Challenges & Solutions
2021

www.mathekalender.de

Содержание

1 Trip Planning

Author: Jonas Lorenz (TU Berlin)
Project: MATH+ School Activities

Challenge

Every year on Christmas Eve, Santa travels the cities Amsterdam (A), Berlin (B), Chicago (C), Delhi (D), Edinburgh (E), Florence (F), Gothenburg (G), Hong Kong (H), and Innsbruck (I) in a specific order and adapts his whole flight route to it. This year, the elves Madina and Elias have received the job of planning this route and decided to draw lots to get the order of the previously mentioned cities. Now, to write down the routes, which encompass each city exactly once, on the lots, Elias sorts the "route-words" alphabetically. For example, given three cities A, B, and C, Elias would produce the following six tickets:

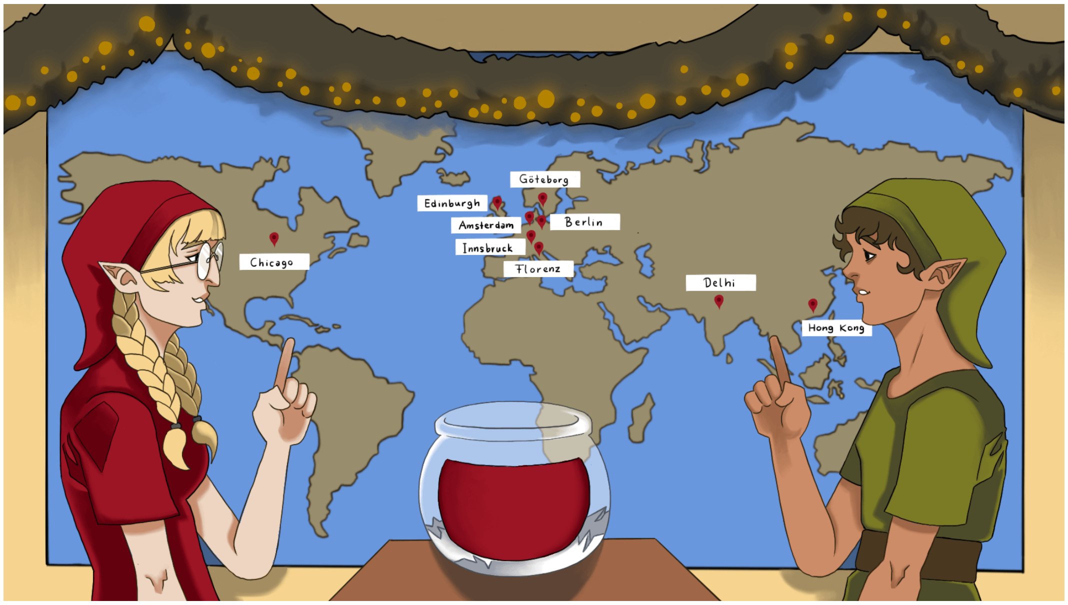
Ticket no. 1: "A-B-C", Ticket no. 2: "A-C-B", Ticket no. 3: "B-A-C",
Ticket no. 4: "B-C-A", Ticket no. 5: "C-A-B", Ticket no. 6: "C-B-A".

After having written the specific "route-words" onto every ticket, the elves throw them into a bowl. Elias shuffles the tickets a little, draws one of them, and reads out the route-word that is written on it:

"H-F-B-D-I-G-A-C-E".

Madina ponders for a while and then proclaims the ticket's number. Quite impressed, Elias confirms her statement.

What is the number of the drawn ticket?



Artwork: Julia Nurit Schönnagel

Possible answers:

1. 308,510
2. 308,511
3. 308,512
4. 308,513
5. 308,514
6. 308,515
7. 308,516
8. 308,517
9. 308,518
10. 308,519

Solution

The correct answer is: 6.

Given n cities, we can easily determine how many different routes of those cities exist when each city is only allowed to appear once in the route. We get n possibilities for the first stop on the route, $n-1$ possibilities for the second, and so on. Thus resulting in

$$n \cdot (n-1) \cdot \ldots \cdot 1 = n!$$

routes. So whenever we fix one of the n cities from our route, the remaining number of routes becomes $(n-1)!$.

To get the correct number of the ticket, we first realize that for every city there are

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$$

different possible Christmas routes. Since Hong Kong is the eighth city in the alphabetical order of our cities, none of the first

$$7 \cdot 8! = 7 \cdot 40{,}320 = 282{,}240$$

tickets starting with a city that comes before Hong Kong alphabetically can be the one we are looking for.
Since Florence is alphabetically the sixth city, we deduce that none of the next $5 \cdot 7! = 25{,}200$ tickets (starting with H-A, H-B, H-C, H-D, H-E) is the drawn ticket. Applying the same logic, we know that the following $6! = 720$ tickets will not be the drawn one, since H-F-A comes before H-F-B.

The next city on the ticket is Delhi. Now the solution gets a bit more tricky: since Berlin was already part of the route, we may not count it again. Thus, we only have to rule out the $2 \cdot 5! = 240$ routes continuing with Amsterdam and Chicago. Similarly, for the next city, Innsbruck, we can safely say that the next $4 \cdot 4! = 96$ tickets cannot be the wanted one, for all the routes starting with H-F-B-D-(A or C or E or G) are not the one on our ticket. Since the next city on our ticket is Gothenburg, we have to rule out another $3 \cdot 3!$ tickets, those being the ones that would continue the route not with Gothenburg but with Amsterdam, Chicago or Edinburgh.

Now, however, we are already finished, because Amsterdam, Chicago, and Edinburgh appear in the route in alphabetical order. Therefore, they are on the ticket with the lowest number which also starts with H-F-B-D-I-G.

Summarising, we know that none of the first

$$282{,}240 + 25{,}200 + 720 + 240 + 96 + 18 = 308{,}514$$

tickets are the drawn one. Hence, it has to be the following one, the ticket with the number 308,515.

2 Heads or Tails

Author Jacques Resing (TU Eindhoven)
Project: 4TU.AMI

Challenge

In the coin collection of Santa Clause, you will find a coin from the Republic of Heddonia. If you flip this coin, it shows heads with a probability of 3/4 and tails with a probability of 1/4. The coin collection also contains another coin from the Kingdom of Tailland, which shows tails with a probability of 3/4 and heads with a probability of 1/4.

Santa Clause flips both coins simultaneously; one coin comes up heads and the other one tails.

What is the probability p that the coin that comes up heads is the one from the Republic of Heddonia?

Artwork: Friederike Hofmann

Possible answers:

1. $p \leq 0.55$ holds.

2. $0.55 < p \leq 0.60$ holds.
3. $0.60 < p \leq 0.65$ holds.
4. $0.65 < p \leq 0.70$ holds.
5. $0.70 < p \leq 0.75$ holds.
6. $0.75 < p \leq 0.80$ holds.
7. $0.80 < p \leq 0.85$ holds.
8. $0.85 < p \leq 0.90$ holds.
9. $0.90 < p \leq 0.95$ holds.
10. $0.95 < p$ holds.

Solution

The correct answer is: 8.

The probability that the coin from Heddonia shows heads and the coin from Tailland shows tails is $p_1 = (3/4) \cdot (3/4) = 9/16$.

The probability that the coin from Heddonia shows tails and the coin from Tailland shows Heads is $p_2 = (1/4) \cdot (1/4) = 1/16$.

Since p_1 is exactly nine times as large as p_2, the probability we are looking for is $p = 9/10$.

