therefore, all elves together produce

$$\sum_{i=1}^{12} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{12} = \frac{86\,021}{27\,720} \approx 3{,}1032 \approx 3.$$

Thus, per day, on average (rounded), **3** gifts are finished.

Since Arvo's gift will be finished in any case, we can translate the event "only one gift will be finished" to

$$n_1 = 1, \quad n_i = 0 \text{ für } i \geq 2;$$

i. e. Arvo's gift is finished, and all other elves decide to have a chat. We can now compute $\mathbb{P}(n_1 = 1,\ n_i = 0 \text{ for } i \geq 2)$ very easily, since the elves' decisions are independent from each other:

$$\begin{aligned}\mathbb{P}(n_1 = 1,\ n_i = 0 \text{ für } i \geq 2) &= \mathbb{P}(n_1 = 1) \cdot \prod_{i=2}^{12} \mathbb{P}(n_i = 0) \\ &= \mathbb{P}(n_1 = 1) \cdot \mathbb{P}(n_2 = 0) \cdot \ldots \cdot \mathbb{P}(n_{12} = 0),\end{aligned}$$

where $\mathbb{P}(n_i = 0)$ is the complementary probability of $\mathbb{P}(n_i = 1)$:

$$\mathbb{P}(n_i = 0) = 1 - \mathbb{P}(n_i = 1) = 1 - \frac{1}{i} = \frac{i-1}{i}.$$

Thus, we obtain

$$\begin{aligned}\mathbb{P}(n_1 = 1,\ n_i = 0 \text{ für } i \geq 2) &= \mathbb{P}(n_1 = 1) \cdot \prod_{i=2}^{12} \mathbb{P}(n_i = 0) = 1 \cdot \prod_{i=2}^{12} \frac{i-1}{i} \\ &= 1 \cdot \frac{2-1}{2} \cdot \frac{3-1}{3} \cdot \ldots \cdot \frac{11-1}{11} \cdot \frac{12-1}{12} \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{10}{11} \cdot \frac{11}{12} \\ &= \frac{1}{12}.\end{aligned}$$

Hence, the probability of only one gift being completed is $\mathbf{\frac{1}{12}}$.

4 Shipping Presents

Author: Clara Stegehuis (Universiteit Twente)
Project: European Women in Mathematics – The Netherlands (EWM-NL)

Artwork: Till Hausdorf

Challenge

Ada Lovelace was one of the first female computer scientists and is an old friend of Santa. She was also one of the first people to recognize that computers cannot only solve difficult equations and approximate numbers, but can also be used to implement algorithms. Now, Santa needs Ada's help to develop an algorithm for the following difficult task.

Santa is packaging and sending presents arranged in finite lines in his storage space. Presents can *only* be sent if they are properly packaged (P). However, for some presents the packaging material has been damaged (D).

When Santa sends out a properly packaged present, the adjacent presents change their status:

- P→D: If the present was properly packaged, then Santa—being the clumsy fella we know and love—manages to damage it.
- D→P: If the present was damaged, then Santa's little helpers—who are always at his side—repair it so that it is properly packaged afterwards.

The presents are sent one after another. Once a present is sent, it leaves a hole in the line. The presents next to the hole will not be regarded as being adjacent to one another. Yesterday's first two shippings of Santa's presents are depicted below:

	D	P	D	P	P	D	P
↪	D	P	P	–	D	D	P
↪	D	D	–	–	D	D	P

The initial line in the example consists of seven presents. In the first step, the 4$^{\text{th}}$ present (in red) is sent. Hence, the status of two adjacent packages (in blue) is changed from D to P and P to D, respectively. In the second step, the 3$^{\text{rd}}$ package (again in red) is sent. It has only one neighbour (again in blue), whose status is then changed from P to D.

Of course, Santa would like to send out all packages of any given line; that is, for a given initial line of presents, he wants the above process to end with an empty line. Although *Santa can choose the order in which he sends out the presents*, he sometimes gets stuck with only damaged presents (which he cannot send). Santa believes that his friend Ada Lovelace will be able to create an algorithm that solves his shipping problem.

For which initial lines of presents does there exist a shipping algorithm that ends with an empty line?

Possible answers:

1. For every initial line of presents that contains at least one properly packaged present.
2. For every initial line of presents that contains at least one properly packaged present, and where properly packaged presents are only adjacent to damaged packages.
3. For every initial line of presents that contains at least one properly packaged present, and where damaged presents are only adjacent to properly packaged packages.
4. For every initial line of presents with an odd number of properly packaged presents.
5. For every initial line of presents that contains at least one properly packaged present and with an odd number of damaged presents.
6. For every initial line of presents with an odd number of presents that contains at least one properly packaged present.
7. For every initial line of presents with an even number of properly packaged presents.
8. For every initial line of presents with at least three adjacent properly packaged presents.
9. For every initial line of presents with properly packaged presents at both ends of the line.
10. For every initial line of presents that contains at least one properly packaged present and with at least three adjacent damaged presents.

Project reference:

EWM-NL is the national association of women working in the field of mathematics in The Netherlands. Established in 2013, EWM-NL has several hundred members from academia, industry, and society at large. EWM-NL organises several events per year, typically open to all, offers career support grants and maintains a mentor network.

More information: https://www.ewmnetherlands.nl/home/mission/

Solution

The correct answer is: 4.

We will prove that

4. For every initial line of presents with an odd number of properly packaged presents.

is indeed true:

First, suppose that there is **only one properly packaged present**. If it is at the end of the line, sending will result in a shorter line with one properly packaged present at one end, since its damaged neighbour will be repaired. Now, Santa is able to send out this repaired package, which leads to an even shorter line of presents with one properly packaged present at one end. Successively, Santa will be able to send out all presents:

$$\begin{array}{ccccccc} & \text{P} & \text{D} & \text{D} & \text{D} & \cdots & \text{D} \\ \hookrightarrow & - & \text{P} & \text{D} & \text{D} & \cdots & \text{D} \\ \hookrightarrow & - & - & \text{P} & \text{D} & \cdots & \text{D} \\ \vdots & & & & & & \\ \hookrightarrow & - & - & - & - & \cdots & \text{P} \\ \hookrightarrow & - & - & - & - & \cdots & - \end{array}$$

If it is *not* at one of the line's ends, sending it will result in *two* new lines starting or ending with a properly packaged present. Now, Santa can treat the two lines separately and successively send out all packages as in the above case:

$$\begin{array}{cccccccccc} & \text{D} & \cdots & \text{D} & \text{D} & \text{P} & \text{D} & \text{D} & \cdots & \text{D} \\ \hookrightarrow & \text{D} & \cdots & \text{D} & \text{P} & - & \text{P} & \text{D} & \cdots & \text{D} \end{array}$$

Now, suppose there is **an odd number of properly packaged presents that is greater than 1**. Santa can send the first properly packaged present from the left. This results in two shorter lines: one to its left with only one properly packaged present at its end and one to its right. Since the line on the left can be shipped completely (as shown above), we will now focus on the right line, which still has an odd number of properly packaged presents. This number is at most equal to the number of properly packaged presents of the initial line. This can be seen using the following reasoning:

Case 1. There is a damaged present to the of right the first properly packaged present. After sending the properly packaged present, we will obtain a shorter line to its right containing the same odd number of properly packaged presents:

$$\begin{array}{cccccccl} & \text{D} & \cdots & \text{D} & \text{D} & \text{P} & \text{D} & \text{continuing line with an even \# of p. p. presents} \\ \hookrightarrow & \text{D} & \cdots & \text{D} & \text{P} & - & \text{P} & \text{continuing line with an even \# of p. p. presents} \end{array}$$

Case 2. There is a properly packaged present to the right of the first properly packaged present. After sending the properly packaged present, we will obtain a shorter line to its right containing two properly packaged presents less:

$$\begin{array}{cccccccl} & \text{D} & \cdots & \text{D} & \text{D} & \text{P} & \text{P} & \text{continuing line with an odd \# of p. p. presents} \\ \hookrightarrow & \text{D} & \cdots & \text{D} & \text{P} & - & \text{D} & \text{continuing line with an odd \# of p. p. presents} \end{array}$$

We will now treat this new line on the right as above. Gradually, we will receive shorter and shorter lines with a non-increasing, odd number of properly packaged presents. Therefore, we eventually will end up with a line containing only one properly packaged present, which can be shipped completely.

Now, we will show that, if a line contains an even number of properly packaged presents, we are never able to send it out completely:

Suppose we have **an even number of properly packaged presents**. By an analogous argument as above, shipping a properly packaged present will result in at least one shorter line with an even number of properly packaged presents. Hence, regardless of how we choose to ship the presents, we will never be left with only one properly packaged present to send out.

Finally, we are able to give two easy counterexamples to the remaining nine statements:

- The initial line of presents D P D P D provides a counterexample to answers 1, 2, 3, 5, 6, and 7.
- The initial line of presents P D D D P P P is a counterexample to answers 8, 9, and 10.

Remark: This challenge was adapted from a problem of James Tanton's book "Solve this!". More information: `www.jamestanton.com`

5 Lights On!

Author: Lotte Weedage (Universiteit Twente)
Project: 4TU.AMI

Artwork: Frauke Jansen

Challenge

In the Christmas village of Santa Claus, the elves take decorating very seriously. Every year, the Christmas village is decorated with red and green lights. To be precise, each house should be decorated with exactly two lights: *either* two red lights *or* two green lights. Decorating a house with a green light and a red light is not allowed.

Little elf Alfie is going to decorate his own house and decides to choose two red lights. He found a box in his house with 4 green and 4 red lights. Alfie knows that exactly half of the lights in the box are red and the other half are green, but there is one problem: he is color blind and cannot see the difference between the red and the green lights. Since this is not the first time Alfie encounters this problem, he is prepared and uses his Red Color Testing Machine (RCTM). The RCTM has two containers: when Alfie puts a light in each of the containers, the device makes a sound if and only if both lights are red.

Alfie tests pairs of lights in the RCTM until he hears a sound. If Alfie happens to pick two red lights in his first attempt, he hears a sound after the first test and is done. But in the worst-case scenario, more tests are needed before the RCTM makes a sound.

What is the minimum number of tests that Alfie needs to perform in order to *guarantee* that the RCTM makes a sound?

Possible answers:

1. 4
2. 5
3. 6
4. 7
5. 8
6. 9
7. 10
8. 14
9. 16
10. 28

Solution

The correct answer is: 4.

Let us call the eight lights A, B, C, D, E, F, G, and H. In the best case, we test two lights, say A and B, and they are both red. Then, we are finished after only one test. However, we are looking for the minimum number of tests we need do to be *guaranteed* that we will find two red lights (i. e. minimum number of tests in the worst-case scenario).

First, we show that in the worst-case scenario, no more than seven tests are needed.

We divide the eight lights into three groups: ABC, DEF, and GH. We start with the first group and test all lights with each other, which amounts to three comparisons: AB, BC, and AC. If we are lucky and the red-color-tester makes a sound, we are done (after three tests). Otherwise, we know that there is at most one red light in the first group ABC, so there are at least three red lights among the groups DEF and GH.

We continue with the second group of three lights DEF and test all pairs; i. e. we perform the three tests DE, EF, and DF. Again, if the red-color-tester makes a sound, we are done (and have performed at most six tests). If we are unlucky, and the red-color-tester remains silent, there is at most one red light in the group DEF. Hence, there are at least two red lights in the group GH. But then both lights in group GH must be red. Putting them in the red-color-tester will produce a sound.

So we know that seven comparisons (AB, BC, AC, DE, EF, DF, and GH) are enough *to be sure* that the red-color-tester makes a sound.

But how do we prove that seven comparisons is the *minimum* number of tests we need to do to be sure that the red-color-tester makes a sound? For this, we use a graph-theoretic approach:

We consider the eight lights in Alfie's box to be the *nodes* of a graph. When Alfie tests a pair of lights, we draw a line (an *edge*) in the graph between the two corresponding nodes. The graph in Figure 3 is an example of a situation where Alfie has performed six tests.

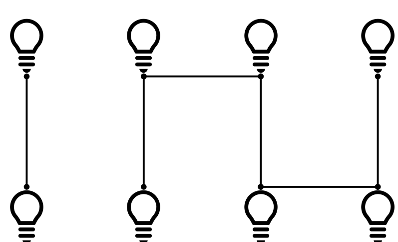


Figure 3: A graph with eight nodes, corresponding to the light bulbs, and six edges, corresponding to six tests with the red-color-tester.

Figure 4 shows that if Alfie is unlucky, none of those six tests would have produced a sound. After all, there is no edge between any of the nodes corresponding to red lights; we say that the red lights in Figure 4 form an *independent set* of size 4.

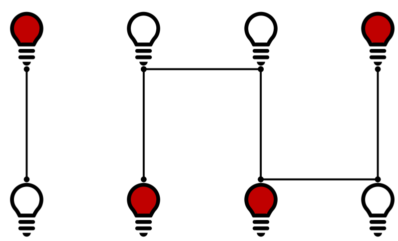


Figure 4: Since there is no edge between the four red lights, these four nodes form an *independent set*.

The question is if we could have drawn the six edges in such a way, that there is no independent set of size 4 in the resulting graph. By trying several options, you might be able to convince yourself that this is not possible. No matter how we draw six edges between the eight nodes, there will *always* be an independent set of size 4 in the resulting graph.

To formally prove this, let G be a graph with eight nodes and six edges. Let S be a *largest* independent set in G, that is, there is no independent set in G containing *more* nodes than S. If S contains at least four nodes, then we are done. Suppose S contains at most three nodes. Then, there are at least five nodes of G that are *not* in S. Denote this set of nodes by T. For each node in T, there is at least one edge between that node and the nodes in S—otherwise, adding the node to S would result in an independent set with *more* nodes than S, which is not possible by the definition of S. So there are at least five edges between S and T. Since G has exactly six edges in total, there is at most one edge connecting nodes in T. But since T contains at least five nodes, there must be four of them that form an independent set. This proves that G contains an independent set of size 4.

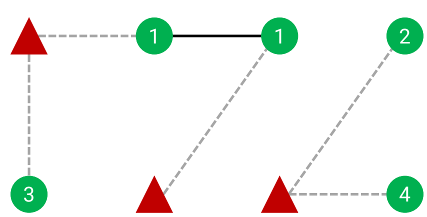


Figure 5: Examplary situation for the proof: S contains at most three nodes (red triangles), and T contains at least five nodes (green circles). All nodes in T need to have a common edge (grey, dashed) with a node from S. This amounts to at least five edges, leaving only one edge (black, solid) to connect nodes in T. Hence, there is an independent set of four nodes in T.

We conclude that six tests is not enough to guarantee that the red-color-tester makes a sound, implying that seven tests is the correct answer.

6 Wuthering Roads

Author: Marvin Lücke (ZIB)
Project: *Concentration Effects and Collective Variables in Agent-Based Systems* (EF 4-8)

Arwork: Julia Nurit Schönnagel

Challenge

To make sure all the Christmas presents will be ready on time, the diligent Christmas elf Gilfi works intently in Santa's gift factory. One evening, he totally forgets the time: "Oh, dear! Now, it's already 6:50 pm, and I promised my elf family that I'd be home in time for dinner at 7:00 pm." Gilfi quickly sets off. But as he leaves the gift factory, he is startled: a huge storm has approached the North Pole!

The little elf boldly advances at a speed of 100 meters per minute. This way, he will soon make the 500 meters to his home. But every minute, with probability $p = 0.2$, Gilfi is caught by a squall and whirled 200 meters backwards. Consequently, every minute he is *either* 100 meters closer to his destination *or* 200 meters farther away than before.

With what probability $q \in [0, 1]$ does Gilfi manage to be home in time for dinner?

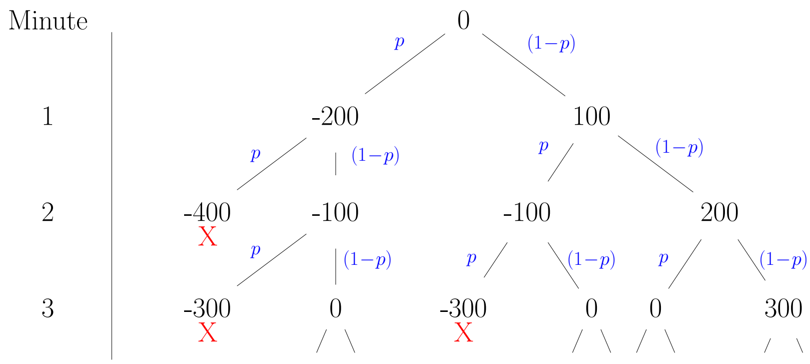
Possible answers:

1. $q < 0,1$
2. $0.1 \leq q < 0.2$
3. $0.2 \leq q < 0.3$
4. $0.3 \leq q < 0.4$
5. $0.4 \leq q < 0.5$
6. $0.5 \leq q < 0.6$
7. $0.6 \leq q < 0.7$
8. $0.7 \leq q < 0.8$
9. $0.8 \leq q < 0.9$
10. $0.9 \leq q$

Solution

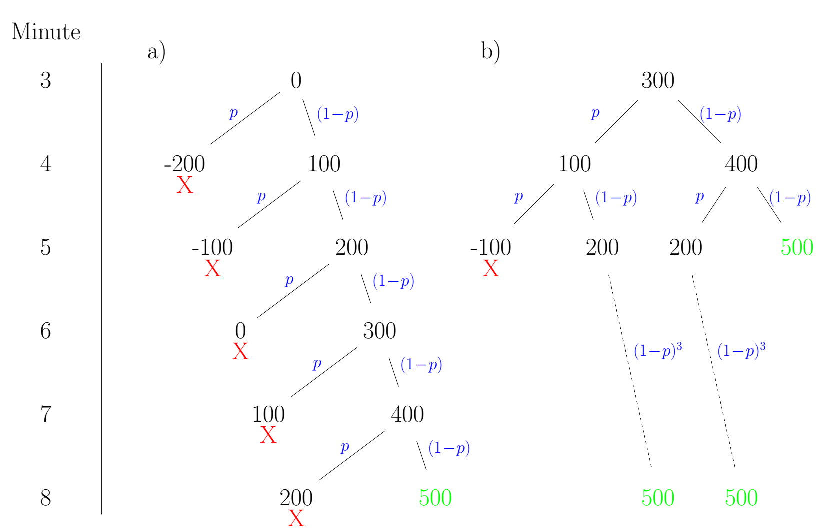
The correct answer is: 4.

The easiest way to solve the task is to write down the event tree. For example, after the first minute, the elf may have covered either 100 meters (with probability $(1-p)$) or -200 meters (with probability p):



We are interested in those scenarios where the elf makes it home in time for dinner. The big red X in the event tree marks scenarios in which the elf will not be able to be home in time. For example, if he is at -400 meters after 2 minutes, he could cover at most 800 meters in the remaining 8 minutes and end up at 400 meters—so he is not able to make in time. Therefore, we do not need to look at the branches that follow it.

After 3 minutes, the elf is at 0 meters in three scenarios. Now, instead of drawing the same subtree three times in the above picture, we look at this scenario separately in figure a) below. The other case, that the elf is at 300 meters after 3 minutes, is shown in b):



We note that there are only six scenarios, in which the elf will make it home in time. We set

$$\xi_k = \begin{cases} 1, & \text{at minute } k\text{, the elf makes -200 meters} \\ 0, & \text{at minute } k\text{, the elf makes 100 meters} \end{cases}$$

to denote these six scenarios by $(\xi_1, \xi_2, \dots) =$

- $(0,0,0,0,0)$ with probability $(1-p)^5$,
- $(0,1,0,0,0,0,0,0)$ with probability $(1-p)\cdot p\cdot(1-p)^6 = p\cdot(1-p)^7$,
- $(0,0,1,0,0,0,0,0)$ with probability $(1-p)^2\cdot p\cdot(1-p)^5 = p\cdot(1-p)^7$,
- $(1,0,0,0,0,0,0,0)$ with probability $p\cdot(1-p)^7$,
- $(0,0,0,0,1,0,0,0)$ with probability $(1-p)^4\cdot p\cdot(1-p)^3 = p\cdot(1-p)^7$,
- $(0,0,0,1,0,0,0,0)$ with probability $(1-p)^3\cdot p\cdot(1-p)^4 = p\cdot(1-p)^7$.