3 Cinnamon Stars

Author Frits Spieksma (TU Eindhoven)
Project: 4TU.AMI

Challenge

On the table, there are an empty red bowl, an empty black bowl, and 16 cinnamon stars. To pass the time, Ruprecht plays a game. In each move, Ruprecht *either* takes a cinnamon star from the table and puts it into one of the bowls, *or* he takes a cinnamon star from one of the bowls and puts it back on the table. Ruprecht sticks to the following rules:

- At the end of every move, the red bowl contains at least as many cinnamon stars as the black bowl.
- If at the end of a move the red bowl contains exactly R cinnamon stars and the black bowl exactly B cinnamon stars, then Ruprecht is not allowed to have exactly R cinnamon stars in the red bowl and B cinnamon stars in the black bowl at the end of any of the later moves.

What is the maximal number M of moves that Ruprecht can perform under these rules?

Artwork: Friederike Hofmann

Possible answers:

1. The maximal number is $M = 67$.
2. The maximal number is $M = 68$.
3. The maximal number is $M = 69$.
4. The maximal number is $M = 70$.
5. The maximal number is $M = 71$.
6. The maximal number is $M = 72$.
7. The maximal number is $M = 73$.
8. The maximal number is $M = 74$.
9. The maximal number is $M = 75$.
10. The maximal number is $M = 76$.

Solution

The correct answer is: 6.

We represent a situation with R cinnamon stars in the red bowl and S cinnamon stars in the black bowl by the point (R, S) in the Cartesian coordinate system; its x-coordinate given by R, and its y-coordinate given by S. Then, the possible game situations are described by those points (R, S) that fulfil the conditions $0 \leq R \leq 16$ and $0 \leq S \leq 16$ and $R \geq S$ and $R + S \leq 16$ according to the rule (see Figure 1).

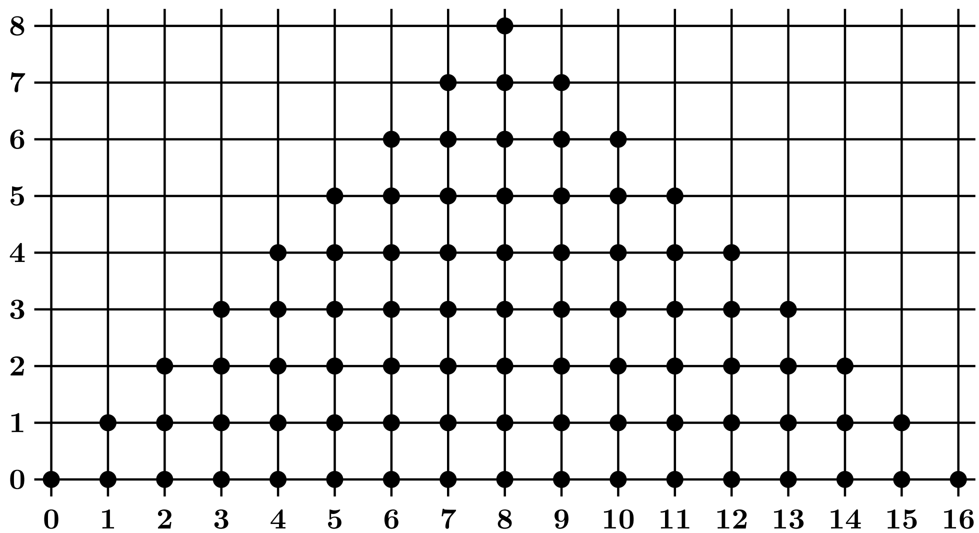


Рис. 1: Points according to the rules (R, S).

Ruprecht's moves are represented by a walk on this set of points:

- If Ruprecht puts a cookie in the red bowl, he goes from (R, S) to $(R + 1, S)$.
- If he puts a cookie in the black bowl, he goes from (R, S) to $(R, S + 1)$.
- If he takes a cookie from the red bowl, he goes from (R, S) to $(R - 1, S)$.
- If he takes a cookie from the black bowl, he goes from (R, S) to $(R, S - 1)$.

Since both bowls are empty at the beginning of the game, Ruprecht starts his walk at $(0, 0)$. Then, he always walks to a horizontally or vertically adjacent grid point without visiting the same point more than once.

Now, we will derive an upper bound for the number of moves. Each move takes Ruprecht either from a point (R, S) with an even coordinate sum $R + S$ to a point with an odd coordinate sum, or from a point with an odd coordinate sum to a point with an even coordinate sum. One easily sees (or counts) that there are 36 different game situations with an odd coordinate sum and 45 game situations with an even coordinate sum. Since Ruprecht starts at the point $(0, 0)$ with an even coordinate sum, he will visit a point with an odd coordinate sum in every odd move. Therefore, Ruprecht can visit at most $1+2\cdot 36 = 73$ points; thus, making at most 72 moves.

Finally, Figure 2 shows us a sequence with exactly 72 admissible moves. Hence, the upper bound 72 can be achieved.

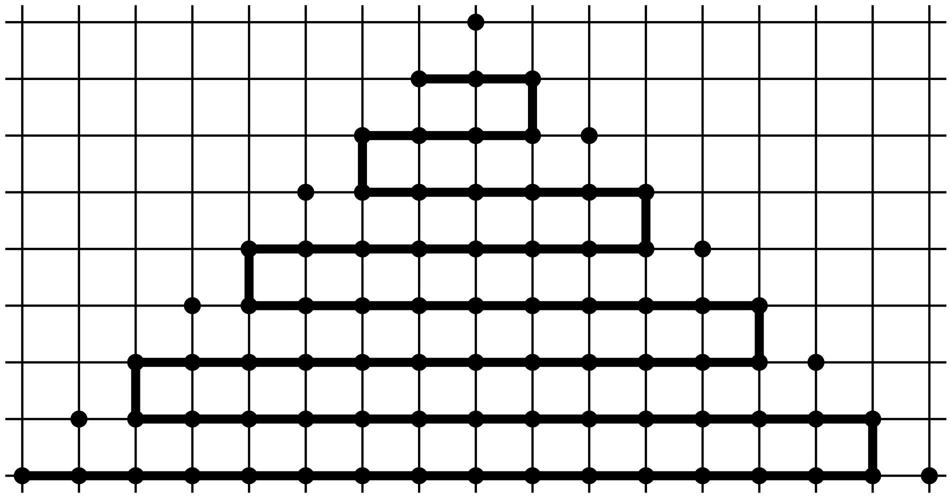


Рис. 2: A sequence of 72 admissible moves.

4 Sawmill

Author: Hennie ter Morsche (TU Eindhoven)
Project: 4TU.AMI

Challenge

The sawing elves Zick and Zack are working in the sawmill of Santa Clause. In the morning, they receive four wooden square-shaped panels, each of dimensions $4\,m \times 4\,m$, and a work schedule. To fulfil their tasks, Zick and Zack always make use of a magic blade that has zero width.

- Part A of their work schedule asks them to cut the first panel into four congruent quadrilaterals such that each quadrilateral has a circumcircle of diameter $\sqrt{8}\,m$.
- Part B asks them to cut the second panel into four congruent quadrilaterals such that each quadrilateral has a circumcircle of diameter $\sqrt{10}\,m$.
- Part C asks them to cut the third panel into four congruent quadrilaterals such that each quadrilateral has a circumcircle of diameter $\sqrt{12}\,m$.
- Part D asks them to cut the fourth panel into four congruent quadrilaterals such that each quadrilateral has a circumcircle of diameter $\sqrt{17}\,m$.

Zick scratches his head and complains, “This work schedule is extremely imprecise. It does not tell us the exact shape of these congruent quadrilaterals.”
Zack also scratches his head and laments, “Perhaps our central administration has once again given us on an impossible task. This has happened before!”

Can you help Zick and Zack?

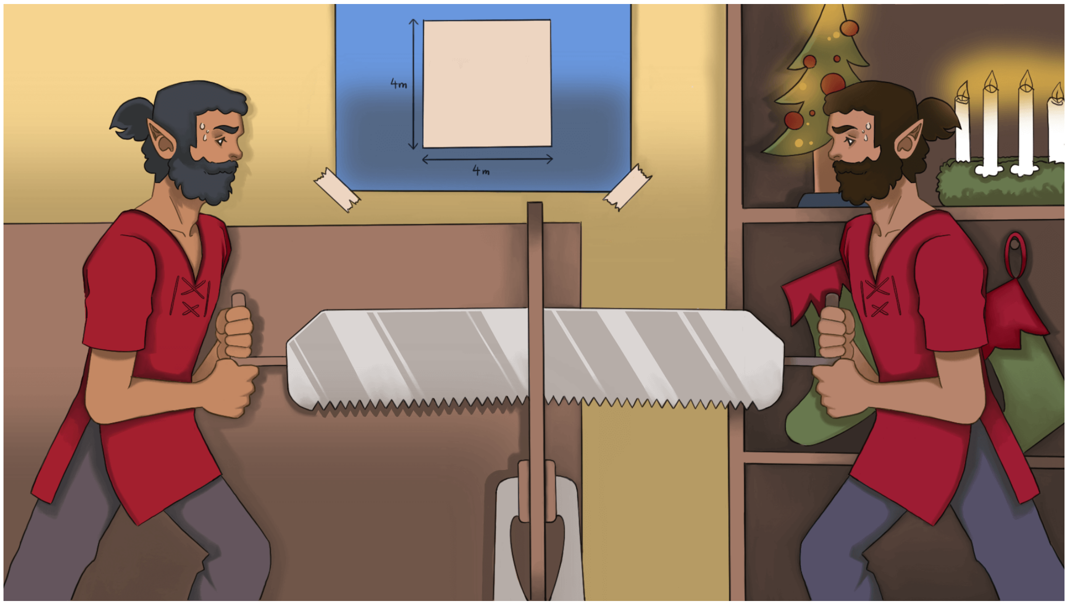


Illustration: Julia Nurit Schönnagel

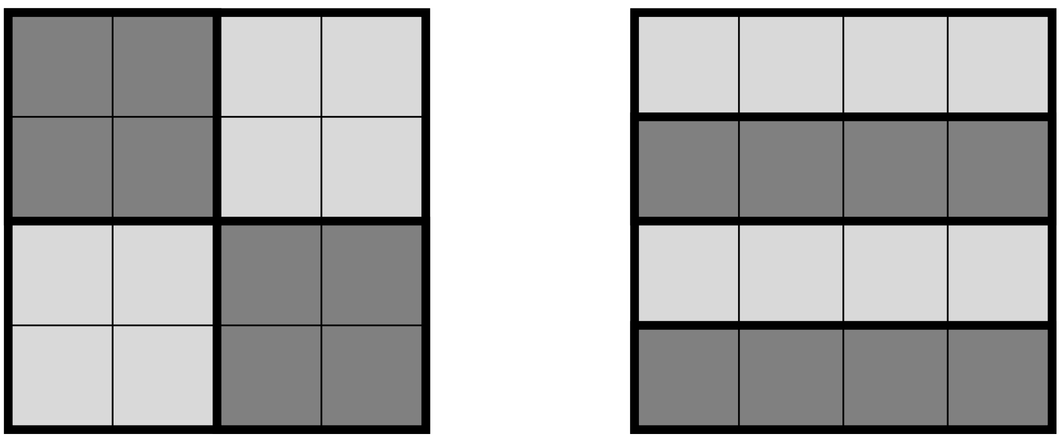
Possible answers:

1. Only parts A and B of the work schedule are feasible.
2. Only parts A and C of the work schedule are feasible.
3. Only parts A and D of the work schedule are feasible.
4. Only parts B and C of the work schedule are feasible.
5. Only parts B and D of the work schedule are feasible.
6. Only parts A, B, C of the work schedule are feasible.
7. Only parts A, B, D of the work schedule are feasible.
8. Only parts A, C, D of the work schedule are feasible.
9. Only parts B, C, D of the work schedule are feasible.
10. All four parts of the work schedule are feasible.

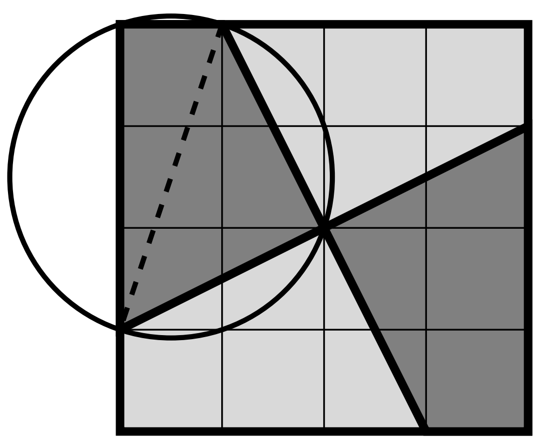
Solution

The correct answer is: 10.

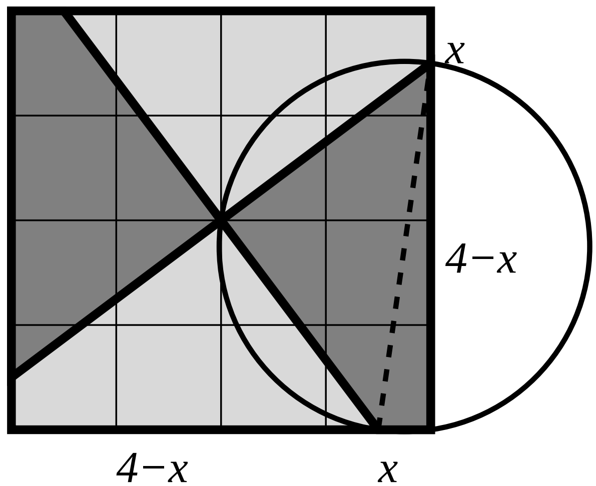
All four sections A, B, C, D of the work plan can be carried out. Thus, the central administration did a good job and did not make a mistake this time.
The following figure shows possible partitions for part A (left) and part D (right). In the left partition for A, each quadrilateral is a square with side length 2. According to the Pythagorean theorem, the diagonal of such a square (and thus the diameter of the circumcircle) is $\sqrt{2^2 + 2^2} = \sqrt{8}$.
In the right partition for D, each quadrilateral is a rectangle with side lengths 4 and 1. According to the Pythagorean theorem, the diagonal of such a rectangle (and thus the diameter of the circumcircle) is $\sqrt{4^2 + 1^2} = \sqrt{17}$.



The following figure shows the partition for part B. Each quadrilateral in the partition is made up of two right-angled triangles that share their hypotenuse. The smaller triangle has side lengths 1, 3, $\sqrt{10}$, and the other triangle has side lengths $\sqrt{5}$, $\sqrt{5}$, $\sqrt{10}$. Therefore, the diameter of the circumcircles of these quadrilaterals (displayed as dashed lines) is $\sqrt{10}$.