The total probability q with which the elf gets home in time is given by the sum of these six scenarios:

$$q = (1-p)^5 + 5\cdot p\cdot(1-p)^7 = \frac{4}{5}^5 + 5\cdot\frac{1}{5}\cdot\frac{4}{5}^7 = \frac{4}{5}^5 + \frac{4}{5}^7 = \frac{25\cdot 4^5 + 4^7}{5^7} = \frac{41{,}984}{78{,}125} \approx 0.54.$$

7 Gift Cube

Author: Rosa Schritt (HU Berlin, Immanuel-Kant-Gymnasium Berlin)
Project: Berlin Network of Schools with a Mathematics and Science Profile

Artwork: Friederike Hofmann

Challenge

Santa's elves are supposed to put 27 cube-shaped gifts in the warehouse. Some of the gifts are wrapped in green, the others in red. Santa wants his warehouse to look neat, but also pretty. Therefore, he gives the Christmas elves the following task:

"Dear elves, please put the 27 gifts together to form a big gift cube of size 3x3x3. On each of the six sides of the cube, you have to create a pattern with exactly one green and two red gifts in each row and each column! This will be the prettiest warehouse we've ever had..."

The elves ponder how to construct the gift cube. After a while, one elf remarks, "We have too many green presents to build the big cube like the one Santa asked for!"

Another elf has an idea, "If we change the pattern so that there are exactly *two* green presents in each row and each column, then we can build the big cube with the 27 presents we have." See Figure 6.

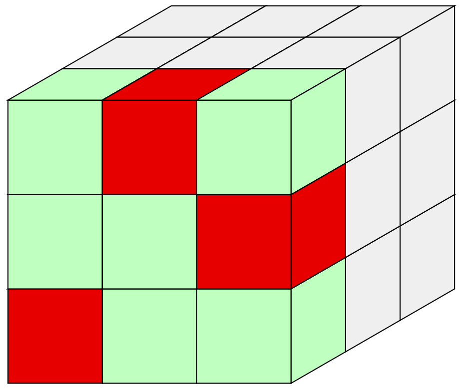


Figure 6: The big cube of 27 small square gifts and a possible admissible pattern of green and red gifts for the front side.

What is the smallest possible and the largest possible number of green gifts in the warehouse of the Christmas elves?

Possible answers:

1. The smallest possible number of green gifts is 10; the largest possible number is 13.
2. The smallest possible number of green gifts is 10; the largest possible number is 15.
3. The smallest possible number of green gifts is 12; the largest possible number is 15.
4. The smallest possible number of green gifts is 12; the largest possible number is 17.
5. The smallest possible number of green gifts is 14; the largest possible number is 17.
6. The smallest possible number of green gifts is 14, the largest possible number is 19.
7. The smallest possible number of green gifts is 16; the largest possible number is 19.
8. The smallest possible number of green gifts is 16; the largest possible number is 21.
9. The smallest possible number of green gifts is 18; the largest possible number is 21.
10. The smallest possible number of green gifts is 18; the largest possible number is 23.

Solution

The correct answer is: 7.

To show that 16 is the smallest possible and 19 the largest possible number of green cubes to build Santa's pattern, we first give an admissible pattern with 16 and 19 green cubes, respectively. Then, we show that there cannot be an admissible pattern with less than 16 or more than 19 green cubes.

We first observe that, except of congruence, there are two possible patterns for an outside face, see Figure 7.

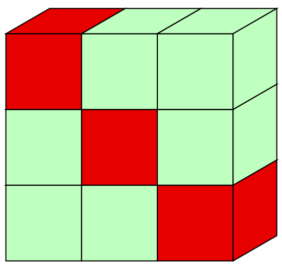


Figure 7: The two, except of congruence, allowed patterns for an outside face of the gift cube.

Using this observation it is easy to verify that, according to Figure 8, we can build an admissible gift cube that has exactly 16 green cubes.

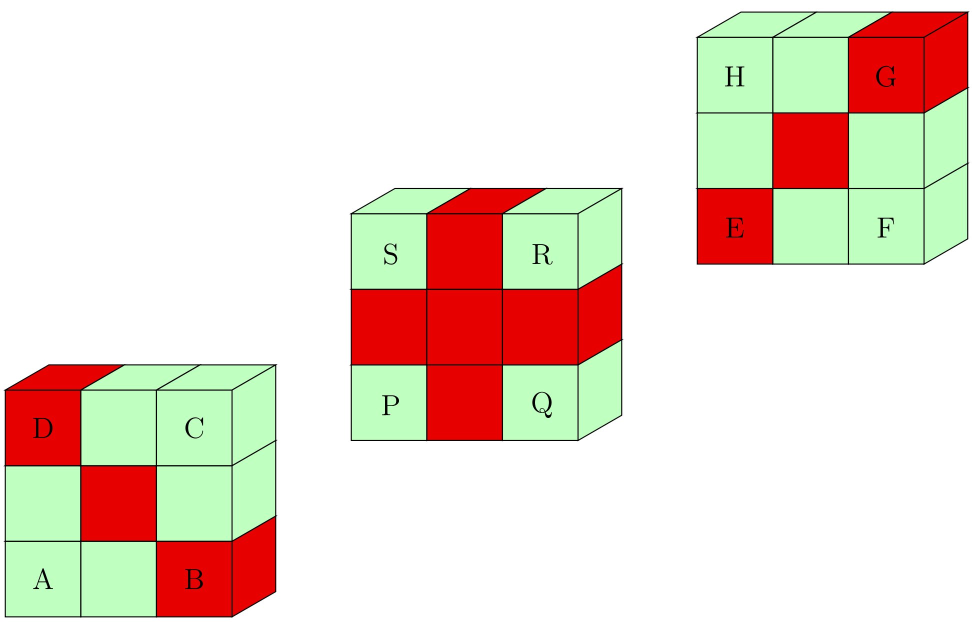


Figure 8: A possible composition of the gift cube with 16 green cubes, divided into three layers: the front layer with corner cubes A, B, C, D; the middle layer with corner cubes P, Q, R, S and the back layer with corner cubes E, F, G, H.

Similarly, Figure 9 shows that we can build a feasible gift cube with 19 green cubes.

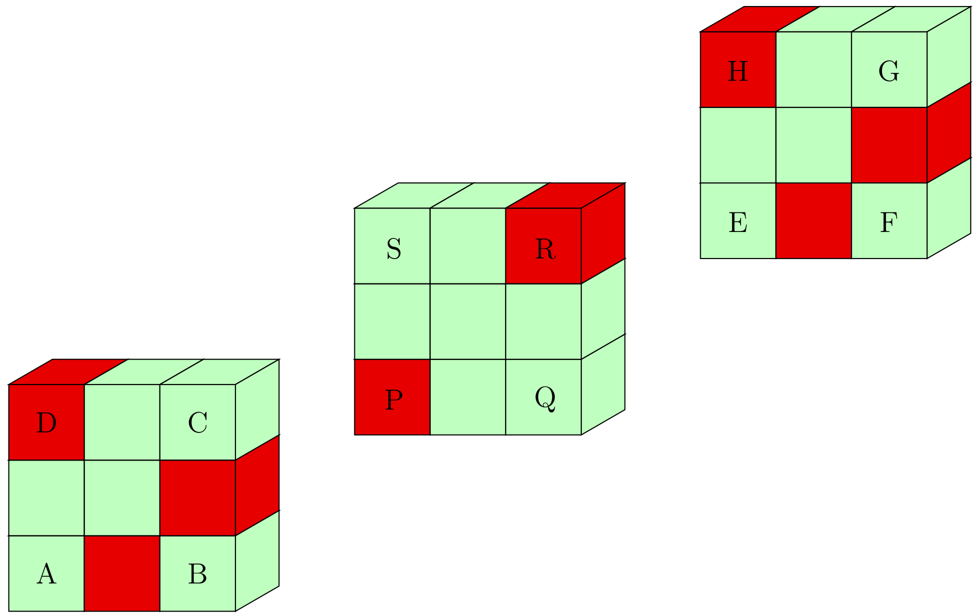


Figure 9: A possible composition of the gift cube with 19 green cubes, decomposed into three layers analogous to Figure 8.

Next, we show by contradiction that the pattern cannot be built with fewer than 16 green cubes. Thus, suppose there is a feasible composition with less than 16 green cubes. It follows immediately that this composition would have to have at least 12 red cubes. There would be exactly 3 red cubes in each of the front and back layers; in addition, the small cube containing the center of the gift cube could possibly be red. Therefore, among the 8 cubes in the square bounded by the corner cubes P, Q, R, S—cf. Figure 8—there would be at least $12-3-3-1=5$ red cubes. Among them there could be only 4 which are none of the corner cubes P, Q, R or S. Hence, at least one of the corner cubes would have to be red. Without loss of generality, we assume that P is red. At least $5-1=4$ red cubes remain. According to the prerequisites, none of them should lie on the path SPQ. This leads to a contradiction since for these at least 4 red cubes only 3 spots are free. Thus, the lower bound of 16 green cubes is proven.

Finally, analogous to the previous proof, we show that we cannot build Santa's pattern with more than 19 green cubes. Thus, suppose there is such a composition with more than 19 green cubes, and hence with less than 8 red cubes. In the front as well as in the back layer there must be 3 red cubes each. Thus, at most 1 red cube would remain for the middle layer. According to the prerequisites, the row with the corner cubes P, Q, as well as the row with the corner cubes R, S, need 1 red cube. Since the two rows are disjoint, 1 red cube is not enough to satisfy this condition; thus the upper bound of 19 green cubes is also proven.

8 The Canal

Author: Attila Karsai (TU Berlin)

Project: *Structured optimal control of port-Hamiltonian network models* (CRC/TRR 154, Project B03)

Artwork: Julia Nurit Schönnagel

Challenge

Santa's little helper Ruprecht was given an important task this year. He is responsible for supplying the cities A, B, and C with presents. Fig. 10 shows the arrangement of the cities, which all lie on a straight line heading east. The cities A and B are 5 kilometers apart, the cities B and C even 15 kilometers.

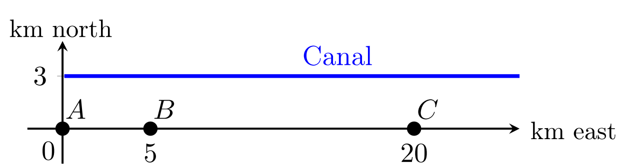


Figure 10: Arrangement of the cities and the frozen canal.

Ruprecht just finished supplying city A with presents and wants to impress Santa by delivering the cities B and C as quickly as possible. Unfortunately, there was heavy snowfall last night, and the streets connecting the cities are buried under a thick layer of snow. Therefore, Ruprecht's progress is much slower than before. However, from earlier visits to this region, Ruprecht remembers that 3 kilometers north of A there is a canal that runs straight east. Fig. 10 also shows this canal. Because of the freezing temperatures, the canal is frozen right now, and Ruprecht can ice-skate on the surface of the canal.

When Ruprecht trudges through the thick layer of snow, he manages to cover 5 kilometers per hour. On the frozen canal, with the help of his skates, he proceeds at 25 kilometers per hour.

We have the following questions:

(a) Does Ruprecht use the frozen canal on the *fastest* way from A to B?

(b) Does Ruprecht use the frozen canal on the *fastest* way from B to C?

(c) Does Ruprecht need more than 1 hour for the *fastest* way from A to B?

(d) Does Ruprecht need less than 1.8 hours for the *fastest* way from B to C?

Possible answers:

1. (a) no, (b) no, (c) no, (d) no.
2. (a) no, (b) no, (c) no, (d) yes.
3. (a) no, (b) no, (c) yes, (d) no.
4. (a) no, (b) no, (c) yes, (d) yes.
5. (a) no, (b) yes, (c) no, (d) no.
6. (a) no, (b) yes, (c) no, (d) yes.
7. (a) no, (b) yes, (c) yes, (d) no.
8. (a) no, (b) yes, (c) yes, (d) yes.
9. (a) yes, (b) yes, (c) no, (d) no.
10. (a) yes, (b) yes, (c) no, (d) yes.

Project reference:

A branch of mathematical optimisation is concerned with *optimal control problems.* Such control problems arise in a variety of real-world applications, ranging from economics over robotics to the control of the power grid of a whole country. In many cases, it can be observed that the optimal control requires a detour to reduce costs. This phenomenon is called *turnpike phenomenon.* The name is reminiscent of an observation from everyday life: when driving a long distance, it is almost always quicker to take a detour via a freeway (turnpike) than to drive slowly on the country road all the time.

Solution

The correct answer is: 6.

Generally, there are two options for the fastest way: either the straight path is the fastest, or there is a faster path which partly runs along the canal. There cannot be an even faster way which is neither straight nor temporarily runs along the canal.
Thus, for a given straight eastward route of length ℓ kilometers, we consider:

- the time $z_{\text{direct}}(\ell)$ required if the direct route is used, and
- the time $z_{\text{canal}}(\ell)$ required for the fastest path that temporarily runs along the canal.

The formula for $z_{\text{direct}}(\ell)$ is easy to determine, since the time required can be calculated by dividing the length by the velocity. Since Ruprecht has to trudge arduously through the snow when walking directly, his speed is 5 kilometers per hour. Therefore,

$$z_{\text{direct}}(\ell) = \frac{\ell}{5} \text{ hours.}$$

To obtain an expression for $z_{\text{canal}}(\ell)$, we have to work a little harder: first, we can assume that a path that temporarily runs along the canal has a shape like that in Figure 11, that is, symmetrical with regards to the center of the path.

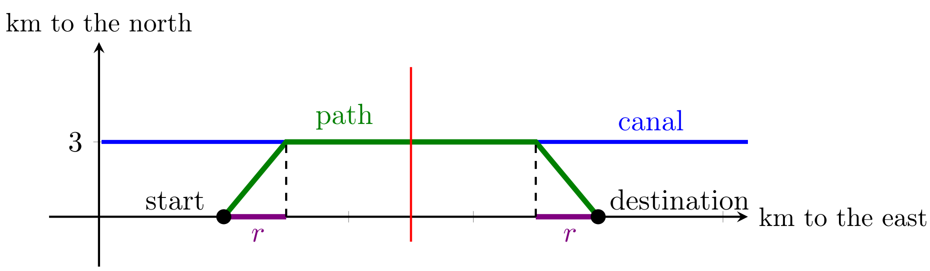


Figure 11: Shape of a path that temporarily uses the canal.

The length of the path only depends on the distance r, and we can calculate it by using the Pythagorean theorem. On the part of the path that lies on the hypotenuse of the triangle, Ruprecht can only make 5 kilometers per hour. This part (of the half way) has a length of $\sqrt{r^2+3^2}$ kilometers, which can be travelled in $(\sqrt{r^2+3^2})/5$ hours. The remaining part (of the half way) has length $\frac{\ell}{2} - r$. On this part, Ruprecht travels with a velocity 25 kilometers per hour; so he needs $(\frac{\ell}{2} - r)/25$ hours for the distance. Thus, in total, the time needed sliding on the canal at a given length r is

$$z_{\text{canal}}(\ell, r) = 2\left(\frac{\sqrt{r^2+3^2}}{5} + \frac{\frac{\ell}{2} - r}{25}\right) \text{ hours.}$$

This time still depends on r. If Ruprecht takes a path along the canal, he will naturally choose the r that gives him the shortest travel time. Consequently,

$$z_{\text{canal}}(\ell) = \min_{r \geq 0} \; z_{\text{canal}}(\ell, r).$$

In the task, the lengths $\ell_1 = 5$ and $\ell_2 = 15$ are given. Now, we want to decide whether

$$z_{\text{canal}}(\ell_i) < z_{\text{direct}}(\ell_i)$$

is true for $i = 1$ or $i = 2$.

For a distance of 5 kilometers, we have

$$z_{\text{direct}}(\ell_1) = \frac{5}{5} \text{ hours} = 1 \text{ hour}.$$

For the detour along the canal, Ruprecht would have to trudge

$$\sqrt{r^2 + 3^2} \geq 3 \text{ kilometers}$$

through the deep snow to reach the canal at all, and then another at least 3 kilometers for the way back. But for these at least 6 kilometers he needs already at least $6/5 = 1{,}25$ hours—so in any case more than he needs on the direct way. Thus, it follows that

$$z_{\text{canal}}(\ell_1) > 1 \text{ hour} = z_{\text{direct}}(\ell_1).$$

Hence, the direct path from A to B is the shorter one (time-wise). This path takes one hour. Hence, the answers to questions (a) and (c) are no.

For a distance of 15 kilometers, we have

$$\begin{aligned} z_{\text{direct}}(\ell_2) &= \frac{15}{5} \text{ hours} = 3 \text{ hours} \\ z_{\text{canal}}(\ell_2) &= \min_{r \geq 0}\ 2\left(\frac{\sqrt{r^2+3^2}}{5} + \frac{\frac{15}{2} - r}{25}\right) \text{ hours.} \end{aligned}$$

We first show that the function

$$f(r) = 2\left(\frac{\sqrt{r^2+3^2}}{5} + \frac{\frac{15}{2} - r}{25}\right)$$

is bounded from *above* by a function $F(r)$ whose minimum we can calculate very easily. To do this, we first note that the first binomial formula gives us

$$\sqrt{r^2+3^2} \leq \sqrt{r^2} + \sqrt{3^2} = r + 3,$$

since $r > 0$ holds. Thus, for all $r > 0$, we obtain

$$\begin{aligned} f(r) = 2\left(\frac{\sqrt{r^2+3^2}}{5} + \frac{\frac{15}{2} - r}{25}\right) &\leq 2\left(\frac{r+3}{5} + \frac{\frac{15}{2} - r}{25}\right) \\ &= \tfrac{2}{15}\left(5r + 15 + \tfrac{15}{2} - r\right) = \tfrac{2}{15}\left(4r + \tfrac{45}{2}\right) \\ &= \tfrac{8}{25}\, r + \tfrac{9}{5} =: F(r). \end{aligned}$$

Since we have shown that $f(r) \leq F(r)$ holds for all $r > 0$,

$$\min_{r \geq 0} f(r) \leq \min_{r \geq 0} F(r) = F(0) = \frac{9}{5} \text{ hours} = 1{,}8 \text{ hours}$$

is true as well. Consequently,

$$z_{\text{canal}}(\ell_2) \leq 1{,}8 \text{ hours} \leq 3 \text{ hours} = z_{\text{direct}}(\ell_2).$$

Thus, the time-wise shortest path from B to C runs along the frozen canal. This path takes less than 1.8 hours. Thus, the answers to questions (b) and (d) are yes.

9 Wish List Optimization

Authors: Max Klimm, Martin Knaack (TU Berlin)

Project: *Combinatorial network flow methods for instationary gas flows and gas market problems* (CRC/TRR 154, Project A07)

Artwork: Friederike Hofmann

Challenge

Before Christmas, all children send Santa their wish list of presents they would like to receive.

Santa has prepared a bag with gifts for each child,whose size depends on how good the child has been in the past year. While filling the bag, Santa works through each child's wish list from top to bottom. More precisely:

- Santa tries to pack the first item on the list first, then the second, and so on until the end of the list.
- Whenever an item fits in the bag, it goes in the bag; otherwise, it is not packed and Santa continues with the next item on the list.

- A set of items fits in the bag if and only if the sum of their sizes is less than or equal to the size of the bag.

Nasti, Manu, Jona, and Uli want the following items for Christmas this year, each of them bringing them a given joy:

	Size	Joy
Warm socks	2	4
Candle	4	5
Bobble hat	6	8
Flute	24	20
Woolen sweater	16	10

The children know the size of each item and the joy they bring, but they do *not* know how big the bag will be that Santa chooses for them. Of course, they want to maximize the total joy of the gifts in their bag. However, they have used different strategies to write their wish list. The children's wish lists are the following:

Nasti's wish list

1. Flute
2. Woolen sweater
3. Bobble hat
4. Candle
5. Warm socks

Manu's wish list

1. Warm socks
2. Candle
3. Bobble hat
4. Woolen sweater
5. Flute

Jona's wish list

1. Warm socks
2. Bobble hat
3. Candle
4. Flute
5. Woolen sweater

Uli's wish list

1. Flute
2. Bobble hat
3. Candle
4. Warm socks
5. Woolen sweater

Santa revealed to us that all the children will get a gift bag of the same size and that this size is a positive integer. Which of the following statements is correct?

Possible answers:

1. The larger the gift bag, the bigger the total joy of Nasti's gifts.
2. The total joy of Nasti's gifts is always at least as big as Manu's.
3. The total joy of Nasti's gifts is always at least as big as Uli's.
4. The total joy of Manu's gifts is always at least as big as Jona's.
5. The total joy of Manu's gifts is always at least as big as Uli's.
6. The total joy of Jona's gifts is always at least as big as Nasti's.
7. The total joy of Jona's gifts is always at least as big as Manu's.
8. The total joy of Uli's gifts is always at least as big as Nasti's.
9. The total joy of Uli's gifts is always at least as big as Jona's.
10. None of the statements 1 to 9 is correct.