Finally, the following figure shows the partinion for part C. If we choose $x = 2-\sqrt{2}$, each of the four small quadrilaterals is composed of two right-angled triangles that share their hypotenuse. One triangle has side lengths $2-\sqrt{2}$, $2+\sqrt{2}$, $\sqrt{12}$, and the other triangle has side lengths $\sqrt{6}$, $\sqrt{6}$, $\sqrt{12}$. Therefore, the diameter of the circumcircles of these quadrilaterals (shown by dashed lines) is $\sqrt{12}$.



5 Soccer

Author	Gerhard Woeginger (TU Eindhoven)
Project:	4TU.AMI

Challenge

Six soccer teams from the cities Icetown, Frostville, Glacierhampton, Coldbury, Snowham, and Winterfield have participated in the Christmas soccer tournament. During this tournament, each team played exactly one match against each of the other five teams. A victory scored three points, a draw one point, and a loss zero points.

Icetown won the tournament. Frostville ended up in second place, two points behind Icetown. Glacierhampton reached the third place, two points behind Frostville. At the fourth position was Coldbury, two points behind Glacierhampton. Snowham was two points behind Coldbury. Winterfield finished last, two points behind Snowham.

Which of the following ten statements **necessarily** follows from the above information?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. Icetown beat Frostville and Glacierhampton.
2. Icetown beat Coldbury and Winterfield.
3. Frostville lost against Glacierhampton and won against Snowham.
4. Glacierhampton beat Coldbury and Snowham.
5. Glacierhampton beat Frostville and Coldbury.
6. Coldbury lost against Icetown and Glacierhampton.
7. Coldbury lost against Snowham and won against Winterfield.
8. Snowham lost against Icetown and Coldbury.
9. Winterfield lost against Icetown and Frostville.
10. Winterfield lost against Coldbury and played draw against Glacierhampton.

Solution

The correct answer is: 10.

First, let us examine the total number N of all points awarded in all 15 games of the Christmas tournament. Since either 2 or 3 points were awarded in each game, one has $30 \leq N \leq 45$.

If Winterfield (W) earned w points, then Icetown (I), Frostville (F), Glacierhampton (G), Coldbury (C), Snowham (S) earned $w+10$, $w+8$, $w+6$, $w+4$ and $w+2$ points, respectively. Therefore,

$$N = 6w + 30 \qquad \text{with} \qquad 0 \leq w \leq 2.$$

We will examine the cases $w = 0, 1, 2$ separately:

- $w = 0$: Since $N = 6 \cdot 0 + 30 = 30$, all games would have ended in a draw and Winterthal would have had $5 \cdot 1 = 5$ points at the end of the tournament. Since this contradicts $w = 0$, this case is impossible.
- $w = 1$: Since $N = 6 \cdot 1 + 30 = 36$, in this case exactly 9 games would have ended in a draw and exactly 6 games with win/loss.

  Now, we have a closer look at the 9 games between the three teams C, S, W on one side and the three teams I, F, G on the other side, i. e.

  $$\text{C–I}, \quad \text{C–F}, \quad \text{C–G}, \qquad \text{S–I}, \quad \text{S–F}, \quad \text{S–G}, \qquad \text{W–I}, \quad \text{W–F}, \quad \text{W–G}.$$

  Of these 9 games, at least 3 would have ended in a draw and would have given the group C/S/W at least 3 points. However, after all 15 games, the group C/S/W would have scored $(w+4)+(w+2)+w = 5+3+1 = 9$ points. Therefore, in the three games that C, S, W would have played among themselves, a maximum of 6 points could have been earned. Thus, these three games would all have ended in a draw. Consequently, Winterthal would have played draw at least twice and thus would have scored a total of at least 2 points. Since this contradicts $w = 1$, this case is also impossible.
- In the only remaining case, $w = 2$ and $N = 42$ apply, and Icetown, Frostville, Glacierhampton, Coldbury, Snowham, Winterfield scored 12, 10, 8, 6, 4, and 2 points, respectively.

Obviously, **Winterfield** has drawn two games, won no game, and lost three games.
Since $12 \equiv 0 \mod 3$, **Icetown** either played draw none or three of its games. If the team had drawn three times, it would have won at most twice and thus scored at most $3 \equiv 1 + 2 \equiv 3 = 9 < 12$. Thus, Icetown ended no game in a draw, won four times, and lost once.
Analogously, we calculate that

- **Frostville** ended only one game in a draw, won three times, and lost ones;
- **Glacierhampton** ended only two games in a draw, won twice, and lost ones;
- **Coldbury** ended no game in a draw, won twice, and lost three times.

It remains to be clarified whether **Snowham** played a draw once or four times. Since only three of the five opposing teams (Frostville, Glacierhampton, and Winterfield) played to a draw, Snowham could not have drawn against four teams. Hence, Snowham ended only one game, won once and lost three times. Summarisingly, we yield the following table:

Icetown:	3	3	3	3	0
Frostville:	3	3	3	1	0
Glacierhampton:	3	3	1	1	0
Coldbury:	3	3	0	0	0
Snowham:	3	1	0	0	0
Winterfield:	1	1	0	0	0

From this table, we can extract the following information:

- Since none of Coldbury' matches ended in a draw and Winterfield never won, Winterfield **neccessarily** lost the game C–W against Coldbury.
- Suppose that the game G–W did *not* end in a draw. Then, both of the games G–F *and* G–S must have ended in a draw. Now, however, Winterfield could not have played a draw against any team. This contradiction shows that the game G–W **neccessarily** ended in a draw.

Thus, statement 10 is correct.

The information given in the task is not sufficient to find a unique final table. However, the following two possible final tables illustrate that none of the remaining nine statements 1 to 9 follows necessarily from the given information:

	I	F	G	C	S	W
I	–	0	3	3	3	3
F	3	–	0	3	3	1
G	0	3	–	3	1	1
C	0	0	0	–	3	3
S	0	0	1	0	–	3
W	0	1	1	0	0	–

	I	F	G	C	S	W
I	–	3	3	0	3	3
F	0	–	1	3	3	3
G	0	1	–	3	3	1
C	3	0	0	–	0	3
S	0	0	0	3	–	1
W	0	0	1	0	1	–

6 T-shirts

Author: Hajo Broersma (Universiteit Twente)
Project: 4TU.AMI

Challenge

33 pixies are standing in a circle. Each of the pixies is wearing either a red or a green T-shirt. If two pixies are direct neighbors in the circle, at most one of them is wearing a red T-shirt. If two pixies in the circle are separated by exactly 15 other pixies, at most one of the two is wearing a red T-shirt.

What is the largest possible number of pixies with a red T-shirt?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. The largest possible number of pixies with a red T-shirt is 7.

2. The largest possible number of pixies with a red T-shirt is 8.

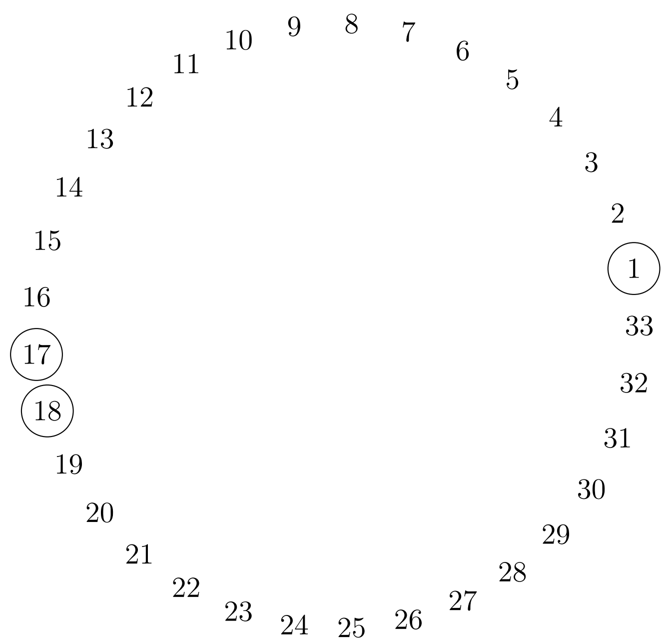
3. The largest possible number of pixies with a red T-shirt is 9.
4. The largest possible number of pixies with a red T-shirt is 10.
5. The largest possible number of pixies with a red T-shirt is 11.
6. The largest possible number of pixies with a red T-shirt is 12.
7. The largest possible number of pixies with a red T-shirt is 13.
8. The largest possible number of pixies with a red T-shirt is 14.
9. The largest possible number of pixies with a red T-shirt is 15.
10. The largest possible number of pixies with a red T-shirt is 16.

Solution

The correct answer is: 5.

We number the elves from 1 to 33 to not have to come up with 33 different names. The elves with the numbers n and $n+1$ stand beside one another for $n \in \{1, 2, 3, \ldots, 32\}$. The elves 33 and 1 do so as well.
First, we realize that for each elf exactly two other elves are "15 elves apart", and these two stand directly next to each other. For elf 1 these are elf 17 and 18, for elf 2 these are elf 18 and 19, ..., for elf 33 these are 16 and 17. Hence, there can be *at most* one elf with a red T-shirt in each of these sets of 3 elves. This becomes apparent when looking at the elves 1, 17 and 18:

> If elf 1 is wearing a red T-shirt, neither elf 17 nor elf 18 can wear a red T-shirt, since they are each "15 elves apart" from 1. If elf 17 is wearing a red T-shirt, 1 cannot wear a red T-shirt, since the elf is "15 elves apart" from 17. Elf 18 cannot wear a red T-shirt as the neighbor of 17. The same argument holds true if elf 18 is wearing a red T-shirt.



Now, we can divide the 33 elves into eleven such disjoint groups. For example, in the following way:

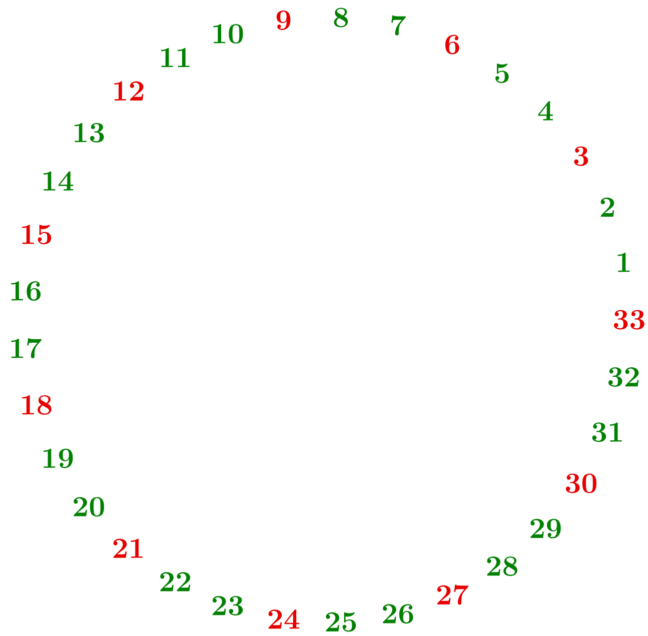
1 – 2 – 18	3 – 19 – 20	44 – 5 – 21	6 – 22 – 23
7 – 8 – 24	9 – 25 – 26	10 – 11 – 27	12 – 28 – 29
13 – 14 – 30	15 – 31 – 32	16 – 17 – 33	

In each of these groups of three at most one elf can wear a red T-shirt. Therefore, there can be a maximum of eleven elves with a red T-shirt.

In fact, we can find a way to distribute eleven red T-shirts among the elves. For example, if the elves with the numbers

$$3, \quad 6, \quad 9, \quad 12, \quad 15, \quad 18, \quad 21, \quad 24, \quad 27, \quad 30, \quad 33$$

wear a red T-shirt, we get a distribution compatible with the setup:



7 Kandinsky

Author: Cor Hurkens (TU Eindhoven)
Project: 4TU.AMI

Challenge

The Grinch offers a painting with the title *Solar Eclipse Number 8* for sale, which (according to the Grinch) might possibly be the work of Wassily Wassilyevich Kandinsky (see Fig. 3).

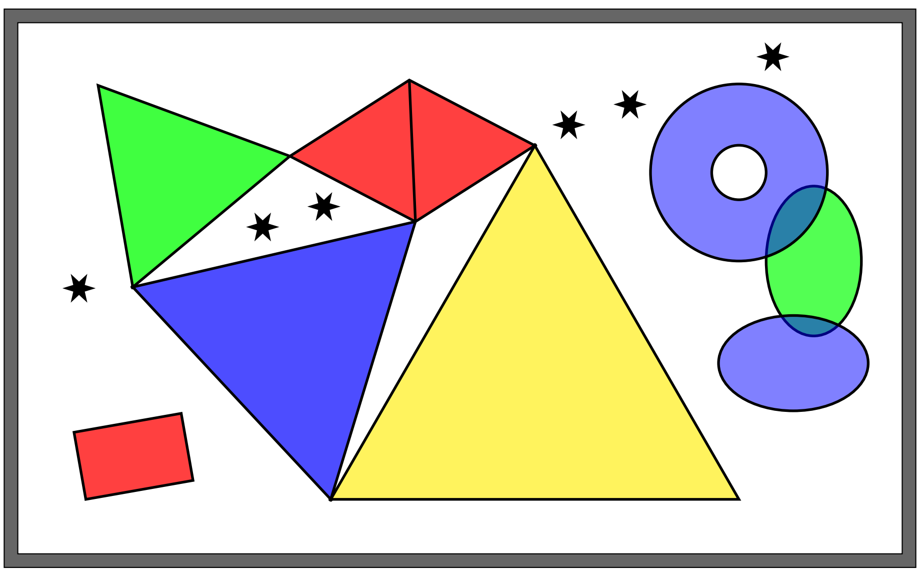


Рис. 3: The painting *Solar Eclipse Number 8*

The authenticity-check-elf Austin has carefully examined the painting and has come to the following conclusions:

- The yellow, the blue, the green and the two red triangles in the painting are equilateral; all angles in these triangles are 60°.
- The center points of the six little black stars lie on a common straight line.
- The red quadrilateral in the lower left corner of the painting is a rectangle. The third digit behind the decimal point in the decimal representation of the area of this rectangle (measured in square meters) is 4.

- The green and the blue ellipses at the right margin of the painting are congruent.
- The two red triangles are congruent and each have an area of $4/3$ square meters.
- The areas of the yellow and the green triangles add up to an even integer number of square meters.

We would like to know: what is the third digit behind the decimal point in the decimal representation of the area of the blue triangle (measured in square meters)?