Project reference

The project deals with capacity utilization in hydrogen networks. The above items then correspond to bookings in the network and their joy to the respective economic benefits of the hydrogen transport. Due to the non-linear gas physics and the complexity of the hydrogen network, the conditions on which items can be packed are much more complicated than in this task. How the items can be packed in a way that in addition the total pleasure is maximized is the subject of research in this project.

Solution

The correct answer is: 7.

The following table shows the joy of Nasti, Manu, Jona and Uli depending on the size of the gift bag:

Size	Nasti	Manu	Jona	Uli
0 - 1	0	0	0	0
2 - 3	4	4	4	4
4 - 5	5	4	4	5
6 - 7	8	9	9	8
8 - 9	12	9	12	12
10 - 11	13	9	12	13
12 - 13	17	17	17	17
14 - 15	17	17	17	17
16 - 17	10	17	17	17
18 - 19	14	17	17	17
20 - 21	15	17	17	17
22 - 23	18	17	17	17
24 - 25	20	17	17	20
26 - 27	24	17	17	24
28 - 29	25	27	27	25
30 - 31	28	27	27	28
32 - 33	32	27	27	32
34 - 35	33	27	27	33
36 - 37	37	27	37	37
38 - 39	37	27	37	37
40 - 41	30	27	37	37
42 - 43	34	27	37	37
44 - 45	35	27	37	37
46 - 47	38	27	37	37
48 - 49	42	27	37	37
50 - 51	43	27	37	37
52 - ∞	47	47	47	47

Using these values, we can now check the statements:

1. is false: at size 14, Nasti's joy is 17 and at size 16 Nasti's joy is 10.
2. is false: at size 6, Nasti's joy is 8 and Manu's joy is 9.
3. is false: at size 16, Nasti's joy is 10 and Uli's joy is 17.
4. is false: at size 8, Manu's joy is 9 and Jona's joy is 12.
5. is false: at size 4, Manu's joy is 4 and Uli's joy is 5.
6. is false: at size 4, Jona's joy is 4 and Nasti's joy is 5.

7. is correct. Jona and Manu always have the same presents in the bag, except for the following sizes:

   From 8 to 11 (integers only), Manu has warm socks and the bobble hat (joy 9) and Jona has the candle and the bobble hat (joy 12) in the bag.

   From 36 to 51 (integers only), Manu has the wool sweater and the three small gifts (joy 27) and Jona has the flute and the three small gifts (joy 37) in the bag.

8. is false: at size 22, Uli's joy is 17 and Nasti's joy is 18.

9. is false: at size 6, Uli's joy is 8 and Jona's joy is 9.

10. is wrong, since statement 7 is correct.

10 Elections at the North Pole

Author: Emil Junker (HU Berlin)

Artwork: Frauke Jansen

Challenge

The ten elves Arne, Bea, Coco, Dante, Enzo, Fina, Greta, Henri, Ida, and Joris work in the gift wrapping department of the Christmas administration office at the North Pole.

Every year before Christmas, from among the employees of the gift wrapping department, the so-called wrapping committee is formed. The wrapping committee is responsible for overseeing the proper wrapping and decoration of all Christmas gifts. On the one hand, it is a great honour for the elves to be part of the wrapping committee. After all, they get to decide which colours and patterns to use for the wrapping paper. On the other hand, working on the wrapping committee also involves a lot of stress and overtime, and not everyone is up for this task. Hence, the election of the members of the wrapping committee has to follow a particular procedure:

The ten elves in the department are each given a ballot on which they write the names of all the elves they think should be on the committee. Everyone is allowed to write as many names as they wish on their ballots—each name, however, not more than once. It is also allowed to write one's own name on the ballot if one wants to be a member of the committee.

After all elves have filled out their ballots, the public count begins for which the following rules apply:

1. Elves who write their own name on their ballot will be elected to the committee if and only if they are proposed by *at least three other elves.*
2. Elves who do not write their own name on their ballot will be elected to the committee if and only if they are proposed by *at least five other elves.*

It is a well-known secret that Santa does not approve of this election process. He would prefer to choose the wrapping committee himself, but it is mandatory for the elves to vote on it themselves. Nonetheless, Santa wants to make sure that there are at least a few capable elves on the wrapping committee. Thus, one day before the election date, he asks all ten department elves who they will vote for and receives the following answers:

(a) Arne would like the committee to consist of himself, Dante, Greta, and Joris.

(b) Bea says she will propose only herself as a member.

(c) Coco plans to elect Arne, Bea, and Joris as members of the committee.

(d) Dante thinks that the committee should consist of Coco, Enzo, and Greta.

(e) Enzo says he will vote for himself, Fina, Henri, and Joris.

(f) Fina thinks Dante, Greta, and Joris should establish the committee.

(g) Greta wants the committee consist of herself, Coco, Enzo, and Henri.

(h) Henri plans to suggest himself, Dante, Greta, and Joris for the committee.

(i) Ida thinks that only herself, Coco, and Joris are suitable members.

(j) Joris plans to choose Bea, Enzo, Henri, and Ida for the committee.

"Oh no," thinks Santa Claus after he gets the answers. "As it seems, there will be far too few competent elves on the committee. This will jeopardize the schedule of the gift deliveries! I'm going to have to intervene!"

Fortunately, Santa knows his employees very well and, in particular, how to manipulate them: he will bribe them with delicious nut cookies from the Christmas bakery. Elves who have been bribed will each vote exactly as Santa tells them to. Elves who have *not* been bribed will vote exactly as they have announced, i. e. as stated in the above list (a)–(j).

Santa wants the packing committee to consist of the elves Arne, Coco, Dante, Enzo, and Henri. In addition, Santa does not want the elves Bea, Fina, Greta, Ida, and Joris on the committee.

What is the minimum number of elves that Santa *needs to* bribe to achieve this goal?

Possible answers:

1. The minimum number of elves that Santa needs to bribe is 1.
2. The minimum number of elves that Santa needs to bribe is 2.
3. The minimum number of elves that Santa needs to bribe is 3.
4. The minimum number of elves that Santa needs to bribe is 4.
5. The minimum number of elves that Santa needs to bribe is 5.
6. The minimum number of elves that Santa needs to bribe is 6.
7. The minimum number of elves that Santa needs to bribe is 7.
8. The minimum number of elves that Santa needs to bribe is 8.
9. Santa doesn't have to bribe anyone. If all the elves vote as announced, the wrapping committee already consists of the members Santa wants.
10. Santa's goal is impossible to reach, no matter how many elves he bribes.

Project reference:

In the context of his master thesis, Emil Junker deals with so-called *group identification problems.* There, one considers a set of individuals (in our example the elves) and wants to determine which subset of them is "qualified" in a certain way. Each individual in the set has an opinion about which individuals are qualified and which are not. Subsequently, there are several different rules that one can use to evaluate. Specifically, Emil Junker is concerned with the manipulability of group identifications through bribery- or control-based attacks. Here, an outside individual (Santa) tries to achieve a certain goal; for example, that at the end of the evaluation certain individuals are classified as qualified. They may try to achieve this goal, for example, by bribing individuals or by adding additional eligible individuals. Of particular interest to Emil Junker is the feasibility and computational complexity of such attacks.

Solution

The correct answer is: 5.

Let us first analyse the initial situation. For simplicity, we use the following notation:

- When an elf X writes elf Y's name on his note, we say *X qualifies Y*.
- When an elf X does *not* write elf Y's name on his note, we say *X disqualifies Y*.

According to rules (1) and (2), an elf who qualifies himself needs three more qualifications from other elves to be elected. An elf who disqualifies himself needs five qualifications from other elves to be elected.

If Santa does not bribe anyone, the situation is as follows (see Fig. 12):

- Arne (A) qualifies himself and receives only one qualification from Coco. Arne is therefore not a member of the committee.
- Bea (B) qualifies herself and receives only two qualifications from Coco and Joris. Bea is therefore not a member of the committee.
- Coco (C) disqualifies herself and receives three qualifications from Dante, Greta, and Ida. Coco is therefore not a member of the committee.
- Dante (D) disqualifies himself and receives three qualifications from Arne, Fina, and Henri. Dante is therefore not a member of the committee.
- Enzo (E) qualifies himself and receives three qualifications from Dante, Greta, and Joris. Enzo is therefore an elected member of the committee.
- Fina (F) disqualifies herself and receives one qualification from Enzo. Fina is therefore not a member of the committee.
- Greta (G) qualifies herself and receives four qualifications from Arne, Dante, Fina, and Henri. Fina is therefore an elected member of the committee.
- Henri (H) qualifies himself and receives three qualifications from Enzo, Greta, and Joris. Henri is therefore an elected member of the committee.
- Ida (I) qualifies herself and receives one qualification from Joris. Ida is therefore not a member of the committee.
- Joris (J) disqualifies himself and receives six qualifications from Arne, Coco, Enzo, Fina, Henri, and Ida. Joris is therefore an elected member of the committee.

Thus, if Santa does not intervene, the committee will consist of Enzo, Greta, Henri, and Joris. Now, Santa has to make sure that Arne, Coco, and Dante are elected into the committee and that Greta and Joris are not elected into the committee.
For an elf X, in order to be elected into the committee, enough other elves have to be bribed to qualify X (of course, only elves who have not yet qualified X are eligible for this), or X can be bribed to qualify himself (if that is not already the case).

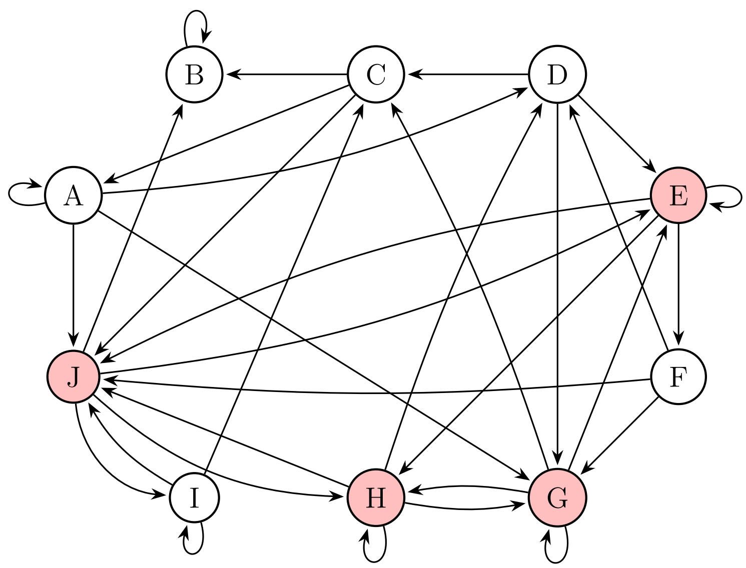


Figure 12: Overview of the initial situation. An arrow from an elf X to an elf Y means: X qualifies Y. Elves whose fields are highlighted in light red are in the committee according to the rules.

For an elf X, in order not to be elected into the committee, enough other elves must be bribed to disqualify X (of course, only elves who have previously qualified X are eligible for this), or X has to be bribed to disqualify himself (if that is not already the case).
This leads to the following observation (see Fig. 13):

- For Arne to become a member of the committee, two other elves must be bribed to qualify Arne.
- For Coco to become a member of the committee, two other elves must be bribed to qualify Coco, or Coco must be bribed to qualify herself.
- For Dante to become a member of the committee, two other elves must be bribed to qualify Dante, or Dante must be bribed to qualify himself.
- In order to prevent that Greta is elected into the committee, two other elves must be bribed to disqualify Greta, or Greta must be bribed to disqualify herself.
- In order to prevent that Joris is elected into the committee, two other elves must be bribed to disqualify Joris.

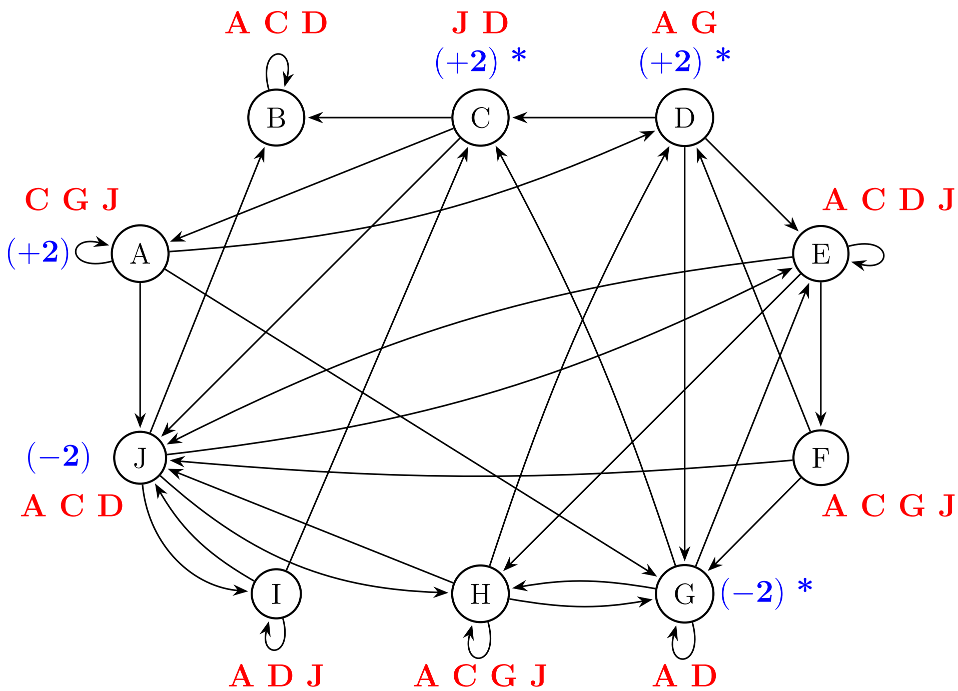


Figure 13: The number in blue indicates in each case how many additional qualifications (positive) or disqualifications (negative) Santa must get an elf in order to achieve his goal. A * means that Santa can alternatively bribe the elf himself. Listed in red are the elves that Santa could provide an additional (dis)qualification if he bribed the corresponding elf.

There are several ways Santa can fulfil the above conditions by bribing three elves. We give one here (see Fig. 14):

Step 1: Santa bribes Coco and makes sure Coco qualifies themself (so Coco is now in the committee). He also makes sure Coco disqualifies Joris (so Joris now only has five qualifications), and that she qualifies Dante (so Dante now has four qualifications).

Step 2: Santa bribes Greta and makes sure Greta disqualifies themself (so Greta is now no longer in the committee). He also makes sure Greta qualifies Arne and Dante (Arne now has two qualifications, in addition to their own; and Dante has five qualifications now, so Dante is in the committee).

Step 3: Santa bribes Ida and makes sure Ida qualifies Arne (so Arne now has three qualifications, in addition to their own, and thus, is in the committee). He also makes sure Ida disqualifies Joris (so Joris now only has four qualifications and is no longer in the committee).

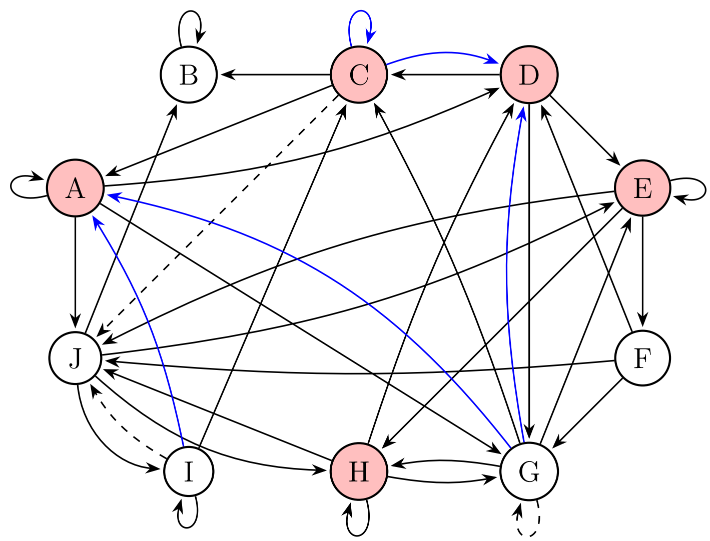


Figure 14: The situation after Coco, Greta, and Ida were bribed. The blue arrows are new, the dashed arrows are gone.

Now, we show that it is not enough to bribe only two elves. Let us assume that there is a way to fulfil the above conditions by bribing only two elves. We distinguish four cases, and lead each of them to a contradiction:

Case 1: One of the two bribed elves is Greta.

Since Joris is already disqualified by Greta in the initial situation, we cannot give Joris any additional disqualification by bribing Greta. Thus, Joris still needs two more disqualifications. However, since we have already bribed Greta, we can only bribe one more elf. Consequently, it is impossible to get Joris the two disqualifications he needs.

Case 2: One of the two bribed elves is Dante.

Since Joris is already disqualified by Dante in the initial situation, Joris still needs two more disqualifications. As in case 1, this is impossible.

Case 3: One of the two bribed elves is Coco.

Since Greta is already disqualified by Coco in the initial situation, we cannot get Greta an additional disqualification by bribing Coco. Hence, Greta still needs two more disqualifications or has to disqualify herself. Since we have already bribed Coco, we can only bribe one more elf. Consequently, we have to bribe Greta to disqualify herself. However, Joris is qualified by at least five elves (namely Arne, Enzo, Fina, Henri, and Ida) and would still be a member of the committee.

Case 4: Neither Coco, nor Dante, nor Greta are among the two bribed elves.

In this case, Arne, Coco, and Dante each need two additional qualifications, and Greta and Joris each need two additional disqualifications. It is easy to see that it is impossible to get all five elves two additional (dis)qualifications each with only two bribes.

11 Chocolate Appetite at Night

Authors: Christian Hercher, Michael Schmitz (Europa-Universität Flensburg)

Artwork: Friederike Hofmann

Challenge

Selma, the elf girl, has a beautiful Advent calendar filled with delicious sweets. On a hook rail, hang packages labelled with the numbers 1 to 24 *in random order*. Each morning, Selma removes its well-padded contents from the package marked with the respective number. Then, she seals it properly and puts it back on its hook.

During the night from 10 to 11 December, however, it happens: Selma wakes up with a ravenous appetite for chocolate and sneaks through the sleeping house to her Advent calendar. *Randomly*, she takes off four packages in the dark and takes them back to her room—apparently unnoticed. Because the packages contain only small almost weightless items and because of the darkness, Selma cannot tell whether she is picking empty or filled packages. After raiding the packages, Selma neatly closes and returns them to the four empty hooks in the dark. In doing so, she does *not* pay attention to which package she puts back on which hook.

(a) What is the probability that Selma robbed only empty packages?

(b) What is the probability that the four numbers of the packages are consecutive numbers?

Selma's mother has heard something at night and becomes suspicious. When she takes a critical look at the Advent calendar, she might notice that not all the packages are on the same hook as they were the day before. If by chance all the packages hang in the right place, the mother does not notice anything and Selma has nothing to worry about. If there are exactly k packages in the wrong place, Selma's mother thinks it is only fair that Selma has to return $k \cdot 20\,\%$ of her Santa St. Nick's Day's candy.

(c) What is the probability that exactly three packages are hanging on the wrong hook?

(d) On average, how many packages are back in their old place after such an operation (i. e. what is the *expectation value* of the number of packages hung back correctly)?

(e) On average, what fraction of her St. Nick's Day's candy does Selma have to return?

Possible answers:

1. (a) 0.02 (b) 1/506 (c) 1/3
2. (a) 0.02 (b) 0.002 (d) 5/4
3. (a) 120/253 (b) 0.002 (e) 1
4. (a) 5/253 (c) 1/3 (d) 1
5. (a) 5/253 (c) 0 (e) 3/5
6. (a) 120/253 (d) 1 (e) 60 %
7. (b) 0.002 (c) 1/2 (d) 2
8. (b) 0.002 (c) 1/3 (e) 1
9. (b) 1/253 (d) 2 (e) 3/5
10. (c) 0 (d) 5/4 (e) 1

Solution

The correct answer is: 4.

We determine the probabilities for all five questions:

(a) In total, there are $\binom{24}{4}$ ways to choose 4 of the 24 packages, and there are $\binom{10}{4}$ ways to choose 4 from the 10 empty packages. Thus, the probability we are looking for is

$$\frac{\binom{10}{4}}{\binom{24}{4}} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{24 \cdot 23 \cdot 22 \cdot 21} = \frac{5}{253}.$$

(b) There are 21 different possibilities of 4 consecutive numbers, namely:

$$\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \ldots, \{21, 22, 23, 24\}.$$

Hence, the probability we are looking for is

$$\frac{21}{\binom{24}{4}} = \frac{21 \cdot 4 \cdot 3 \cdot 2}{24 \cdot 23 \cdot 22 \cdot 21} = \frac{1}{506}$$

(c) We denote the original positions of the removed packages by a, b, c, d. There are in total $4! = 24$ possible *permutations* (i. e. arrangements) of the packages a, b, c, d. These are

$$\begin{array}{llllll} (a,b,c,d), & (a,b,d,c), & (a,c,b,d), & (a,c,d,b), & (a,d,b,c), & (a,d,c,b), \\ (b,a,c,d), & (b,a,d,c), & (b,c,a,d), & (b,c,d,a), & (b,d,a,c), & (b,d,c,a), \\ (c,a,b,d), & (c,a,d,b), & (c,b,a,d), & (c,b,d,a), & (c,d,a,b), & (c,d,b,a), \\ (d,a,b,c), & (d,a,c,b), & (d,b,a,c), & (d,b,c,a), & (d,c,a,b), & (d,c,b,a). \end{array}$$

If exactly three packages are in the wrong place after putting them back on the hook rail, exactly one package is in the right place. If, for example, a is at the correct position, then there are only two possibilities that none of the three remaining packages also were put back correctly, namely (a, d, b, c) and (a, c, d, b) (above in blue). Since, in addition to a, b (top in red), c (top in green), or d (top in cyan) can also be at the correct position, there are in total $4 \cdot 2 = 8$ possible permutations of the packages a, b, c, d in which exactly one package is in its original place. Thus, the probability we are looking for is $\frac{8}{24} = \frac{1}{3}$.

(d) We determine the expected value of the correctly returned packages. This value indicates how many packages are correctly hung back on average. To do this, we first calculate the probabilities p_0, p_1, p_2, p_3, p_4 that exactly 0, 1, 2, 3, or 4 packages were correctly put back, using the list of all 24 possible arrangements of the hung back packages above.

0) In the nine cases

$$\begin{array}{l} (b,a,d,c), (b,c,d,a), (b,d,a,c), \\ (c,a,d,b), (c,d,a,b), (c,d,b,a), \\ (d,a,b,c), (d,c,a,b), (d,c,b,a) \end{array}$$

none of the four packages is in the right place, i. e. $p_0 = \frac{9}{24} = \frac{3}{8}$.

1) In (c) we have already calculated that $p_1 = \frac{1}{3}$ holds.
2) In the six cases

$$(a, b, d, c), (a, c, b, d), (a, d, c, b), (b, a, c, d), (c, b, a, d), (d, b, c, a).$$

exactly two packages are back on the right hook, i. e. $p_2 = \frac{6}{24} = \frac{1}{4}$.
3) It is not possible that exactly three packages are in the right place afterwards, since in this case the fourth package is automatically also in the right place, i. e. $p_3 = 0$.
4) Of course, there is only one arrangement in which all packages are at their original place (a, b, c, d), i. e. $p_4 = \frac{1}{24}$.

Thus, in total, we get an expected value of

$$\begin{aligned} E &= 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + 4 \cdot p_4 \\ &= 0 \cdot \tfrac{3}{8} + 1 \cdot \tfrac{1}{3} + 2 \cdot \tfrac{1}{4} + 3 \cdot 0 + 4 \cdot \tfrac{1}{24} \\ &= 0 + \tfrac{1}{3} + \tfrac{1}{2} + 0 + \tfrac{1}{6} \\ &= 1. \end{aligned}$$

Remark: Interestingly, in an analogous way, we obtain that the expected value of randomly correctly returned packages is always equal to 1 for any number of stolen packages.

(e) Since on average one package was correctly put back, on average three packages end up in the wrong position. Since the probability of detection depends proportionally on the number of incorrectly placed packages, the probability of Selma's misdeed being detected is $3 \cdot 20\% = 60\%$.

12 Ten Elves in the Christmas Bakery

Authors: Sarah Hiller, Rupert Klein (FU Berlin)
Project: *A Mathematical Theory of Responsibility in Complex Multi-Agent Decision Problems with Uncertainties* (EF 5-3)

Artwork: Frauke Jansen

Challenge

The Christmas bakery is cluttered and messy. The elves are frantically baking cookies; the kitchen air is thick with flour dust, and the icing is dripping from the gingerbread houses. Santa comes into the bakery and tells the ten baking elves, "At least one of you has dough stuck to their hat!"

The elves look at each other. Everyone can see all the other elves, but not themselves. Since Santa and the elves are deeply honest creatures, they always tell the truth; and they all know that fact about each other. In addition, the baking elves are logical geniuses: they will always—and in no time—make the correct considerations with the information available to them.

- "If you know whether you have dough on your hat, step forward!" Santa continues. None of the elves moves.
- "If you know whether you have dough on your hat, step forward!" Santa repeats word for word. Again, all the elves stay still.
- As Santa repeats himself word for word again, some, but not all, of the elves step forward.
- Finally, on the next repetition of "If you know whether you have dough on your hat, step forward!" all of the remaining elves step forward.

Since Santa came to the bakery, the elves have not talked to each other or otherwise exchanged any secret information. Knowing whether one has dough on one's hat means *either* one knows that one has dough on the hat *or* one knows that one does not have dough on the hat.

How many elves have dough stuck to their hat?

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

Project reference:

This task is a common problem in the field of *Dynamic Epistemic Logic* (DEL), which studies the evolution of private and common knowledge, for example, through public communications.

Motivated by the needs of climate change research, the project EF 5-3 *A Mathematical Theory of Responsibility in Complex Multi-Agent Decision Problems with Uncertainties* aims to formalise the concept of moral responsibility—both backward- and forward-looking—in interactive multi-agent decision scenarios with various levels of uncertainty. We use methods from DEL also in our project to describe unawareness, which is relevant when assigning responsibility.

Solution

The correct answer is: 3.

First, we consider the situation where there are only few dirty elves and then conclude the answer for the general case.

Case 1: Among the ten elves there is **exactly one dirty elf**.

We will call this elf Dorothy. Dorothy sees nine other elves, all of whom are clean. Thus, after Santa's first statement ("At least one of you has dough stuck to their hat!"), Dorothy immediately knows that she herself is dirty and can step forward after the first question.

Let us call one of the nine clean elves Susie. Susie sees the dirty elf Dorothy and eight clean elves at the beginning. So after Santa's first statement, Susie cannot be sure whether there are in total one or two elves with dough on their hats. However, after Dorothy steps right forward in response to the question, Susie knows that Dorothy sees nine clean elves, and thus Susie herself must be clean as well. Otherwise, Dorothy would be in the situation that Susie is actually in, and would still be unsure. Hence, Susie can also step forward after the second question.

Of course, the same is true for all other clean elves. Therefore, all clean elves can step forward together after the second question.

Case 2: Among the ten elves there are **exactly two dirty elves**.

Let us call these two dirty elves Dorothy and Dobby. Dorothy sees eight clean elves and Dobby. So she does not know if there is a total of one or two dirty elves. However, when Dobby does not step forward after Santa's first question, Dorothy knows that Dobby must see at least one other dirty elf—namely herself. Thus, after the second question, Dorothy knows she must be dirty and steps forward. The same is true for Dobby.

Again, let us call one of the clean elves Susie. Susie sees seven clean elves plus Dorothy and Dobby. So she does not know if in total two or three of the elves are dirty. However, after Dorothy and Dobby have come forward in the second round, she knows that these two must have seen eight clean elves, otherwise they could not have known yet that they themselves are dirty. Thus, Susie must be clean and can step forward in the next round. The same applies again to all other clean elves.

...

Case n: Among the ten elves there are **exactly n dirty elves**.

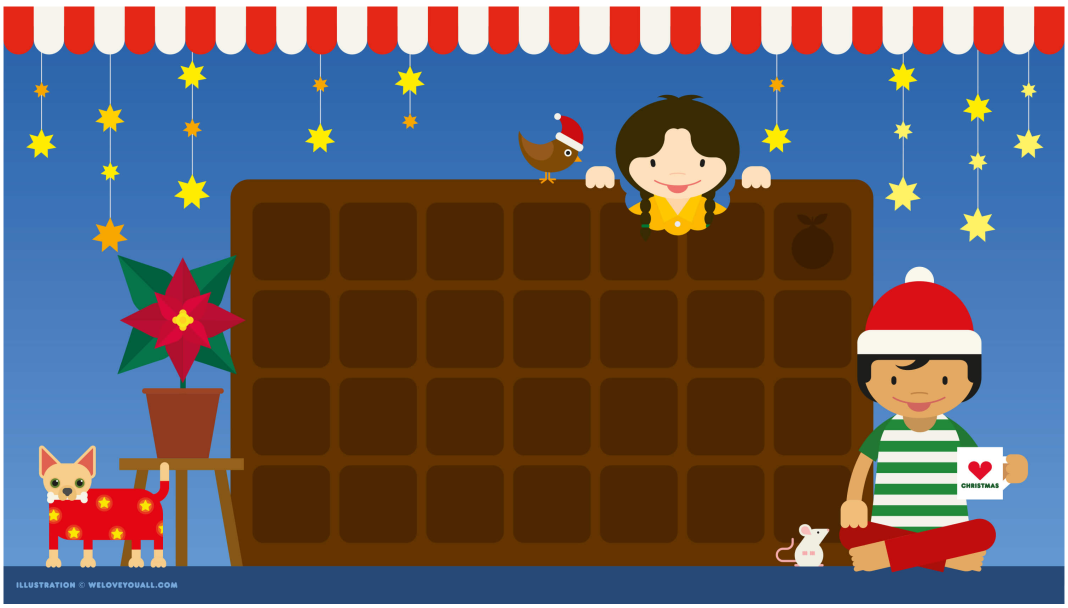
We can apply the argument iteratively, and thus see that if n elves have dough on their hats, the dirty elves will step forward in round n, and the clean elves will step forward one round later, in round $n + 1$.

Since in our example the dirty elves stepped forward in round 3, we now know that exactly three elves have dough stuck to their hats.

13 The Chocolate Game

Authors: Olaf Parczyk, Silas Rathke (FU Berlin)

Project: *Learning Extremal Structures in Combinatorics* (EF 1-12)



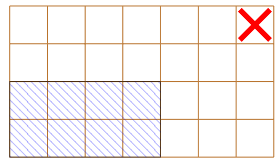
Artwork: Friederike Hofmann

Challenge

The supersmart elves Atto and Bilbo have a big bar of chocolate with n columns and m rows, where m and n are positive integers. We denote by (i, j) the piece in the i-th column and the j-th row of the chocolate. The piece (n, m) is filled with orange jelly and is therefore disgusting (this fact need not be proved).

Atto and Bilbo play the following game: *starting with Atto*, they take turns making moves. A move consists of choosing a piece (i, j) that is still available and eat up all the pieces (i', j') with $i' \leq i$ and $j' \leq j$. Whoever has to eat the piece (n, m) with the orange jelly loses.

Example: A possible move for $n = 7$ and $m = 4$ might look like this. If a player chooses the piece $(4, 2)$ in his turn, he must eat the pieces shaded in blue:



The piece $(7, 4)$ with the orange jelly is marked with a red X. In this example, a game, which Atto loses in the end, could look like this (see Figure 15):

1. Atto (blue) starts with the piece $(4, 2)$.
2. Then, Bilbo (orange) chooses the piece $(3, 4)$.
3. Now, Atto chooses $(6, 4)$.
4. After that, Bilbo chooses $(7, 3)$.
5. Finally, Atto must eat the piece $(7, 4)$ with the orange jelly in his last move.

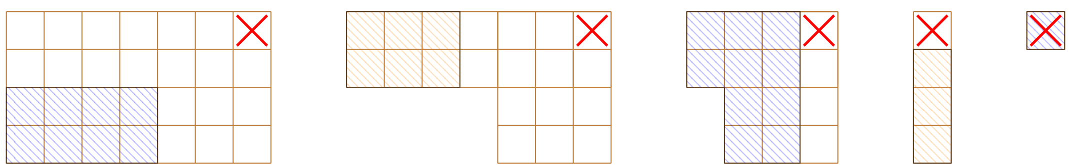


Figure 15: A possible game with a chocolate bar of size 7×4.

Depending on m and n, there is *either* a strategy that can guarantee a win for Atto, *or* a strategy that can guarantee a win for Bilbo.

If we choose m and n independently and randomly from the set $\{1, 2, 3, \ldots, 10^6\}$, all numbers being chosen with equal probability, what is the probability p that Atto has a winning strategy?

Possible answers:

1. $p < 10\%$
2. $10\% \leq p < 20\%$
3. $20\% \leq p < 30\%$
4. $30\% \leq p < 40\%$
5. $40\% \leq p < 50\%$
6. $50\% \leq p < 60\%$
7. $60\% \leq p < 70\%$
8. $70\% \leq p < 80\%$
9. $80\% \leq p < 90\%$
10. $90\% \leq p \leq 100\%$

Project reference:

In the project EF 1-12 *Learning Extremal Structures in Combinatorics*, we use approaches from the field of artificial intelligence and machine learning to find new structures with certain properties. This also helps us to find new winning strategies for combinatorial games, quite similar to the one from this task.

Solution

The correct answer is: 10.

Interestingly, to the authors' knowledge, no one has yet been able to give an explicit winning strategy for all possible (m, n). However, we know that almost always Atto must have a winning strategy:

Claim: For $(m, n) = (1, 1)$, Bilbo has a winning strategy. In all other cases, Atto has one.

Proof:

1. If $(m, n) = (1, 1)$, Atto is forced in his first move to eat the piece with the disgusting orange jelly.
2. Let now $(m, n) \neq (1, 1)$. We want to prove that Atto has a winning strategy in this case. We will prove the claim by contradiction, i. e. we assume that Bilbo has a winning strategy. That is, no matter for which moves Atto decides, Bilbo always has an answer move that will allow him to win.

   In particular, Bilbo has an answer if Atto chooses the square $(1, 1)$ in his first move. Let (i, j) be such a response move by Bilbo to Atto's move $(1, 1)$ that will allow him to win. But this also means that Atto could have chosen (i, j) in his first move and would have won.

   This is a contradiction yielding that there can be no winning strategy for Bilbo for $(m, n) \neq (1, 1)$.

Thus, the probability we are looking for is

$$p = \frac{10^6 \cdot 10^6 - 1}{10^{12}} = \frac{999\,999\,999\,999}{1\,000\,000\,000\,000} = 0.9999999999.$$

14 Magical Reindeer Breeding

Authors: Christian Kuchler, Falk Hante (HU Berlin)
Project: *Decision-Making for Energy Network Dynamics* (AA 4-7)

Artwork: Friederike Hofmann

Challenge

To deliver Christmas gifts around the world, Santa uses his enchanted sledge. Each year, the sledge is dragged by a herd of magical reindeer. Due to the heavy load, the reindeer can only drag the sledge once and need to retire afterwards. So Santa is required to breed a new herd of reindeer every year.

For that purpose, Santa uses a special enclosure in which the current population magically doubles its current size after each day. The enclosure is surrounded by a fence. This fence currently possesses a length of 112 meters and cannot be enlarged, because all elves in the workshops have to craft presents until Christmas. Furthermore, due to new regulations on reindeer-appropriate husbandry, Santa has to arrange the enclosure in the shape of a rectangle (see Fig. 16) and has to ensure that the reindeer each are given space of at least 3 square meters. These regulations are enforced by the Department of Reindeer Welfare, and inspections can happen at *any time*.

The breeding process itself cannot be interrupted once it has been started, and Santa begins the breeding with Rudolph, his most famous and beloved reindeer.

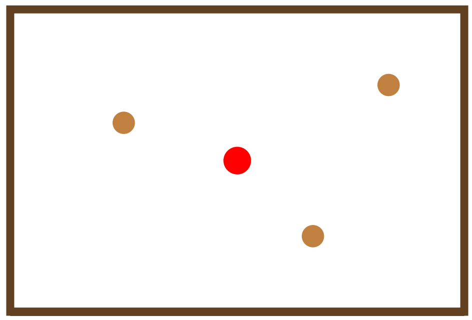


Figure 16: The rectangular enclosure, Rudolph (red), and three of his offspring (brown).

At which day should Santa start the breeding process to maximize the number of reindeer on 24 December while abiding all the above regulations at any time?

Possible answers:

1. 1 December
2. 6 December
3. 10 December
4. 12 December
5. 14 December
6. 16 December
7. 18 December
8. 20 December
9. 22 December
10. 24 December

Project reference:

The move towards sustainable energy networks, with due consideration of the complexities arising from the integration of rather unsteady renewable resources, causes major operational challenges for network providers. The research agenda of project AA 4-7 *Decision-Making for Energy Network Dynamics* tackles fundamental questions concerning control and optimization in context of this application. The above challenge is an example of a constrained optimization problem.

Solution

The correct answer is: 6.

The problem is a *constrained optimization task*. We want to

$$\text{maximise } 2^z \text{ for } z \in \mathbb{N}_0 \text{ such that } \begin{cases} x \geq 0, \\ y \geq 0, \\ 2x + 2y = P, \\ 3 \cdot 2^z \leq xy. \end{cases}$$

Here, the variables x and y denote the horizontal and vertical length of the fence, respectively. The parameter $P > 0$ describes the given perimeter of the fence. In our case it is given as 112 meters. The unknown z gives the number of breeding days.

It can be seen that the area of the enclosure restricts Santa's reindeer population. Therefore it makes sense to first maximize its area xy; that is,

$$\text{maximize } xy \text{ for } (x, y) \in \mathbb{R}^2 \text{ such that } \begin{cases} x \geq 0, \\ y \geq 0, \\ 2x + 2y = P. \end{cases}$$

We can do this by elimination: $2x + 2y = P$ is equivalent to $y = \frac{P}{2} - x$, and our problem becomes

$$\text{maximize } x \cdot \left(\frac{P}{2} - x\right) \text{ for } x \in \mathbb{R} \text{ such that } \begin{cases} x \geq 0, \\ \frac{P}{2} - x \geq 0. \end{cases}$$

The function $A(x) = x \cdot \left(\frac{P}{2} - x\right) = -x \cdot \left(x - \frac{P}{2}\right)$ is quadratic in x. Thus, its global maximum is at $x_{max} = \frac{P}{4}$ with a value of

$$A(x_{max}) = \frac{P}{4} \cdot \left(\frac{P}{2} - \frac{P}{4}\right) = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16}.$$

Indeed, $x_{max} = \frac{P}{4}$ is a feasible point, since

$$x_{max} = \frac{P}{4} \geq 0 \quad \text{and} \quad \frac{P}{2} - x_{max} = \frac{P}{2} - \frac{P}{4} = \frac{P}{4} = x_{max} \geq 0.$$

Note that this solution reproduces the known property that a square maximises the area of a rectangle with a given perimeter.

For the given parameter $P = 112$, we obtain $x_{max} = \frac{112}{4} = 28$ and

$$A = x_{max}^2 = 28^2 = 784.$$

Therefore, the original task is reduced to

$$\text{maximize } 2^z \text{ for } z \in \mathbb{N}_0 \text{ such that } 2^z \leq \frac{784}{3} = 261 + \frac{1}{3}.$$

The goal function 2^z is monotonic increasing. Hence, we are looking for the largest number $z \in \mathbb{N}_0$ such that $2^z \leq \frac{784}{3}$. Since $2^8 = 256$, but $2^9 = 512$, this largest number is given by $z_{max} = 8$ and we obtain the following breeding schedule:

Day	16th	17th	18th	19th	20th	21th	22th	23th	24th
Population	1	2	4	8	16	32	64	128	256

In summary, if Santa starts the breeding process on 16 December, he will maximize the reindeer population on 24 December while abiding all regulations. He will end up with 256 reindeer.

15 Lost and Found?

Author: Svenja M. Griesbach
Project: *Information Design for Bayesian Networks* (AA3-9)

Artwork: Julia Nurit Schönnagel

Challenge

To get in shape for the 24 December, Santa has been hiking in the mountains for a few days. When he arrived at the main chalet with the red door, however, he noticed that he had lost a few wish lists in one of the smaller chalets. Of course, he has to find them as quick as possible. Unfortunately, he does not remember exactly in which of the six chalets he left the wish lists; he only knows that they must be in one of them. Since Santa did not stop at all of the chalets during his hike, he can at least rule out two of the six. For each of the other four chalets, he can at least say how probable it is that the wish lists lie there.