Artwork: Friederike Hofmann

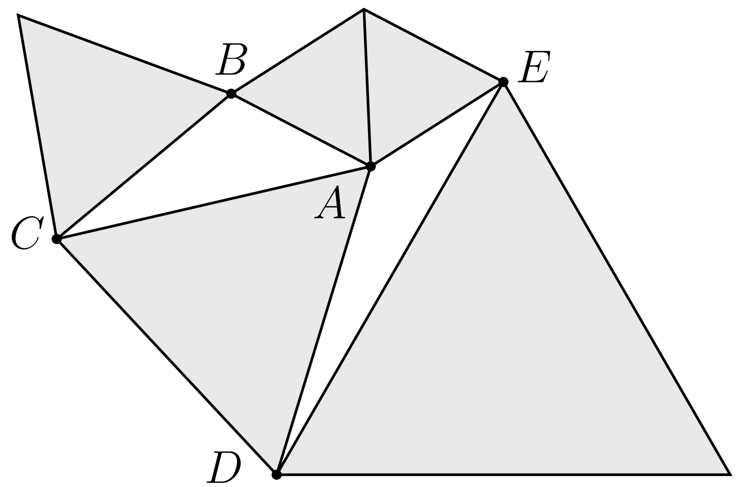
Possible answers:

1. The third digit behind the decimal point is 1.
2. The third digit behind the decimal point is 2.
3. The third digit behind the decimal point is 3.
4. The third digit behind the decimal point is 4.
5. The third digit behind the decimal point is 5.
6. The third digit behind the decimal point is 6.
7. The third digit behind the decimal point is 7.
8. The third digit behind the decimal point is 8.
9. The third digit behind the decimal point is 9.
10. There is not enough information in the problem statement that would allow to uniquely determine this third digit behind the decimal point.

Solution

The correct answer is: 6.

In our solution, we only need the first, fifth and sixth of Echnaton's conclusions. The following figure introduces the five points A, B, C, D, E.



First, we look at the triangle ABC. The angle $\alpha = \angle CAB$ denotes the inner angle at A. The law of cosines gives

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB|\,|AC|\cos(\alpha). \tag{1}$$

Next, we consider the triangle ADE and let $\beta = \angle EAD$ denote the inner angle at A. Again, the law of cosines further yields

$$|DE|^2 = |AE|^2 + |AD|^2 - 2|AE|\,|AD|\cos(\beta). \tag{2}$$

Adding the five angles adjacent to A, we get

$$\alpha + 60 + 60 + \beta + 60 = 360.$$

Therefore, we know that $\alpha + \beta = 180$ and conclude $\cos(\beta) = \cos(180 - \alpha) = -\cos(\alpha)$. Using $|AE| = |AB|$ and $|AD| = |AC|$, we rewrite (2) to

$$|DE|^2 = |AB|^2 + |AC|^2 + 2|AB|\,|AC|\cos(\alpha). \tag{3}$$

Adding (1) and (3) yields

$$|BC|^2 + |DE|^2 = 2|AB|^2 + 2|AC|^2. \tag{4}$$

Since an equilateral triangle with sidelength s has an area of $\frac{1}{4}\sqrt{3}\,s^2$, we conclude from (4) through multiplication by $\frac{1}{4}\sqrt{3}$ that

$$\text{Area}(\triangle_{\text{green}}) + \text{Area}(\triangle_{\text{yellow}}) = 2 \cdot \text{Area}(\triangle_{\text{red}}) + 2 \cdot \text{Area}(\triangle_{\text{blue}}). \tag{5}$$

holds true. Since the left side of equation (5) is an even number $2N$ and the area of the red triangle equals $4/3$, we get $N - 4/3$ for the area of the blue triangle. Hence, the third digit behind the decimal point is 6.

8 Gingerbread

Authors: Stefan Felsner (TU Berlin)
Xueyi Guo (TU Berlin)
Project: MATH+ School Activities

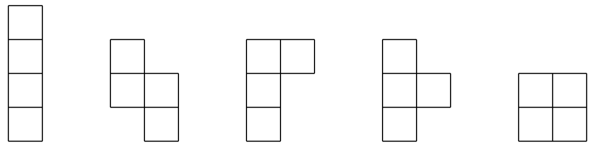
Challenge

Luca, the cheekiest child at the North Pole, has broken two walls of a gingerbread house into pieces. Originally, each wall was made of 6×6 square-shaped gingerbread tiles. Santa is not pleased about this at all and says: “Luca, if you don’t fix the walls, you won’t get a present on Christmas Eve.”
But Luca has already mixed the pieces of the house with other gingerbread. Now, the child does not know exactly which pieces are needed to repair the two walls.

A wall can only be built out of one set (see below). The pieces may be rotated and/or mirrored. However, you are of course not allowed to cut them into smaller pieces. In each of the given sets, the number under each piece indicates how many copies of that piece are contained in the set.

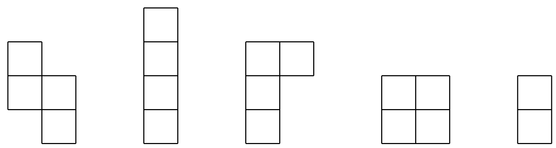
Find out which two sets are needed to repair the gingerbread house.

Set 1:



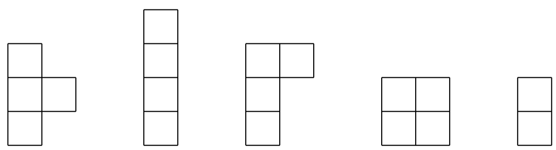
2 2 2 1 2

Set 2:



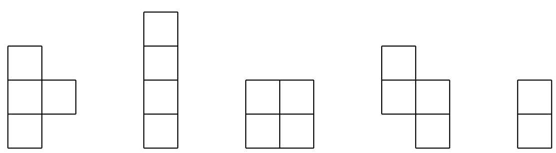
2 1 2 2 4

Set 3:



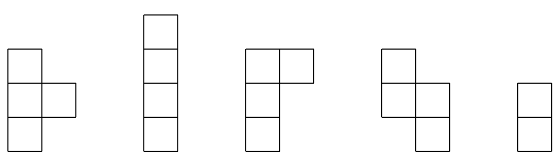
1 2 2 2 4

Set 4:



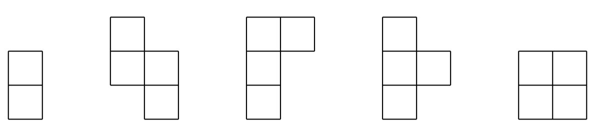
1 2 2 2 4

Set 5:



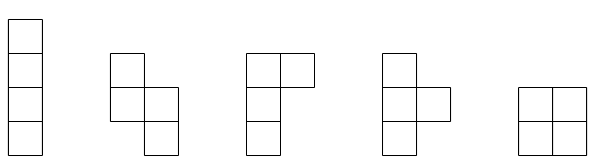
1 2 2 2 4

Set 6:



2 2 3 1 2

Set 7:



1 2 2 2 2

Artwork: Friederike Hofmann

Possible answers:

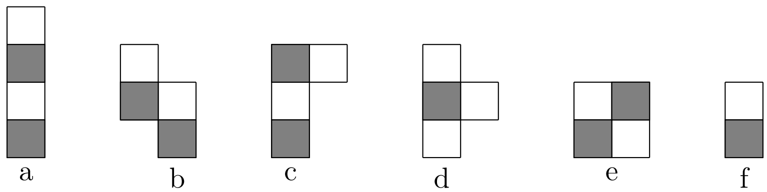
1. Set 1 and 2.
2. Set 1 and 4.
3. Set 1 and 6.
4. Set 2 and 3.
5. Set 2 and 5.
6. Set 2 and 7.
7. Set 3 and 4.
8. Set 3 and 6.
9. Set 4 and 5.
10. Set 4 and 7.

Solution

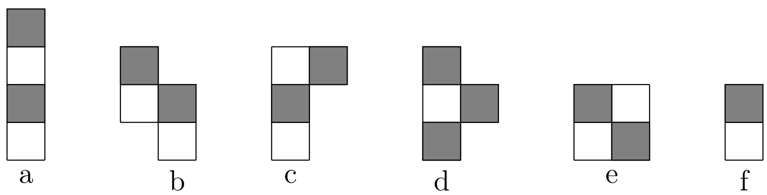
The correct answer is: 6.