Since Santa has no time to lose, he immediately calls Rudolph with his super-fast sleigh to help him. Just as the two are about to set off, they notice that a thick layer of snow is blocking the slopes. So first, the slopes have to be ploughed before they can be used by their sled. For this purpose, only one single magic snowplow is available, which can clear the slopes *one after* *the other, but in completely arbitrary order*. Thus, there is no extra time needed in addition to clearing the individual slopes. On his map (see Fig. 17), Santa can read off how long it takes to plough each of the slopes. Once a slope is cleared of snow, Santa and Rudolph can dash from one end of that slope to the other in their lightning-fast sleigh without wasting any time.

The two start in the main chalets with the red door. As soon as Santa finds the wish lists in one of the chalets, they stop the search as well as the clearing of the slopes.

In what order should Santa have the slopes ploughed in order to find the wish lists *on average* as quickly as possible?

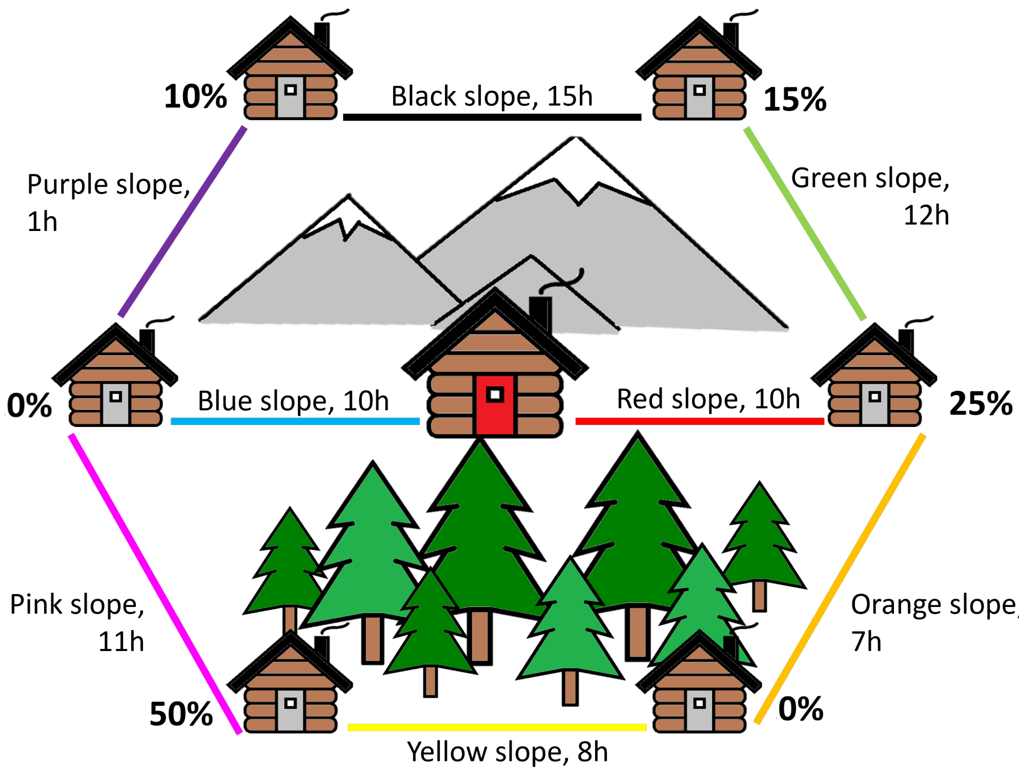


Figure 17: Santa's map with the main chalet with the red door, the surrounding chalets (each with the probability that the wish lists lie there), and the slopes (with the respective ploughing time) between the chalets.

Possible answers:

1. blue, pink, red, green, purple
2. blue, pink, purple, red, green
3. blue, purple, black, green, pink
4. blue, purple, pink, red, green
5. blue, red, purple, green, pink
6. red, green, orange, yellow, blue, purple
7. red, blue, purple, pink, green
8. red, blue, pink, purple, green
9. red, orange, yellow, green, blue, purple
10. red, orange, yellow, blue, purple, black

Project reference:

The mathematical problem behind this puzzle is called *expanding search*, where the search can be continued at any previously reached node of a graph at any time.

In project AA 3-9 *Information Design for Bayesian Networks*, we consider very similar networks to Santa's map: *Bayesian networks* are directed graphs consisting of weighted edges (here the slopes with their respective ploughing time) and nodes (the chalets) with a conditional probability distribution (represented by the probabilities).

Solution

The correct answer is: 9.

Since this is a so-called *NP-hard problem*, there is no way around calculating all possible paths. As an example: the exact calculation for the path given in answer no. 1 is:

$$\begin{aligned}
&\mathbb{P}(\text{list not yet found}) * \text{ duration for blue slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for pink slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for red slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for green slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for purple slope}\\
=&100\% * 10\,\text{h}\\
&\quad + (100\% - 0\%) * 11\,\text{h}\\
&\quad + (100\% - 50\%) * 10\,\text{h}\\
&\quad + (100\% - 50\% - 25\%) * 12\,\text{h}\\
&\quad + (100\% - 50\% - 25\% - 15\%) * 1\,\text{h}\\
=&100\% * 10\text{h} + 100\% * 11\text{h} + 50\% * 10\text{h} + 25\% * 12\text{h} + 10\% * 1\,\text{h}\\
=&29.1\,\text{h}
\end{aligned}$$

We claim that 9 is the correct answer and calculate its expected value:

$$\begin{aligned}
&\mathbb{P}(\text{list not yet found}) * \text{ duration for red slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for orange slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for yellow slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for green slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for blue slope}\\
&\quad + \mathbb{P}(\text{list not yet found}) * \text{ duration for purple slope}\\
=&100\% * 10\,\text{h}\\
&\quad + (100\% - 25\%) * 7\,\text{h}\\
&\quad + (100\% - 25\%) * 8\,\text{h}\\
&\quad + (100\% - 25\% - 50\%) * 12\,\text{h}\\
&\quad + (100\% - 25\% - 50\% - 15\%) * 10\,\text{h}\\
&\quad + (100\% - 25\% - 50\% - 15\%) * 1\,\text{h}\\
=&100\% * 10\text{h} + 75\% * 8\text{h} + 75\% * 7\text{h} + 25\% * 12\text{h} + 10\% * 10\text{h} + 10\% * 1\,\text{h}\\
=&25.35\,\text{h}
\end{aligned}$$

To do this, we show that the expected value for any other order (not just the ones listed) is strictly greater than 25.35 h.

To reduce the number of possibilities, we first make the following observations:

1. If we plough a path that, even before visiting all the chalets where the wish lists might be, has an expected value greater than 25.35 h, we no longer need to compute all the

expected values of the paths that result from that path, since the expected value is monotonically increasing with respect to addition.

2. If a path contains slopes that, if omitted, still visits all the chalets where the wish lists might be, it cannot be optimal.

Last, we compute the expected values of all remaining sequences:

Order	Duration
1. blue, pink, red, ...	26 h
blue, pink, yellow, orange, ...	28,5 h
blue, pink, yellow, purple, ...	25.5 h
blue, pink, purple, black, ...	27,5 h
2. blue, pink, purple, red, ...	25.5 h
blue, purple, black, red, ...	32 h
3. blue, purple, black, green, ...	33.5 h
blue, purple, black, pink, ...	32,75 h
blue, purple, red, green, ...	27.8 h
blue, purple, red, black, ...	29.75 h
blue, purple, red, orange, yellow, ...	29.75 h
blue, purple, red, orange, black, ...	34.3 h
blue, purple, red, orange, green, ...	32.35 h
blue, purple, red, pink, ...	27,15 h
blue, purple, pink, black, ...	26.9 h
blue, purple, pink, yellow, orange, ...	26.9 h
blue, purple, pink, yellow, red, ...	28.1 h
blue, purple, pink, yellow, black, ...	30.1 h
blue, purple, pink, red, black	27.15 h
4. blue, purple, pink, red, green	26.7 h
blue, red, pink, ...	28.25 h
blue, red, orange, yellow, ...	31.25 h
blue, red, orange, green, ...	34.25 h
blue, red, orange, purple, ...	26 h
blue, red, purple, black, ...	30.5 h
5. blue, red, purple, green, ...	28.55 h
blue, red, purple, pink, ...	27.9 h
blue, red, purple, orange, yellow, ...	30.5 h
blue, red, purple, orange, black, ...	35.03 h
blue, red, purple, orange, green, ...	33.1 h
blue, red, green, ...	29 h

Order	Duration
8. red, blue, pink, ...	25.75 h
red, blue, orange, yellow, ...	28.75 h
red, blue, orange, green, ...	31,75 h
red, blue, orange, pink, ...	31 h
red, blue, orange, purple, black, ...	33.25 h
red, blue, orange, purple, green, ...	31.3 h
red, blue, orange, purple, yellow, ...	28.7 h
red, blue, purple, black, ...	28 h
red, blue, purple, green, ...	26.05 h
7. red, blue, purple, pink, ...	25.4 h
red, blue, purple, orange, yellow, ...	28 h
red, blue, purple, orange, black, ...	32.55 h
red, blue, purple, orange, green, ...	30.6 h
red, blue, green, ...	26.5 h
red, green, black, ...	28 h
red, green, blue, black, ...	34 h
red, green, blue, purple, ...	25.6 h
red, green, blue, pink, ...	31.6 h
red, green, blue, yellow, ...	29.8 h
6. red, green, orange, yellow, ...	29 h
red, orange, blue, yellow, ...	28.75 h
red, orange, blue, purple, black, ...	33.25 h
red, orange, blue, purple, green, ...	31.3 h
red, orange, blue, purple, yellow, ...	28.7 h
red, orange, blue, green, ...	31.75 h
red, orange, green, black, ...	31.25 h
red, orange, green, blue, ...	30.25 h
red, orange, green, yellow, ...	29.05 h
red, orange, yellow, green, black	25.75 h
9. red, orange, yellow, green, blue, purple	**25,35 h**
red, orange, yellow, green, pink, purple	25.45 h
red, orange, yellow, pink, purple, black	26.5 h
red, orange, yellow, pink, purple, green	26,05 h
red, orange, yellow, pink, green, ...	27 h
10. red, orange, yellow, blue, purple, black	26.25 h
red, orange, yellow, blue, purple, green	25.8 h
red, orange, yellow, blue, green, ...	26.75 h

16 O Christmas Tree

Authors: Susanne Brinkhaus (HU Berlin, Käthe-Kollwitz-Gymnasium)
Christoph Werner (HU Berlin)
Project: *School@DecisionTheatreLab*
(BUA Experimental Lab for Science Communication)

Artwork: Frauke Jansen

Challenge

Santa Claus has chosen a Christmas tree with very special lighting this year. The Christmas tree is divided into 40 small squares, and in each square there is exactly one light that is either on or off (see Fig. 18).

The special feature: the lights on the tree can light up each other. A light that is off is lit up if the lights in *at least* two horizontally or vertically adjacent squares are already burning (see Fig. 18). The newly lit lights can subsequently ignite further lights. A light cannot be lit across diagonally adjacent squares. In addition, once a light is on, it continues to burn and does not go out.

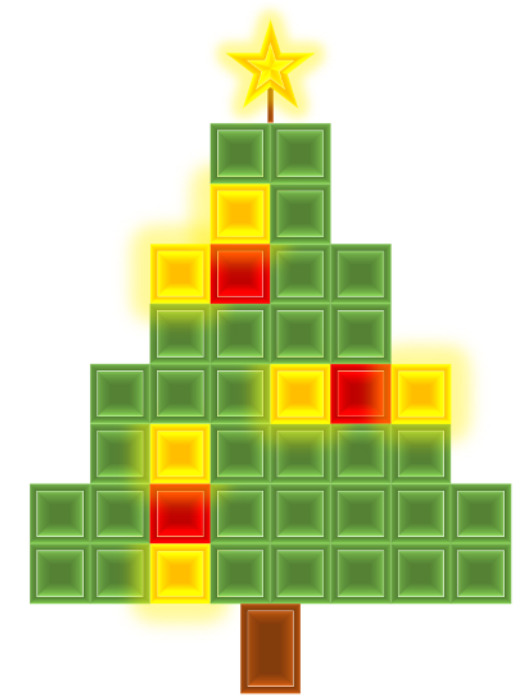


Figure 18: Santa's Christmas tree. An example with burning lights (yellow squares) and the lights ignited by them (red squares).

What is the smallest possible number of already burning lights on Santa's Christmas tree needed, such that eventually every light on the tree is lit up?

Possible answers:

1. The smallest possible number of already burning lights is 6.
2. The smallest possible number of already burning lights is 7.
3. The smallest possible number of already burning lights is 8.
4. The smallest possible number of already burning lights is 9.
5. The smallest possible number of already burning lights is 10.
6. The smallest possible number of already burning lights is 11.
7. The smallest possible number of already burning lights is 12.
8. The smallest possible number of already burning lights is 13.
9. The smallest possible number of already burning lights is 14.
10. The smallest possible number of already burning lights is 15.

Project reference:

The operating principle of the Christmas tree corresponds to that of a *cellular automaton*. In a way, *agent-based models* are generalization of such cellular automata and can be used to model complex dynamical systems in which global properties depend on the behaviour of many individual agents. In the project *School@DecisionTheatreLab*, funded by MATH+ and BUA, such agent-based models for socially relevant topics are mathematically analysed and used as a starting point for political discussions.

Solution

The correct answer: 3.

We want to show that, to gradually light up the entire tree, we need at least eight lit squares at the beginning. In Figure 19, we see a start configuration with eight lit squares which indeed lightens up the entire tree after 28 steps. The sequence of illumination is shown at the end of the solution in Figure 20.

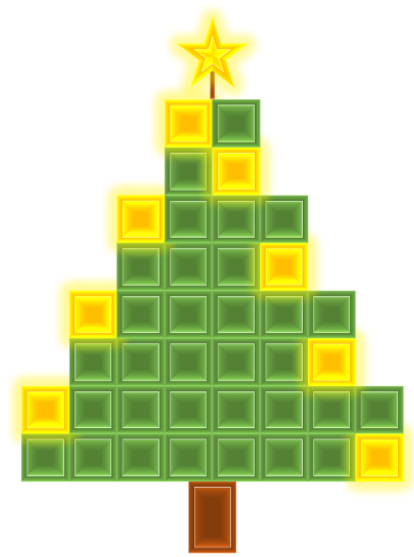
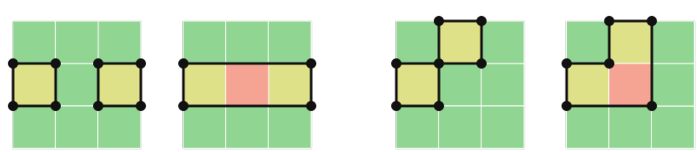


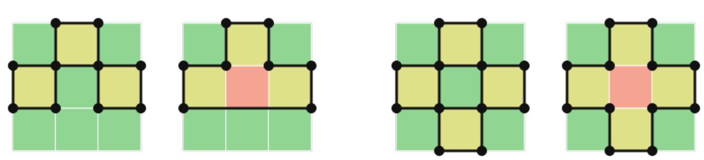
Figure 19: A possible start configuration such that all squares are lit at the end.

We now need to show that there is no start configuration with less than eight lit squares that allows us to light up the entire tree. To do so, we use a so-called *invariance argument*. We claim that the perimeter of all luminous fields cannot increase from one step to the next one:

- If a field that is not yet burning is lit by two adjacent fields, the perimeter of the lit region remains constant:



- If a field that is not yet lit is lit by at least three adjacent fields, the perimeter of the lit region actually decreases:



If a single square box has side length 1, then the perimeter of the given Christmas tree is 32. Hence, the perimeter of the lit region in the start configuration must also be at least 32. But this is only possible with at least eight (non-adjacent) squares.

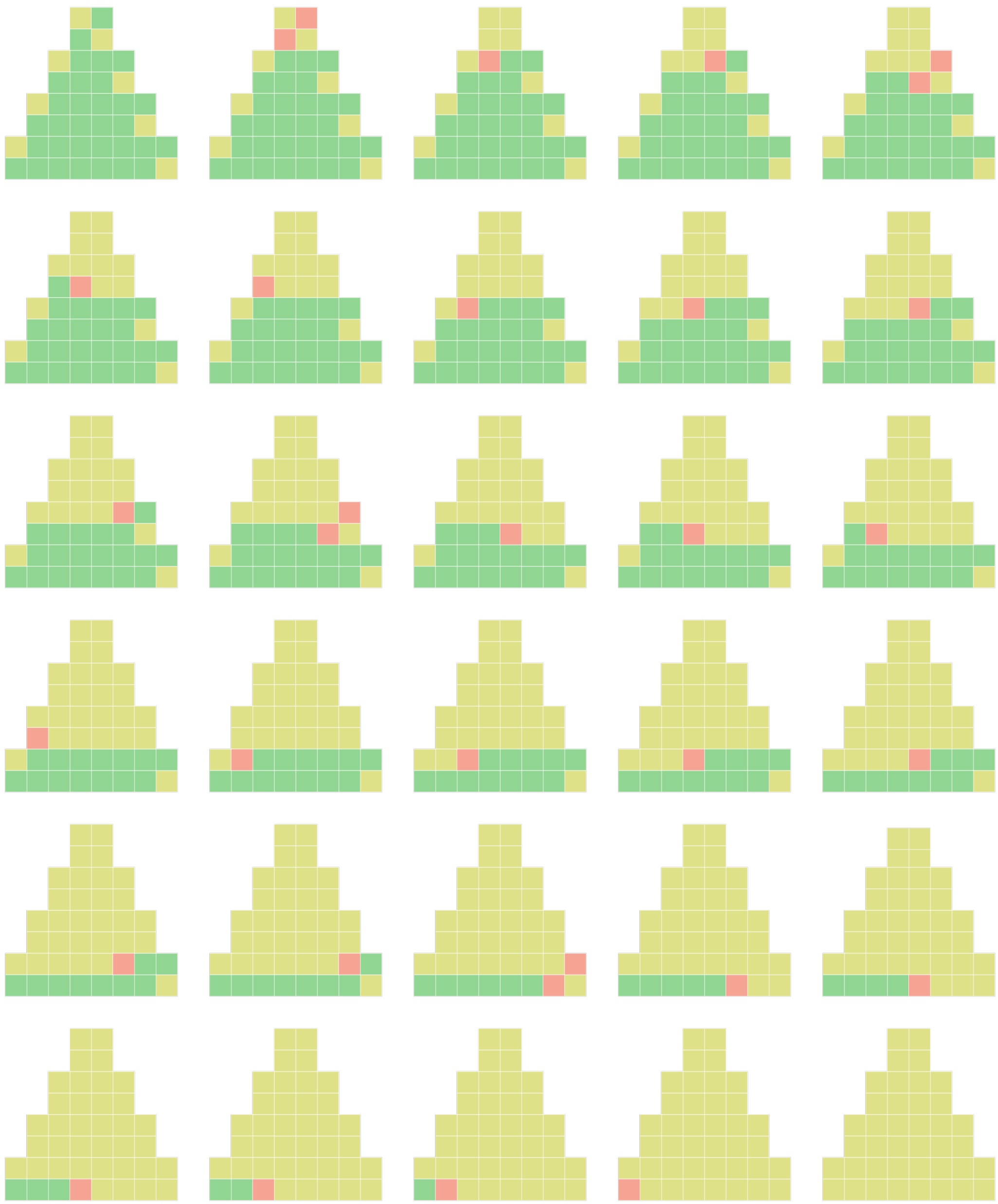


Figure 20: From left to right and from top to bottom: In 28 steps, the start configuration from Fig. 19 (top left) gradually illuminates the whole tree (bottom right).

17 Rowdy Reindeer

Author: Jesse van Rhijn (Universiteit Twente)
Project: 4TU.AMI

Artwork: Julia Nurit Schönnagel

Challenge

Santa has a stable for his reindeer, which is divided into nine enclosures as shown in Figure 21.

Santa's elves have ordered red-nosed and brown-nosed reindeer to fill these nine enclosures. Unfortunately, something went wrong: instead of nine reindeer they accidentally ordered twelve. To make it even worse, the reindeer are very territorial: the enclosures of any two brown-nosed reindeer cannot share a wall without the reindeer fighting, and the same holds for the enclosures of any two red-nosed reindeer. Fortunately, a reindeer will never fight with a reindeer whose nose has a different color; so the enclosures of a red-nosed reindeer and a brown-nosed reindeer can share a wall. Also, there will never be any fighting between reindeer whose enclosures only meet in a corner, or do not meet at all.