If we imagine that the walls are patterned like a chessboard, then there are 18 black and 18 white squares. Now we can think about the different ways the individual parts can be patterned.

For each part there are two possibilities, namely



or



The **first set** contains parts a, b, c, d and e. If we now look at the number of black/white squares, we see that the parts a, b, c and e all have an even number of black/white squares each and d has an odd number. Since the number of part d is odd, the total number of black/white squares is therefore odd. However, a 6×6 chessboard always has an even number of black and white squares. Accordingly, Luca cannot build a perfect wall from the first set.

Now we apply the same logic to the **second set**. We can already see that there is one piece in the second set that has an odd number of black/white squares, and that is piece f. But in this set, piece f exists four times. So in the set there is a total of $2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 4 \cdot 1 = 18$ black/white squares. This is consistent with the given number of black/white squares on a 6×6 chessboard. Hence Luca might build a wall with this set.

In the **third set** there are two parts that have an odd number of black/white squares, namely part d and part f. Part d exists once and part f four times. So we get an odd number of black/white squares again. Unfortunately, Luca cannot build a perfect wall from this set either.

The **fourth, fifth and sixth set** also contain the parts d and f. In each of these sets, part d is contained once and part f an even number of times. Therefore, these three sets are also not suitable for repairing the second wall.

Now we look at the **seventh set**. Part d exists twice here. Thus, we can colour it according to possibility one *and* according to possibility two. This gives us

$$1 \cdot 2 + 2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 + 2 \cdot 2 = 18$$

black and 18 white squares. This agrees with the total number needed. Luca might be able to repair the second wall with this set.

Finally, we show that the two sets 2 and 7 can each be assembled into a wall of 6×6 square gingerbread. One way to assemble the walls from set 2 and 7 is given in figure 4.

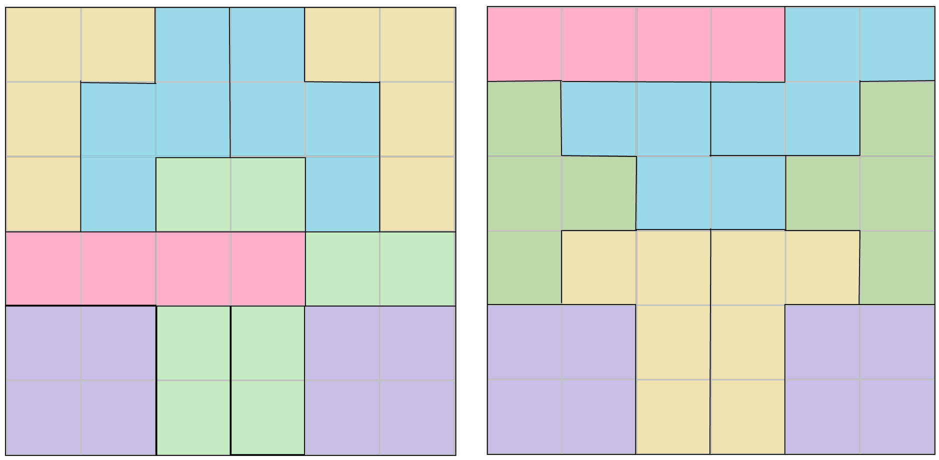


Рис. 4: On the left, a wall that can be built from set 2. On the right, a wall that can be built from set 7.

Hence, Luca was lucky and gets a present from Santa Claus this year.

9 Two Rooks

Authors Aart Blokhuis (TU Eindhoven),
Gerhard Woeginger (TU Eindhoven)
Project: 4TU.AMI

Challenge

The three very intelligent elves, Reiko, Spaxo, and Yello, are looking at Santa filled with anticipation. Santa says, “You can find a blue and a yellow rook, which do not threaten each other by standard chess rules, on my chess board. Those rooks are positioned on **two** of the following twenty squares:

a1, a3, a4, a8, b1, b5, b8, c3, d2, d4, d7, e2, f1, f3, f5, f6, f7, g7, h4, h7.”

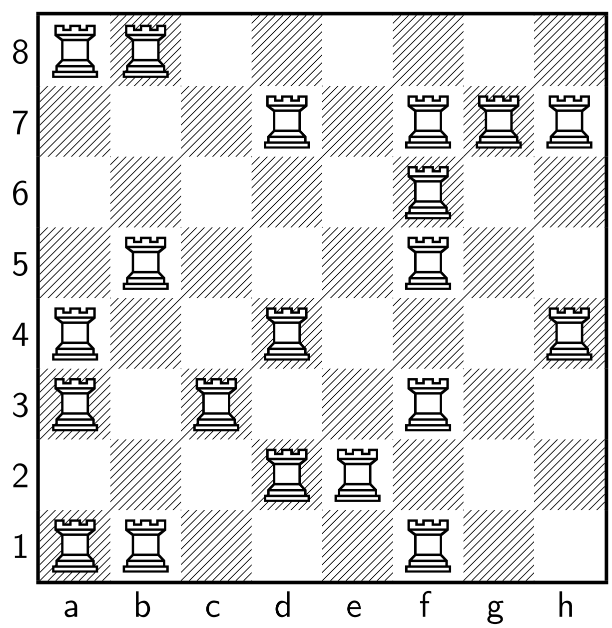


Рис. 5: The twenty possible squares.

Santa continues, “Reiko knows the rank (1, 2, 3, 4, 5, 6, 7, or 8) of the blue rook. Spaxo knows the file (a, b, c, d, e, f, g, or h) of the blue rook. Yello knows the square occupied by the yellow rook. However, that is everything you know about the positions of the rooks.”

Thereupon, the elves make the following statements. After each statement, the elves contemplate for one minute before moving on.

1) Spaxo says, “I cannot tell the square of the blue rook.”
2) Then, Spaxo says, “I know that Reiko cannot tell the square of the blue rook either."
3) After that, Reiko says, “I still do not know where to find the blue rook."
4) Now, Spaxo says, “I still do not know where to find the blue rook."
5) Then, Spaxo says, “I know that Yello cannot tell the square of the blue rook."
6) Afterwards, Yello says, “Before Spaxo’s last statement, I did not know the file of the blue rook.”
7) Then, Reiko says, “Before Spaxo’s last statement, I did not know the file of the blue rook.”
8) Finally, Yello says, “Eureka! Now, I know the square of the blue rook!”

Now, we want to know: Which square is occupied by the blue rook?

Artwork: Friederike Hofmann

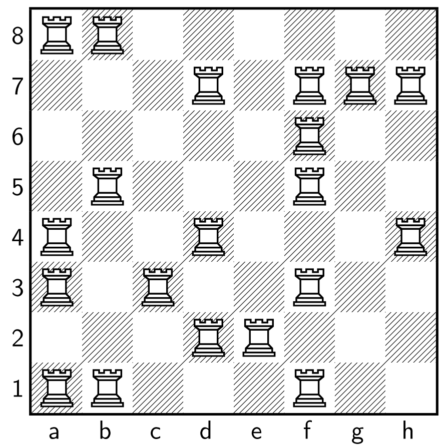
Possible answers:

1. The blue rook stands either on a1 or d7.
2. The blue rook stands either on a3 or e2.
3. The blue rook stands either on a4 or f1.
4. The blue rook stands either on a8 or f3.
5. The blue rook stands either on b1 or f5.
6. The blue rook stands either on b5 or f6.
7. The blue rook stands either on b8 or f7.
8. The blue rook stands either on c3 or g7.
9. The blue rook stands either on d2 or h4.
10. The blue rook stands either on d4 or h7.