The elves cannot change the total number of reindeer they have already ordered, but they can still choose the number of red-nosed and brown-nosed reindeer they receive. To accommodate

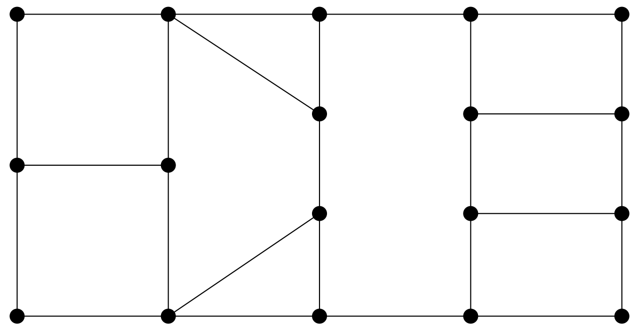


Figure 21: Santa's stable.

the twelve reindeer, the elves need to create exactly three new enclosures within the existing stable. For this purpose, they to are instructed to build *exactly three new straight walls*. Each of the new walls must be put in between two points where already at least two walls meet. These 18 points are marked in Figure 21. Furthermore, the new walls are neither aloud to cross the old walls nor other new ones.

In how many ways can the elves build the three new straight walls, such that it is possible to give each of the twelve reindeer its own enclosure without any reindeer fighting?

Possible answers:

1. Zero ways, i. e. it is not possible build the three walls as required.
2. One way.
3. Four ways.
4. Five ways.
5. Six ways.
6. Eight ways.
7. Nine ways.
8. Twelve ways.
9. Fourteen ways.
10. Sixteen ways.

Solution

The correct answer is: 9.

To create an admissible new enclosure, it helps to look at the points where different enclosures meet in the interior. For instance, look at the point marked B in Figure 22 below.

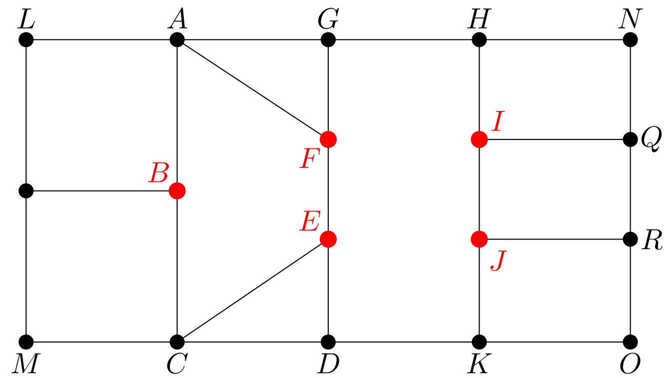


Figure 22: Santa's original stable. The problem points B, E, F, I, J are marked in red.

In B, three enclosures meet. If we try to put reindeer in the empty enclosures surrounding B, you immediately run into a problem: for instance, suppose we were to put a red-nosed reindeer in the large enclosure to the right of B. Since that enclosure shares a wall with the two enclosures to the left of B, we must put brown-nosed reindeer in those two enclosures to prevent the red-nosed reindeer from fighting. But the two enclosures to the left of B share a wall, leading to a conflict between the two brown-nosed reindeer.

If however four enclosures met in this point, then there would be no problem: going clockwise around the point, assign red-nosed and brown-nosed reindeer to the enclosures alternately. This pattern repeats; whenever an even number of enclosures meet in one internal point, that is, one of the points B, E, F, I, J, we can assign reindeer to those enclosures without risking a conflict. A valid solution is then exactly a division where all internal points are surrounded by an *even* number of enclosures.

We can easily find new divisions by using this property. The next challenge is to make sure we count all distinct possibilities, and do not double-count any. To do so, we proceed systematically. Note that there are exactly five points (B, E, F, I, J) in the interior where an odd number of enclosures meet; we will call these the *problem points*. We must affix at least one wall to each of these problem points.

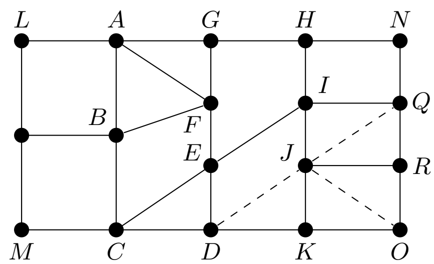
Since there are five such points, and we may only build three walls, each problem point must get exactly 1 additional wall. To see this, note that we cannot affix two walls to one problem point, since this would turn it again into a problem point (odd + 2 = odd). But if we affix three walls to one problem point, then the remaining endpoints of these walls can fix at most three additional points, fixing at most four of the five problem points.

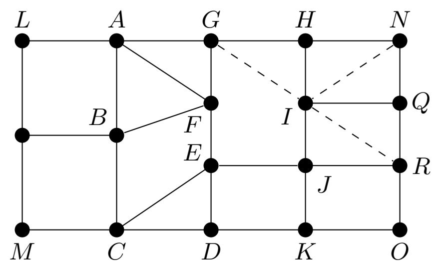
Consider the problem point B. We can fix it in four distinct ways, namely by drawing exactly one of the lines BF, BE, BL, and BM. We will only consider the cases where we draw BF and BL, since, by symmetry, the cases BE and BM will yield additionally the same number of possibilities.

Case *BF*: After drawing BF, the remaining problem points are E, I and J. We can draw two more lines, so one of these lines must be used to connect two points among $\{E, I, J\}$. This can be done in two ways: EI, EJ.

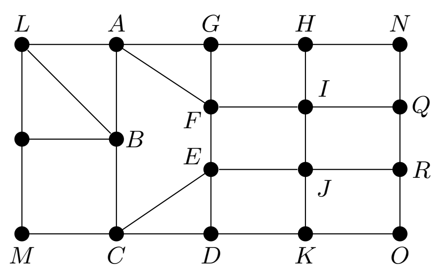
Consider the first the case where we draw EI. Then J is the only problem point that still needs to be fixed. There are three options for the final line that fix problem point J: JD, JO, and JQ.

Next, consider the case where we draw EJ. Then we are again left with three options for the final line: IG, IN, and IR.





Case *BL*: After drawing BL, there are again two lines left to draw. This time, there are four problem points left, so we must use the remaining lines to connect up these points. There is only one valid way to connect these points, which is to connect IF and EJ.



These two cases together yield seven solutions. As mentioned previously, the cases where we draw BE and BM will yield again seven solutions by symmetry. Therefore, the total number of solutions is fourteen.

18 Up and Down

Author: Christian Hercher (Europa-Universität Flensburg)

Artwork: Friederike Hofmann

Challenge

Elf Collean likes to pass her free time by playing with positive integers. In doing so, she has noticed that whenever she considers an odd positive integer n, its triple is odd again; that is, $3n + 1$ is divisible by two. Therefore, she invents the following two rules to calculate a positive integer from a given positive integer n:

1. If the given number n is even, she divides it by two and gets another positive integer, namely $\frac{n}{2}$.

   Since the result is smaller then the starting number, Collean calls this operation a *decreasing step*.

2. If the given number n is odd, she first multiplies it by three, then adds one, and finally divides everything by two. The result $\frac{3n+1}{2}$ is, as we have seen above, another positive integer.

Here, the result is bigger than the starting number. Hence, Collean calls this operation an *increasing step*.

In this way, starting with any positive integer, Collean produces more and more positive integers. For example, if she starts with the number 11, the next number she gets is 17, then 26, 13, 20, 10, 5, 8, 4, 2, and finally 1. Then, something peculiar happens: after 1, according to her rules, she gets 2 again, then 1 again, then 2 again, and so on.

Collean wonders if this is always the case: no matter what number you start with, do you always eventually end up with this loop from 1 to 2 and back? All the examples she calculates point in this direction. Of course, this is far from being a proof ...

However, as a mathematically interested elf, she is not satisfied with stating such a conjecture. Although its proof is probably still beyond her reach, she is perhaps able to make some related observations and then prove or disprove them. Thus, she postulates the following claims concerning her two rules:

Claim A: If a positive integer n can be written as $a \cdot 2^k - 1$ (where a and k are positive integers), then starting with n one can perform at least k *increasing steps* in a row.

Claim B: There is a positive integer n which is followed only by *increasing steps*.

Claim C: Let n be an odd positive integer that is first followed at least one *increasing* and then two *decreasing steps* in a row, resulting in a number m. Then, this number m could have also been obtained by starting with $\frac{n-1}{2}$.

Claim D: If there exist starting numbers that will never reach 1, then the remainder of the smallest of these numbers when divided by 6 is 1 or 3.

Which of the above claims are correct?

Possible answers:

1. All four claims are correct.
2. A, B, and C are correct, D is wrong.
3. A, B, and D are correct, C is wrong.
4. A, C, and D are correct, B is wrong.
5. B, C, and D are correct, A is wrong.
6. A and B are correct, C and D are wrong.
7. B and C are correct, A and D are wrong.
8. A is correct, B, C, and D are wrong.
9. D is correct, A, B, and C wrong.
10. All four claims are wrong.

Solution

The correct answer is: 4.

Claims A, C, and D are all true, while claim B is false. To see this, we turn our attention to each claim and (dis-)proof its correctness.

Claim A is true: If a number can be written as $a \cdot 2^k - 1$ and if $k > 0$, then the number is odd and an increasing step follows. As a result we get

$$\frac{3 \cdot (a \cdot 2^k - 1) + 1}{2} = \frac{3a \cdot 2^k - 2}{2} = 3a \cdot 2^{k-1} - 1,$$

a number of similar shape. However, the exponent of the power of two has become smaller by one. If this exponent is larger than zero, an increasing step follows, which reduces the exponent of the power of two by 1 again. We can now perform increasing steps until the exponent is reduced to zero. Thus, the starting number $a \cdot 2^k - 1$ is followed by at least k increasing steps and statement A is true.

Claim B is false: First, we note that the converse of statement A also holds: if a number n is directly followed by k increasing steps, then n is of the form $a \cdot 2^k - 1$. Thus, it is intuitively clear that an infinite number of increasing steps cannot be applied to any number, since we cannot write a number as $a \cdot 2^\infty - 1$. In other words, after sufficiently many increasing steps, we always get an even number. Nevertheless, we will prove the statement formally by *induction* over k:

Base case: For a number to be followed by 1 increasing step, it must be odd, and thus be of the form $a_1 \cdot 2^1 - 1$ with $a_1 \in \mathbb{N}$.

Induction step:

Induction hypothesis: Let the statement hold for k, that is, a number n followed by k increasing steps is of the form $n = a_k \cdot 2^k - 1$.

Claim: If the induction hypothesis holds for k, then it also holds for $k+1$, i.e. a number m followed by $k+1$ increasing steps is of the form $m = a_{k+1} \cdot 2^{k+1} - 1$.

Induction proof: Let m be a number followed by $k+1$ increasing steps. In particular, m is therefore followed by k increasing steps and by the induction hypothesis, m is of the form $m = a \cdot 2^k - 1$. After k increasing steps we obtain (see proof of statement A) the number

$$\tilde{m} = 3^k \cdot a - 1.$$

But since m is followed by $k+1$ increasing steps, $\tilde{m}$ must be followed by one more increasing step, so that $\tilde{m}$ has the form $\tilde{m} = a_1 \cdot 2 - 1$ by the base case. Thus, $3^k \cdot a = a_1 \cdot 2$ holds, yielding a must be divisible by 2 (since $3^k = 3 \cdots 3$ is not divisible by 2 at all). Hence, we can write a as $a = \tilde{a} \cdot 2$ and thus m as

$$m = a \cdot 2^k - 1 = \tilde{a} \cdot 2 \cdot 2^k - 1 = \tilde{a} \cdot 2^{k+1} - 1,$$

whereby the claim follows.

We want to use this statement to show that there cannot be a number only followed by increasing steps. To do so, we give a proof by contradiction: we assume the number n is only followed by increasing steps. In particular, n is followed by some, say k, increasing steps. Thus, the number n can be written as $n = a \cdot 2^k - 1$ and after k increasing steps we get the number $\tilde{n} = 3^k \cdot a - 1$. If a is odd, then $\tilde{n}$ is even. Hence, $\tilde{n}$ is followed by a decreasing step – a contradiction to the assumption that n is followed only by increasing steps. Thus, let a be even, i.e. of the form $a = b \cdot 2^l$. Here, we choose l to be as large as possible, i.e. such that b is not divisible by 2. Thus, $\tilde{n}$ has the form $\tilde{n} = 3^k \cdot b \cdot 2^l - 1$. According to statement A, $\tilde{n}$ is therefore followed by l increasing steps and we obtain the number

$$\tilde{\tilde{n}} = 3^k \cdot 3^l \cdot b - 1.$$

The number $\tilde{\tilde{n}}$ is odd, implying that a decreasing step follows. Thus, n is followed by $k + l$ increasing steps and then by a decreasing step. This is a contradiction to the assumption and we have proven that claim B is false.

Claim C is true: Again, we can construct the general form that starting numbers n must have so that they can be followed by exactly k increasing steps first, and two decreasing steps immediately after: for the k increasing steps we already know that n must be of the form $n = a \cdot 2^k - 1$ and that from this, after applying these k increasing steps, the number $\tilde{n} = a \cdot 3^k - 1$ results. In order that $\tilde{n}$ can be followed directly by two decreasing steps, $\tilde{n}$ must be divisible by 4 i.e. $a \cdot 3^k$ must leave the remainder 1 when divided by 4.
If $k = 2l > 0$ is even, then we can write 3^k as

$$\begin{aligned} 3^k &= 3^{2l} = (3^2)^l = 9^l = (8+1)^l \\ &= 8^l + a_1 \cdot 8^{l-1} + a_2 \cdot 8^{l-2} + \cdots + a_{l-1} \cdot 8 + 1 \\ &= 8 \cdot (a_1 \cdot 8^{l-2} + a_2 \cdot 8^{l-3} + \cdots + a_{l-1}) + 1 \\ &=: 4c + 1 \end{aligned}$$

Therefore, $3^k = 4c + 1$ leaves the remainder 1 when divided by 4 and a must be of the form $a = 4b + 1$, so that

$$a \cdot 3^k = (4b+1) \cdot (4c+1) = 4^2bc + 4b + 4c + 1 = 4(4bc + b + c) + 1$$

also having remainder 1 when divided by 4. Accordingly, in this case n has the form $n = (4b + 1) \cdot 2^k - 1$ and after applying the k increasing and two decreasing steps we obtain

$$\tilde{n} = \frac{(4b+1) \cdot 3^k - 1}{4}.$$

Let us now consider

$$\frac{n-1}{2} = \frac{(4b+1) \cdot 2^k - 1 - 1}{2} = \frac{(4b+1) \cdot 2^k - 2}{2} = (4b+1) \cdot 2^{k-1} - 1.$$

We note that this number is first followed by $k - 1$ increasing steps, yielding the number $(4b+1) \cdot 3^{k-1} - 1$. This number is obviously even, but not divisible by 4 since 3^{k-1} leaves the remainder 3 when divided by 4. This is due to the fact that 3^{k-1} is, similar to 3^k, of the form

$$3^{k-1} = 3^{2l-1} = 3 \cdot 3^{2(l-1)} = 3 \cdot 9^{l-1} = 3 \cdot (8+1)^{l-1} =: 4\tilde{c} + 3.$$

Thus, this number is followed first by a decreasing step and then again by an increasing step, resulting in $\tilde{n}$:

$$\frac{3 \cdot \frac{(4b+1)\cdot 3^{k-1}-1}{2}+1}{2} = \frac{(4b+1)\cdot 3^k - 3}{4} + \frac{1}{2} = \frac{(4b+1)\cdot 3^k - 3 + 2}{4}$$
$$= \frac{(4b+1)\cdot 3^k - 1}{4} = \tilde{n}$$

We can make a similar observation if $k = 2l+1 > 0$ is odd: In this case 3^k has remainder 3 when divided by 4, since 3^k can be written again as $3^k = 4c+3$. The number a must be of the form $a = 4b+3$ so that

$$a \cdot 3^k = (4b+3)\cdot(4c+3) = 4^2bc + 4\cdot 3\cdot(b+c) + 9 = 4(4bc + 3(b+c) + 2) + 1$$

again has remainder 1 when divided by 4. This time, n is of the form $n = (4b+3)\cdot 2^k - 1$. Thus, after k increasing steps, we get

$$\tilde{n} = \frac{(4b+3)\cdot 3^k - 1}{4}.$$

On the other hand , if we start with

$$\frac{n-1}{2} = \frac{(4b+3)\cdot 2^k - 1 - 1}{2} = \frac{(4b+3)\cdot 2^k - 2}{2} = (4b+3)\cdot 2^{k-1} - 1,$$

then we can perform $k-1$ increasing steps obtaining the number $(4b+3)\cdot 3^{k-1} - 1$. This number is again even, but not divisible by 4 (3^{k-1} leaves remainder 1 when divided by 4). Thus, we have to make another increasing step after a decreasing step. Thereby we obtain again

$$\frac{3 \cdot \frac{(4b+3)\cdot 3^{k-1}-1}{2}+1}{2} = \frac{(4b+3)\cdot 3^k - 3}{4} + \frac{1}{2} = \frac{(4b+3)\cdot 3^k - 3 + 2}{4}$$
$$= \frac{(4b+3)\cdot 3^k - 1}{4} = \tilde{n}.$$

We have proven claim C for all even and odd $k > 0$. Thus, the claim is true.

Claim D is true: Suppose that there are starting numbers for which one never ends up at 1. In particular, there is a smallest such number, say n. Then, n cannot be even since otherwise the following number $n/2 < n$ would also be a number with which one would never end up at 1. Contradiction to the minimality of n. Thus, the number n is odd. Furthermore, n does not leave remainder 5 when divided by 6, otherwise it could be written as $n = 6m+5$. Since in this case, the starting number $\tilde{n} = 4m+3 < 6m+5 = n$ would be followed after an increasing step by

$$\frac{3\cdot(4m+3)+1}{2} = \frac{12m+9+1}{2} = 6m+5 = n,$$

so that even with $\tilde{n} < n$ one could never end up at 1 – again a contradiction to the minimality of n. Thus, n can only have remainder 1 or 3 when divided by 6. Therefore, claim D is also true.

Research Reference

These statements, in particular statement C, are useful to prove other properties related to the *Collatz conjecture*: Possibly existing other loops besides from 1, 2, 1, . . . must already have billions of sequence members (see [1]). Moreover, there must be more than 90 ascending and descending passages in these loops (see [2]).

[1] C. Hercher. *Über die Länge nicht-trivialer Collatz-Zyklen*, Die Wurzel **6, 7** (2018).

[2] C. Hercher. *There are no Collatz-m-Cycles with* $m \leq 90$, published in: Journal of Integer Sequences. Preprint: https://arxiv.org/pdf/2201.00406.pdf (2022).

19 Santa Claus Needs Optimal Transport

Author: Fabian Altekrüger (HU Berlin)

Project: *Convolutional Proximal Neural Networks for Solving Inverse Problems* (EF 3-7)

Artwork: Frauke Jansen

Challenge

Every year, Santa encounters the same kind of problem, when he and his helpers organize the transport of the presents to all the children of the world... The elves are supposed to bring the presents from the Christmas factories (F) to secret warehouses (W) on every continent.

F1: This year a secret outpost (F1) in the north of Canada was used to produce 500 loads of presents,

F2: since the main factory (F2) at the north pole has had production problems such that only 400 loads of presents were produced there.

F3: A third factory (F3) at the south pole produced 900 loads of presents.

Although the secret warehouses are already quite well-stocked, some presents are still missing:

W1: in the warehouse in North America (W1) 100 loads are missing;

W2: in South America (W2) 450 loads are missing;

W3: in Africa (W3) 300 loads are missing;

W4: in Australia (W4) 400 loads are missing;

W5: in Europe (W5) 200 loads are missing, and

W6: in Asia (W6) 350 loads are missing.

Since the year was already exhausting enough, and Santa does not want his elves to work overtime, they are searching for an optimal present transport plan from the factories to the secret warehouses minimizing the flight hours. For this purpose, Santa prepares a table containing the travel times (in hours) from each factory to each of the warehouses (see Table 1), taking into account that the factory in Canada (F1) does not produce the correct presents for the warehouses in Africa (W3) and Australia (W4):

	W1	W2	W3	W4	W5	W6
F1	2	4	-	-	5	8
F2	7	8	8	11	3	5
F3	8	6	5	4	9	9

Table 1: The flight durations from each factory (F) to each warehouse (W). No flights are operating from (F1) to (W3) and (W4).

Only 1 load per flight can be transported from one factory to one warehouse. The empty planes are flown back to the factories by a subcontractor called Miracle Transport at a fixed rate, such that there are always enough planes at each factory and the duration of these return trips is not considered in the flight duration Santa wants to minimize.

Still, this task seems a bit to complicated for poor old Santa... Can you help him? What is the minimal total flight duration needed to transport all presents from the factories to the secret warehouses?

Possible answers:

1. 8080 h
2. 8090 h
3. 8100 h
4. 8110 h
5. 8120 h
6. 8130 h
7. 8140 h
8. 8150 h
9. 8160 h
10. 8170 h

Project reference:

Optimal Transport (OT) is a theory that emerged from the analytical description of the classical transportation problem, in which objects (here presents) are to be distributed optimally (i. e., cost minimizing) from different supply points (here the factories) to different demand points (here warehouses). In our example, the costs are given by the flight duration.

As part of project EF 3-7 *Convolutional Proximal Neural Networks for Solving Inverse Problems*, I have been working on the *2-Wasserstein distance*, which is a special case of the OT problem. We use the 2-Wasserstein distance as a regularizer in inverse problems to compare empirical image distributions.

Solution

The correct answer is: 8.

Solution using only elementary math

The problem at hand is an *Optimal Transport problem*; it is particularly easy to solve, since one can follow one's intuition in the distribution of goods from factories to the *nearest* warehouses:

		W1	W2	W3	W4	W5	W6
		100	450	300	400	200	350
F1	500	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	900	8	6	5	4	9	9

1. We start by considering warehouse (W1). Since the nearest factory is (F1), it is most reasonable to transport as much goods as possible from (F1) to (W1), that is all 100 loads needed there. This takes **200 hours** and leaves 400 loads in (F1).

		W1	W2	W3	W4	W5	W6
		0	450	300	400	200	350
F1	400	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	900	8	6	5	4	9	9

2. Next, consider warehouse (W2). Its closest factory is also (F1). Thus, we ship the remaining 400 loads from (F1) to (W2), which takes 1600 hours. The second-nearest factory is (F3), which supplies (W2) with the remaining 50 loads. This takes 300 hours. In total, we need **1900 hours** to fully supply (W2).

		W1	W2	W3	W4	W5	W6
		0	0	300	400	0	150
F1	0	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	850	8	6	5	4	9	9

3. We proceed with warehouse (W3). Its nearest factory is (F3), which supplies all 300 loads needed. This takes **1500 hours**.

		W1	W2	W3	W4	W5	W6
		0	0	0	400	0	150
F1	0	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	550	8	6	5	4	9	9

4. Now, consider warehouse (W4). Its nearest factory is again (F3), which supplies all 400 loads needed. This takes **1600 hours**.

		W1	W2	W3	W4	W5	W6
		0	0	0	0	0	150
F1	0	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	150	8	6	5	4	9	9

5. Let us proceed with warehouse (W5). Its nearest factory is (F2), which can supply all 200 loads needed. This takes **600 hours**.

		W1	W2	W3	W4	W5	W6
		0	0	0	0	0	150
F1	0	2	4	-	-	5	8
F2	200	7	8	8	11	3	5
F3	150	8	6	5	4	9	9

6. At last, focus on warehouse (W6). Its nearest factory is (F2), which can supply 200 loads. This takes **1000 hours**. Then, we need another 150 loads from (F3), which takes 1350 hours. In total, we have a flight duration of **1950 hours**.

		W1	W2	W3	W4	W5	W6
		0	0	0	0	0	0
F1	0	2	4	-	-	5	8
F2	0	7	8	8	11	3	5
F3	0	8	6	5	4	9	9

If we sum the durations from the transports described in 1 to 6, we get a total flight duration of

$$200\,\text{h} + 1900\,\text{h} + 1500\,\text{h} + 1600\,\text{h} + 600\,\text{h} + 2350\,\text{h} = 8150\,\text{h}.$$

But is distribution indeed optimal; that is, is there no other distribution that takes less then 8150 h?

Since warehouses (W1), (W3), (W4), and (W5) are supplied by the nearest factories, we only need to consider (W2) and (W6) for the optimality check. We return to our starting configuration

		W1	W2	W3	W4	W5	W6
		100	450	300	400	200	350
F1	500	2	4	-	-	5	8
F2	400	7	8	8	11	3	5
F3	900	8	6	5	4	9	9

and supply (W6) from its nearest factory (F2). This takes **1750 hours**.

		W1	W2	W3	W4	W5	W6
		100	450	300	400	200	0
F1	500	2	4	-	-	5	8
F2	50	7	8	8	11	3	5
F3	900	8	6	5	4	9	9

Next, we note that we do not save time if we supply warehouse (W1) from factories other than (F1), since it takes much longer from (F2) and (F3) to (W1) than from (F1) to (W2) or (W5). Thus, we cannot supply (W2) solely from its nearest factory (F1). We transport 100 loads from (F1) to (W1), which takes **200 hours**.

		W1	W2	W3	W4	W5	W6
		0	450	300	400	200	0
F1	400	2	4	-	-	5	8
F2	50	7	8	8	11	3	5
F3	900	8	6	5	4	9	9

As in the above case, also (W3) and (W4) can be supplied by their nearest factory (F3), which takes **1500 hours** plus **1600 hours** (see 3. and 4.).

		W1	W2	W3	W4	W5	W6
		0	450	0	0	200	0
F1	400	2	4	-	-	5	8
F2	50	7	8	8	11	3	5
F3	200	8	6	5	4	9	9

For supplying (W2) and (W5), we consider an optimal distribution for the factories: It is best to ship

- 50 loads from (F2) to (W5), which takes 150 hours,
- 150 loads from (F1) to (W5), which takes 750 hours,
- 250 loads from (F1) to (W2), which takes 1000 hours, and
- 200 loads from (F3) to (W2), which takes 1200 hours.

Thus, we need **2000 hours** to supply (W2) and (W5). Again, we obtain a total flight duration of

$$1750\,\mathrm{h} + 200\,\mathrm{h} + 1500\,\mathrm{h} + 1600\,\mathrm{h} + 3100\,\mathrm{h} = 8150\,\mathrm{h}.$$

Mathematically more involved solution using transport matrices

We need to find a *transport matrix* $P \in \mathbb{R}^{3,6}$, where the (i,j)-th entry $P_{i,j}$ describes the number of loads transporting from factory i to warehouse j, that minimizes the total flight duration given by

$$\sum_{i=1}^{3}\sum_{j=1}^{6} C_{i,j}P_{i,j},$$

where $C \in \mathbb{R}^{3,6}$ with

$$C = \begin{bmatrix} 2 & 4 & \infty & \infty & 5 & 8 \\ 7 & 8 & 8 & 11 & 3 & 5 \\ 8 & 6 & 5 & 4 & 9 & 9 \end{bmatrix};$$

that is, (i,j)-th entry $C_{i,j}$ describes the flight duration from factory i to warehouse j.

Above we first constructed the transport matrix

$$P = \begin{bmatrix} 100 & 400 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 200 & 200 \\ 0 & 50 & 300 & 400 & 0 & 150 \end{bmatrix}$$

leading to a flight duration of

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{6} C_{i,j}P_{i,j} &= 2\,\mathrm{h}\cdot 100 + 4\,\mathrm{h}\cdot 400 + 3\,\mathrm{h}\cdot 200 + 5\,\mathrm{h}\cdot 200 + 6\,\mathrm{h}\cdot 50 + 5\,\mathrm{h}\cdot 300 + 4\,\mathrm{h}\cdot 400 + 9\,\mathrm{h}\cdot 150 \\ &= 8150\,\mathrm{h}.\end{aligned}$$

In our second attempt, we derived

$$\hat{P} = \begin{bmatrix} 100 & 250 & 0 & 0 & 150 & 0 \\ 0 & 0 & 0 & 0 & 50 & 350 \\ 0 & 200 & 300 & 400 & 0 & 0 \end{bmatrix}$$

and obtained again a flight duration of

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{6} C_{i,j}\hat{P}_{i,j} &= 2\,\mathrm{h}\cdot 100 + 4\,\mathrm{h}\cdot 250 + 5\,\mathrm{h}\cdot 150 + 3\,\mathrm{h}\cdot 50 + 5\,\mathrm{h}\cdot 350 + 6\,\mathrm{h}\cdot 200 + 5\,\mathrm{h}\cdot 300 + 4\,\mathrm{h}\cdot 400 \\ &= 8150\,\mathrm{h}.\end{aligned}$$

20 The IcePhone 3.14

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

Artwork: Friederike Hofmann

Challenge

For several years now, Santa has been annoyed by the fact that many children and adults wish for a new phone every Christmas, “That’s such a waste! Not sustainable at all!” But many devices actually need to be replaced simply because they fall and break during everyday use. Therefore, Santa has instructed his research department to develop a new indestructible display. The research department at the North Pole is one of the most renowned in the world; so not surprisingly, after only a short time, the researchers are able to present Santa Claus with a phone with an almost unbreakable display made of ice crystal glass, called *Penguin Glass* in the technical jargon: the *IcePhone 3.14* is not only quite pretty, but also can survive falls from very high altitudes.

The research department wants to showcase the device to Santa Claus as spectacularly as possible and assigns two clever elves to determine the highest floor of the 141-story Polar Star Tower from which such a phone can fall without breaking. For experimentation purposes, the research department provides them with two brand-new IcePhones. Of course, the same IcePhone can be dropped several times until it breaks. Since the two elves are brilliant, but not particularly fit, they want to solve the problem with as few drops as possible.

What is the minimum number of drops that is needed to determine the highest safe floor in worst case?

Possible answers:

1. 11
2. 12
3. 13
4. 14
5. 15
6. 16
7. 17
8. 18
9. 19
10. 20

Solution

The correct answer is: 7.

We solve this problem "backwards": instead of directly calculating the minimum number of drops that is needed for a building with 141 floors, we determine the the maximum number of floors $F(n)$ for which the problem can be solved in n drops.

The first drop has to be made from floor n, because if the IcePhone breaks, the elves will need to sequentially test the first $n-1$ floors with the second IcePhone. If the first drop does not break the IcePhone, the second drop has to be made from floor $n+(n-1)$. This way, the elves are prepared for the possibility that if the IcePhone breaks, each of the $n-2$ floors from floor $n+1$ to floor$2n-2$ need to be tested sequentially. On repeating this argument for the remaining $n-2$ drops, we obtain the following formula $F(n)$:

$$F(n) = n + (n-1) + (n-2) + \ldots + 1 = \frac{n(n+1)}{2}.$$

Now, it remains to calculate the smallest number of drops n' such that

$$F(n') = \frac{n'(n'+1)}{2} \geq 141.$$

This number is $n' = 17$, for which $F(n') = F(17) = \frac{17 \cdot 18}{2} = 153$.

21 Elves on Strike

Author: Lara Glessen (TU Berlin)
Projekt: MATH+ School Activities

Artwork: Julia Nurit Schönnagel

Challenge

It's December and there's way too much to do in Santa's gift factory ... The working conditions of the ten elves working for him are even more terrible than during the rest of the year. So they get together and promptly decide to go on strike. Just as one of the elves, Alexis, agrees to work out the schedule for the strike until tomorrow morning, Santa Claus bursts in, puts two and two together and immediately knows what the elves are up to. Anxious to avoid a strike so close to Christmas at all costs, Santa announces that starting tomorrow, all the elves will have to work individually in their offices and will no longer be allowed to talk to each other. He then leave—pleased with himself and believing that he has prevented the planned strike—the gift factory.

But the elves sit together for a while, thinking back and forth about how they can still share the yet-to-be-developed strike schedule tomorrow. That's when Alexis says, "I have a plan! We're all being sent to the copy room over and over again. Santa will make sure that there are never more than one of us in there and that we don't leave any messages for each other. However, there is a secret locker in the wall for which we all have a key! Tomorrow morning before 6 am, I will put the strike schedule into the locker and lock it afterwards. Every time I find the locker unlocked, I will lock it again. Now, we just have to figure out a strategy to make sure that I know all of you have read the schedule. Because as soon as I know that, I'm going to call for an immediate strike over the speakers."

How should Alexis instruct *the other nine elves* to unlock / lock the locker in the copy room so that Alexis at some point knows with absolute certainty that all nine elves have seen the strike schedule, even though they have *no* influence on the order in which they are sent to the copy room?

Possible answers:

1. Every time they are sent to the copy room, they unlock the locker if it is locked. Otherwise, they do nothing.
2. Every time they are sent to the copy room, they lock the locker if it is unlocked. Otherwise, they do nothing.
3. Every time they are sent to the copy room, they unlock the locker if it is locked, and lock it if it is unlocked.
4. The first time they are sent to the copy room, they unlock the locker and then lock it again. Otherwise, they do nothing.
5. Every odd time they are sent to the copy room, they unlock the locker if it is locked. Otherwise, they do nothing.
6. Every even time they are sent to the copy room, they unlock the locker if it is locked. Otherwise, they do nothing.
7. Every odd time they are sent to the copy room, they lock the locker if it is unlocked. Otherwise, they do nothing.
8. The first time they find the locker locked, they unlock it. Otherwise, they do nothing.
9. The first time they find the locker unlocked, they lock it. Otherwise, they do nothing.
10. The first time they find the locker locked, they unlock it. And the first time they find it unlocked, they lock it. Otherwise, they do nothing.

Solution

The correct answer is: 8.

Crucial for solving this puzzle is that not all elves have the same role, but that Alexis plays a special role. First, we show that in scenario 8 Alexis knows with absolute certainty at a particular that all the other nine elves were in the copy room.

So, consider **scenario 8**: the first time an elf (other than Alexis) finds the locker locked, the elf unlocks it; on all other occasions, the elf does nothing. Whenever Alexis finds the locker unlocked, Alexis locks it. Thus, after the ninth time Alexis knows that all the other nine elves must have unlocked it once, since no one unlocks it more than once. So at this point, Alexis can call a strike.

It remains to show that all other answer choices are not goal-directed strategies.

Scenarios 2, 7, and 9: Since there are no elves who ever unlock the locker, nothing happens. Accordingly, Alexis never receives any information about whether elves, or even how many, have been in the copy room.

Scenarios 3 and 4: In these two scenarios, the locker is indeed unlocked and locked. However, the two actions are always performed together during one visit to the copy room. This means that Alexis does not notice anything about the opening or closing and therefore does not receive any information—just as in scenarios 2, 7 or 9.

Scenarios 1, 5, and 6: In these scenarios, Alexis learns that someone was in the copy room when Alexis enters the room and the locker is unlocked. However, Alexis has no way of knowing whether the same or different elves always unlock the locker. In particular, Alexis cannot know when all the elves have unlocked the locker at least once.

Scenario 10: Whether Alexis gets enough information depends on the order in which the elves are sent to the copy room. For simplicity, we number the elves from 1 to 10, assigning number 1 to Alexis, and look at two possible orders:

1) 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 1, 10, 1.
2) 2, 2, 1, 3, 3, 1, 4, 4, 1, 5, 5, 1, 6, 6, 1, 7, 7, 1, 8, 8, 1, 9, 9, 1, 10, 10, 1.

In the first case, after Alexis enters the copy room for the ninth time, Alexis knows that all nine elves have read the plan in the locker. In the second case, however, just as in scenarios 3 and 4, Alexis receives no information at all, because the locker is always locked when Alexis enters the room. Of course, since the elves do not know what order they will be sent to the copy room the next day, they have to come up with a plan that works for all orders. However, we have already found one where Alexis cannot know when all the elves have read the plan. Accordingly, answer 10 is also incorrect.

22 A Special Game

Authors: Alex McDonough (UC Davis), Ulrich Reitebuch (FU Berlin),
Martin Skrodzki (TU Delft)

Artwork: Friederike Hofmann

Challenge

The games department of the Christmas Research Center is constantly developing new games to keep the children of the world from getting bored. This year, it has come up with a game especially for only children: it is a solitaire game played on a board with $n \times 3$ fields.

Each field can hold exactly one token. There are three different types of tokens and n tokens of each type: squares, circles, and diamonds. For each type, the n tokens are labeled with numbers 1 to n.

A placement of all tokens on the board is called *stable* if:

(a) In no row a square is to the right of a circle or a diamond.
In no row a circle is to the right of a diamond.

(b) In columns, the vertical relations on labels as given in Figure 23 hold:

That is, the numbered squares, circles, and diamonds have to be placed according to the inequalities given in Figure 23. For instance,

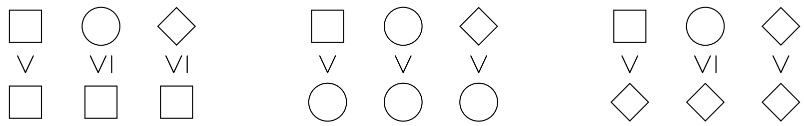


Figure 23: The vertical relations on labels.

- a square numbered i can be below a circle or diamond numbered j with $i \leq j$,
- but a if a circle numbered i is under a square, then the square's label needs to be j with $i < j$.

(c) In rows, the horizontal relations on labels as given in Figure 24 hold, even if the respective tokens are *not* next to each other:

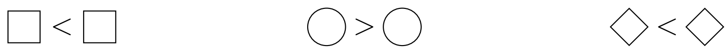
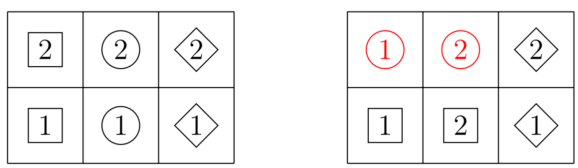


Figure 24: The horizontal relations on labels.

Examples of a stable and an unstable placement for the 2×3 board is depicted in Figure 25:



stable unstable

Figure 25: A stable and an unstable placement on the 2×3 board. The left placement of the tokens is stable; all relations are satisfied. The right placement of the tokens is unstable: while all columns as well as the lower row satisfy all relations, the upper row violates rule (c), since the circle with larger number is placed to the right of the circle with the smaller number (marked in red).

Note that while there are restrictions on labels across different types of tokens when placed *vertically*, there are only restrictions on *horizontal* labels within one token type. That is, if two tokens of the same type are in the same row, they have to be ordered from left to right according to their label (even if they are not next to each other). In rows, two tokens of different types only have to satisfy rule (a), independent of their labeling. That is, a square labeled 5 can be left of a circle labeled 3.

Question: How many *stable* placements of the tokens are there on a 4×3 board?

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. $3 \cdot 4$
6. $3! \cdot 4$
7. $4! \cdot 3$
8. $3! \cdot 4!$
9. $(3 \cdot 4)!/3!^4$
10. $(3 \cdot 4)!$

Project reference:

The presented task is part of an open question in a current combinatorics project. In the project, we investigate an object called the *neighborhood grid*, which has connections to finding relationships in data sets (for more details, see: `https://arxiv.org/abs/1710.03435`). Roughly speaking, the data structure will speed-up the performance of specific algorithms such as particle or cell simulations by providing fast, approximate answers to the question: For a given particle or cell, what are the nearest other particles or cells around it?

Solution

The correct answer is: 1.

There is only one stable placement of the tokens. Checking for this solution amounts to going through all combinatorial possibilities of placing the tokens and verifying that only one solution is valid, namely that of placing all squares in the left column, all circles in the center column, and all diamonds in the right column, sorted by their labels from bottom to top. Next to brute-forcing the solutions, e. g., with the help of a computer, it is helpful to deduce limited positions for certain tokens.

Consider the square with label 1. By rule 1, it has to be left of any circle or diamond; i. e., it cannot be in a row such that it is in the second or third column with circles or diamonds next to it. Furthermore, squares in a row have to be ordered from left to right according to their label. Thereby, the square has to be left of all other squares, as it has the lowest available label.
A similar reasoning holds for the vertical placement. Squares, circles, and diamonds have to satisfy a strict inequality when being placed below a square. But the considered square has the lowest label available. Hence, it has to go in the lowest column. Combining these two observations pins the square in the lower left corner.

Now consider the diamond with label 4. By a similar reasoning as above, since the diamond has the highest label available, no diamond can be right to it. Furthermore, following rules 1 and 2, no squares or circles can be right to it. Hence, the diamond with label 4 has to be in the right column.
Furthermore, following the vertical rules from Figure **??**, a diamond has to satisfy strict inequality to squares and other diamonds. Since the diamond with label 4 has the highest label available, it can either go to the top row or sit below a circle of label 4.

One can now continue with this sort of reasoning, e. g., by placing the circle of label 4, in case it has not been fixed by the choice of placing the diamond of label 4. This provides a growing decision tree as to where the remaining tokens can go on the board.

Apart from checking all combinations of other placements, no solution that holds for general n is known yet (see project connection). Going through all possibilities reveals that a continuation of the stable state shown in Figure **??**, i. e., placing all squares in the left, all circles in the center, and all diamonds in the right column, ordered by their label is the only stable placement possible.

Relation to current research

The combinatorial question we are currently investigating is, whether for any $n \times m$ board, there is a certain class of objects such that they allow only one stable placement.

The presented exercise is a special case for $m = 4$ of the open question, which asks: For any n, is there only one stable placement for the tokens? So far, we were able to check that the answer is yes for up to $n = 7$, which amounts to checking $21! \approx 5.11 \cdot 10^{19}$ different placements. We conjecture that the presented tokens have only the one placement solution for any n; i. e., placing all squares in the left, all circles in the center, and all diamonds in the right column. However, we cannot prove it at this point.

If you have suggestions or prove ideas for how to tackle the problem, please get in touch:
`m.skrodzki@tudelft.nl`.

23 Robin Hood's New Arrows

Author: Jan Marten Sevenster (FU Berlin)
Project: *Quiver Representations in Big Data and Machine Learning* (EF 1-16)

Artwork: Till Hausdorf

Challenge

Robin Hood was given 32 brand new arrows for Christmas last year, which he meticulously ordered in a quiver by fixing them to certain points (see Figure 26). Today, Robin wants to go out into the snowy Sherwood Forest to finally try out the new arrows. However, as he shoulders his bow and quiver, he notices that the tips of three arrows are not looking down, but dangerously pointing upwards. So Robin could hurt himself if he reaches for the arrows.

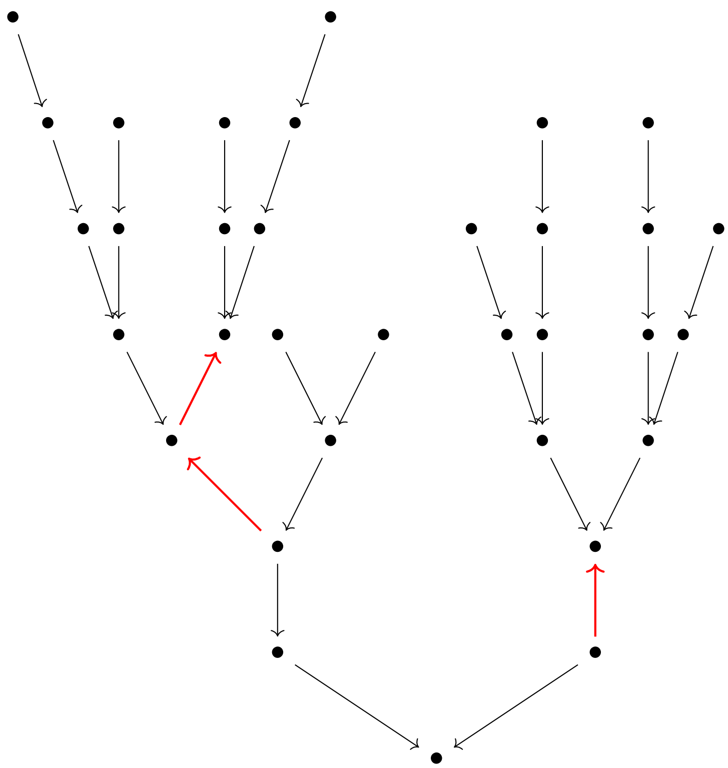


Figure 26: The exact arrangement of the arrows in Robin's quiver. The red arrows are upside down in the quiver.

For reasons only known to experienced archers, you can only change the orientation of the arrows in your quiver by choosing a point that has only arrowheads attached to it and flipping *all* the arrows attached to this point (see Figure 27).

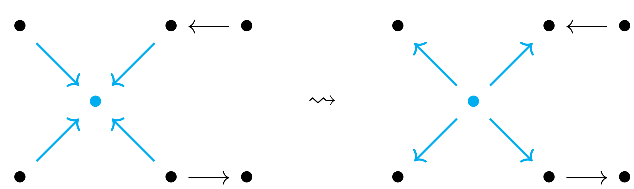


Figure 27: An example of a permitted operation.

What is the units digit of the minimum number of operations needed so that eventually all arrows are pointing down?

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Project reference:

Quiver representations occur in the description and analysis of Big Data or neural networks. In the project EF1-16 *Quiver Representations in Big Data and Machine Learning* algorithmic and algebro-geometric methods are combined to classify quiver representations from Big Data and to analyze the geometry of the moduli space of a neural network. The operations described in the task are so-called *Bernstein-Gelfand-Ponomarev reflections.*

Solution

The correct answer is: 2.

The number of operations required to change an arrow equals the number of operations required to reach a state in which you are allowed to perform an operation on the point that the arrowhead in question is attached to, plus one for performing the operation that corrects your arrow, plus the number of operations required to correct the arrows that are now unjustly flipped.

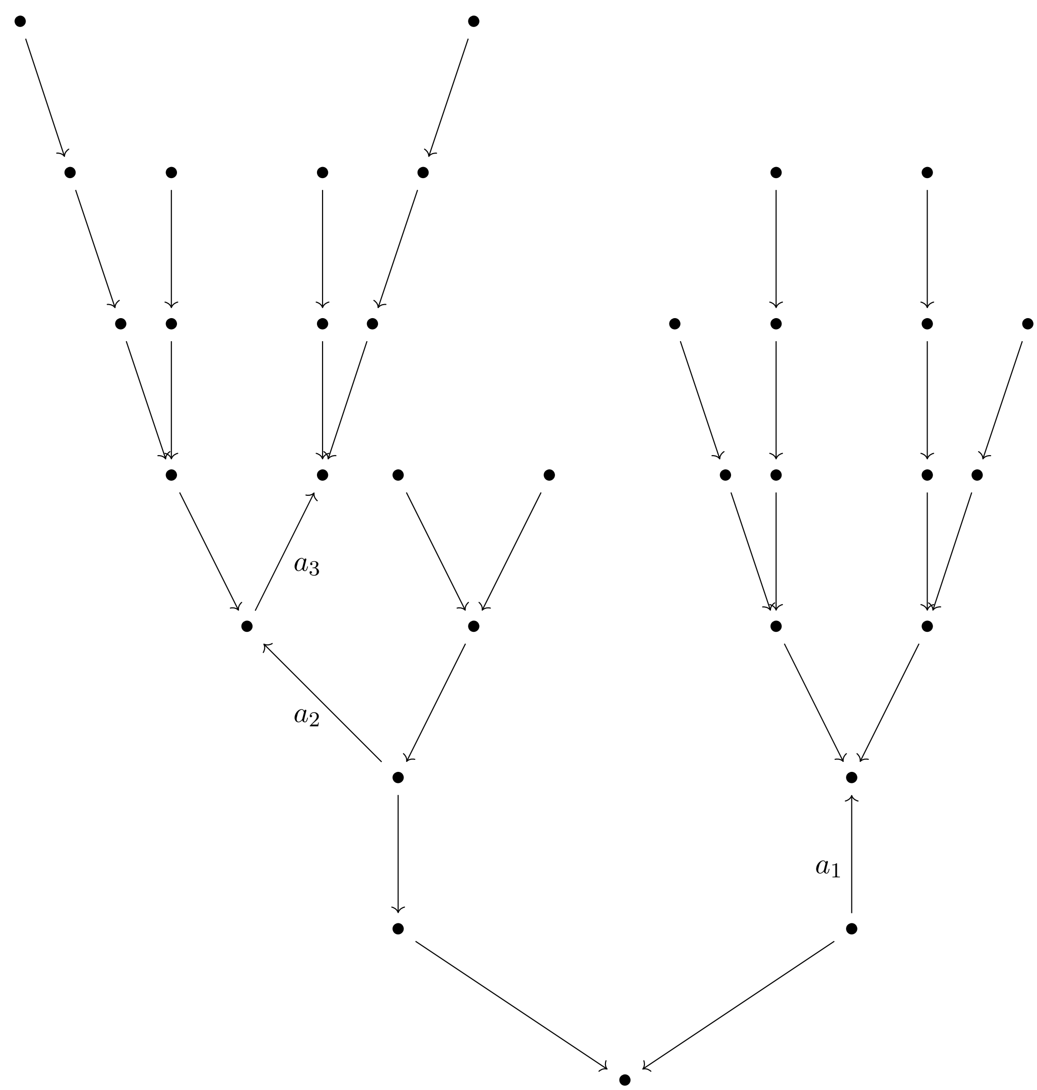


Figure 28: The arrows in the quiver.

The number of flips required to adjust an arrow for which all arrows above it are pointing in the right direction equals the number of points above that arrow, including the point the arrowhead is attached to. This is illustrated for the arrow a_3 in Figure 29:

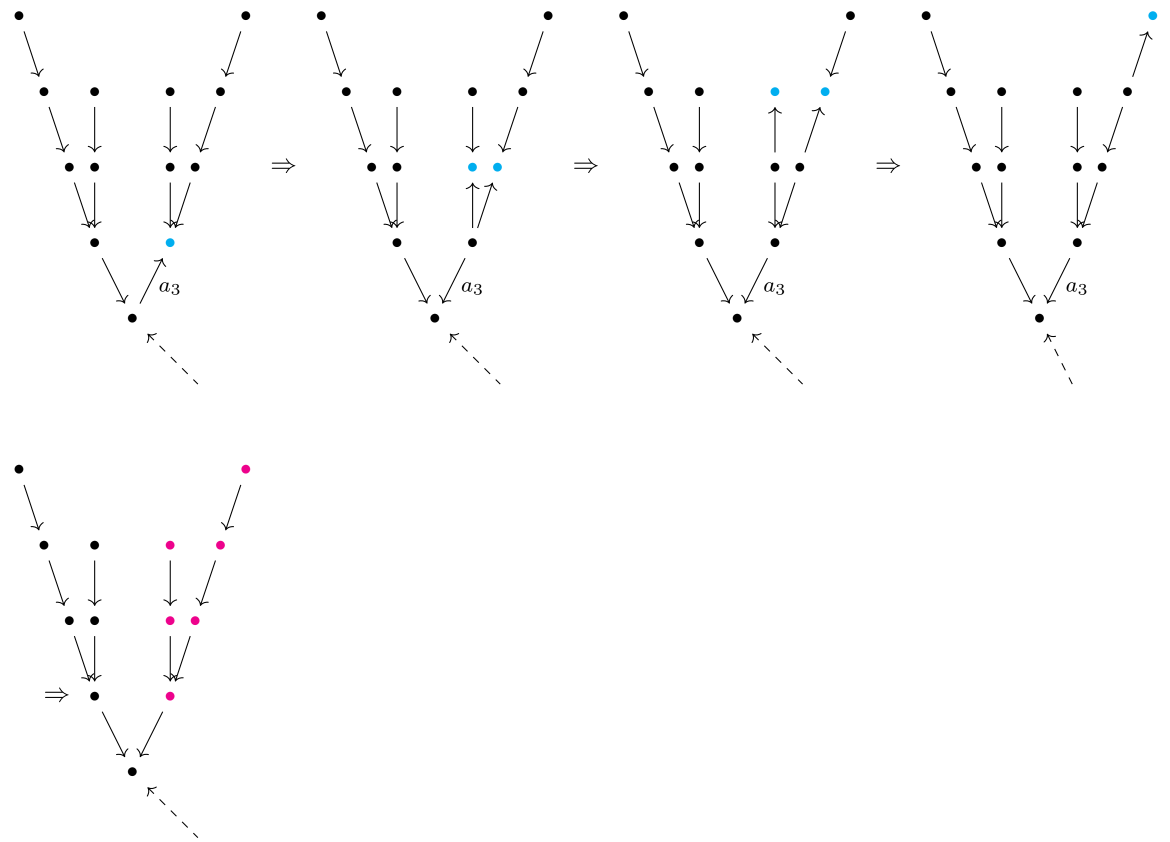


Figure 29: Illustration of how to correct a_3. The points on which an operation is performed in each step are indicated in blue. Starting in the first situation, you perform an operation on the point in blue that the arrow a_3 is pointing towards. The situation you end up in is drawn adjacently to the left. In this situation, it is allowed to perform operations on the two points above the one from the first step and these are now in blue etc. Notice that these operations are allowed because blue points are only connected to arrows pointing towards the blue points. There are six blue points in total, so the number of operations required to adjust a_3 and leave all the other arrows uncorrupted is six. The pink points after the final step are also six. An operation has been performed on all these points exactly once. To safe space in this illustration, the dashed arrow represents a part of the quiver that is not relevant for the performed operations.

Because you first have to flip a_3 in order to be allowed to flip a_1, the total number of operations required equals

$$op_{tot} = op_{a_3} + op_{a_1} + op_{a_2}.$$

So counting the numbers of points, you find

$$op_{tot} = 6 + 13 + 13 = 32.$$

Figure 29 certainly convinces us that $op_{a_3} = 6$ and similarly $op_{a_1} = 13$. However, to see that we indeed cannot adjust the orientation of a_2 in a cleverer way that allows us to adjust a_3 and a_2 with fewer operations than it would take to adjust them individually, we remember the remark in the first paragraph of the solution and reason as follows:

We call the points that the arrows a_2 and a_3 are pointing towards in the initial configuration of Figure 26 p_2 and p_3, respectively. To adjust the arrow a_2, we have to perform an operation on either p_2 or on the other point that a_3 is connected to. But in order to reach a state in which we are allowed to perform an operation on that other point, we must already have flipped the arrow a_2 an odd number of times so that it points in the opposite direction. This can never yield the fastest possible sequence of operations; so we conclude we have to perform an operation on p_2. Since the arrow a_3 is not yet pointing towards the point p_2, we must first flip the arrow a_3 to point in the other direction by performing an operation on the point p_3. After we performed an operation on p_3 and consequently on p_2, the arrow a_3 has been flipped an even number (exactly two) of times; so it is once again pointing in the wrong direction. However, we are no longer allowed to perform an operation on p_3, since the other two arrows that are connected to p_3 are pointing away from p_3 ever since we performed the first operation on p_3. So we first must reach a state in which it is allowed to perform an operation on p_3. Repeating this logic for all the points above p_3, we see that we are still forced to perform the operations that we would have done if we had first performed the six operations required to adjust the orientation of only a_3 (illustrated in Figure 29) and only then started to worry about adjusting a_2, but in a different order.

24 Decorations for New Year's Eve

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

Illustration: Frauke Jansen

Challenge

Christmas elf Annelie would like to craft some decorations for her New Year's Eve party. Since she is both mathematically interested and a talented craftswoman, Annelie has constructed a crescent moon from two circles (see Figure 30), which she wants to saw out and paint: the larger circle l is the incircle of a square $ABCD$ with side length 1. The smaller circle s is the incircle of the triangle MBC, where M is the midpoint of the line segment $\overline{AD}$.

In her workshop, Annelie has enough wood for the task. She only has to get the special fairy dust paint from ELFZON.NP, the fair E-commerce company at the North Pole. Since the fairy dust paint is quite expensive, Annelie wants to buy only as much of it as she really needs. To do this, she calculates the area of the crescent very precisely.

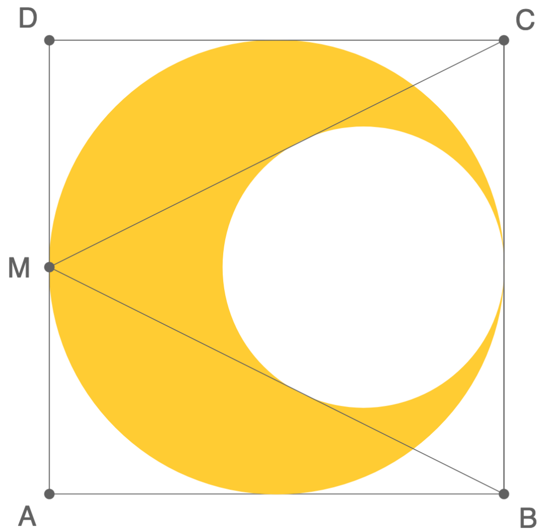


Figure 30: The crescent constructed by Annelie.

Let

- r_s and A_s be the radius and area of the smaller circle s and
- r_l and A_l be the radius and area of the larger circle b.

What is the second decimal place of $\frac{r_s}{r_l} + \frac{A_s}{A_l}$?

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

Solution

The correct answer is: 10.

We show that

$$\frac{A_k}{A_g} + \frac{r_k}{r_g} = 1$$

holds independently of the side length $a > 0$ of the square $ABCD$.

We have

$$r_g = \frac{a}{2} \quad \text{und} \quad A_g = \frac{\pi}{4}\, a^2.$$

We now calculate r_k and A_k. Therefore, we denote the center of the inner circle of the triangle MBC by K. Let N, O, and P be the base points of the perpendiculars of K to $\overline{BC}$, $\overline{MB}$, and $\overline{MC}$ respectively (see Figure 31).

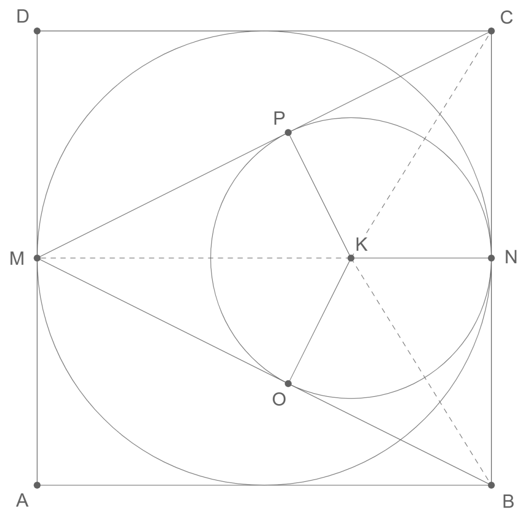


Figure 31: The crescent.

It holds

$$r_k = |\overline{KN}| = |\overline{KO}| = |\overline{KP}|.$$

As the circles g and k are axially symmetric with respect to $\overline{MN}$, we have

$$|\overline{CN}| = |\overline{NB}| = \frac{a}{2}.$$

Further, we know

$$|\overline{PC}| = |\overline{CN}| = \frac{a}{2} \quad \text{und} \quad |\overline{OB}| = |\overline{NB}| = \frac{a}{2},$$

since the respective distances are the tangent sections to the small circle k of C and B and thus of equal length.

If we apply the Pythagorean theorem to the right triangle ABM, we obtain

$$|\overline{MB}| = \sqrt{a^2 + \frac{a^2}{4}} = \frac{\sqrt{5}}{2}\, a$$

yielding

$$|\overline{MO}| = |\overline{MB}| - |\overline{OB}| = \frac{\sqrt{5}}{2}\, a - \frac{a}{2} = \frac{\sqrt{5}-1}{2}\, a.$$

By applying the Pythagorean theorem again to the right triangle MOK, we find

$$\begin{aligned}
|\overline{MK}|^2 &= |\overline{MO}|^2 + |\overline{KO}|^2 \\
\left(|\overline{MN}| - |\overline{KN}|\right)^2 &= |\overline{MO}|^2 + |\overline{KO}|^2 \\
(a - r_k)^2 &= \frac{(\sqrt{5}-1)^2}{4}\, a^2 + r_k^2 \\
a^2 - 2ar_k + r_k^2 &= \frac{5 - 2\sqrt{5} + 1}{4}\, a^2 + r_k^2 \\
-2ar_k &= \left(\frac{3-\sqrt{5}}{2} - 1\right) a^2 \\
r_k &= \frac{\sqrt{5}-1}{4}\, a
\end{aligned}$$

resulting in

$$A_k = \pi r_k^2 = \frac{\pi(3-\sqrt{5})}{8}\, a^2.$$

We now substitute the calculated values into the term we want to calculate and obtain the equation

$$\frac{A_k}{A_g} + \frac{r_k}{r_g} = \frac{\pi(3-\sqrt{5})a^2}{8} \cdot \frac{4}{\pi a^2} + \frac{(\sqrt{5}-1)a}{4} \cdot \frac{2}{a} = \frac{3-\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2} = 1,$$

which holds independently of the value of the side length $a > 0$ of the square $ABCD$.

Remark: This problem was set in the school round of the 41st German Mathematics Olympiad for grades 11-13. Together with its solution, it can be in the book *Die schönsten Aufgaben der Mathematik-Olympiade in Deutschland* (ISBN 978-3-662-63183-6).