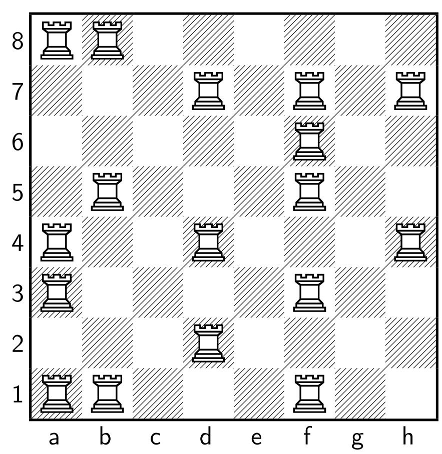
Solution

The correct answer is: 4.

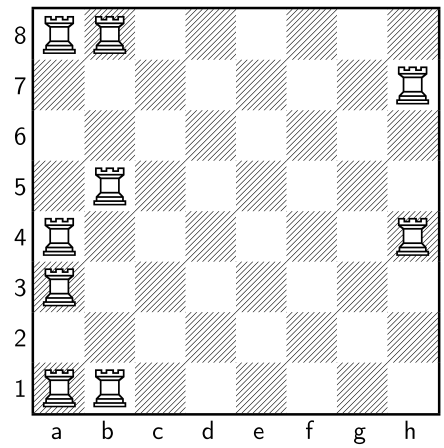
We will work step by step through the elves statements and analyze their consequences. We call a square *active* at a certain point of time if the statements made until then allow for the blue rook to stand on that square. As a reminder, we give the 20 possible positions of the two rooks:



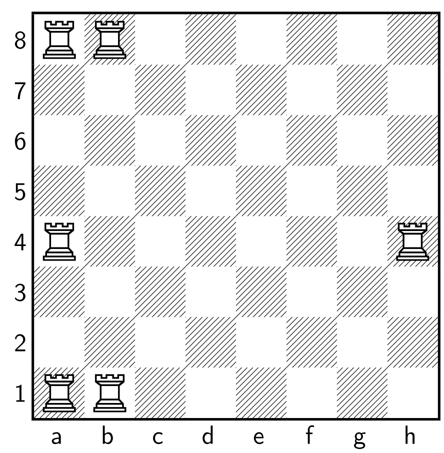
Spaxo: I cannot tell the square of the blue rook. Spaxo only knows the file of the blue rook. If there are two or more different active squares on that file, Spaxo cannot tell where the blue rook is positioned. However, if only one square is active on a file, Spaxo knows the rook has to stand on that ative square. Thus, we analyze that the blue rook cannot stand on file c, file e or file g.



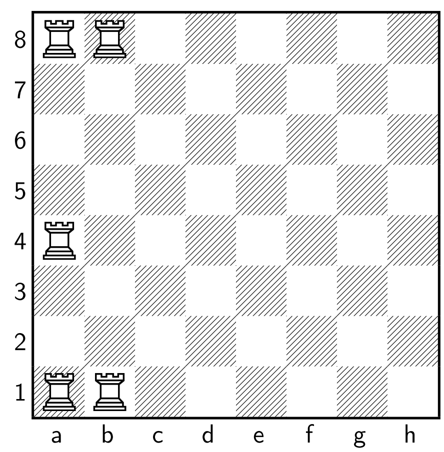
Spaxo: I know that Reiko cannot tell the square of the blue rook either. Reiko only knows the rank of the blue rook. He therefore can know the position of the blue rook if and only if the rank has exactly one active square; only ranks 2 and 6 have exactly one active square. That is why Spaxo has to know that the blue rook cannot be on rank 2 or on rank 6. That lets us know that Spaxo has neither file d (with the active square d2) nor file f (with the active square f6).



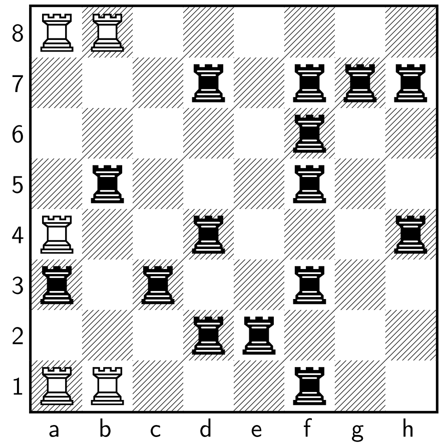
Reiko: I still do not know where to find the blue rook. Since Reiko knows the blue rook's rank, that rank has to encompass at least two active squares. We conclude that the blue rook cannot stand on ranks 3, 5 and 7.



Spaxo: I still do not know where to find the blue rook. Since Spaxo knows the file of the blue rook, that file has to have at least two active squares, eliminating file h from consideration.



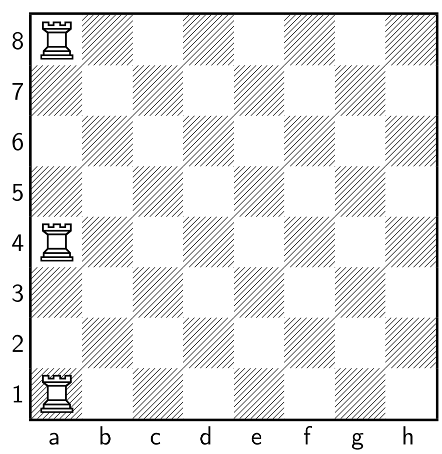
Spaxo: I know that Yello cannot tell the square of the blue rook. Obviously, Yello attained the same knowledge from the discussion as we did; he knows that the blue rook can only stand on one of the five squares a1, a4, a8, b1 or b8.



In which cases can Yello deduce the position of the blue rook from the position of the yellow one?

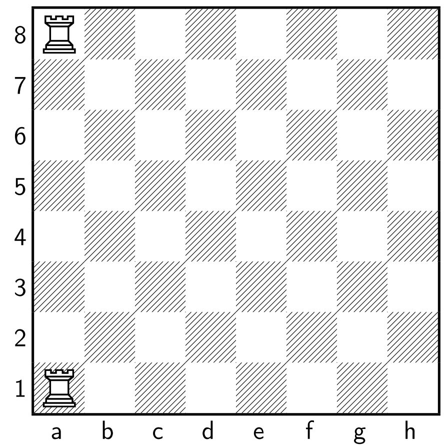
- If the yellow rook stands on a1, the blue rook has to stand on b8.
- Following the same logic, the yellow rook standing on a8 would put the blue rook on b1.

In every other case Yello is left with at least two active candidates for the blue rook. This means Spaxo has to know that Yello has neither square a1 nor square a8. Spaxo can only know this with certainty if he himself has file *a*. The only active squares remaining are thus a1, a4 and a8:

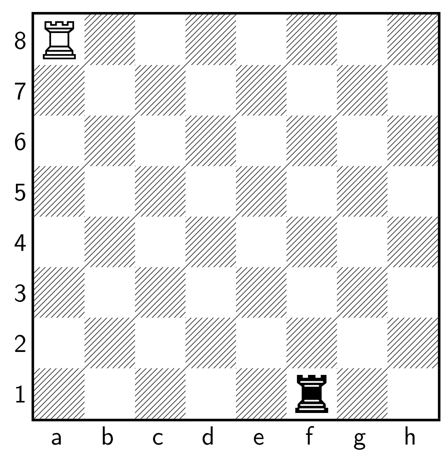


Yello: Before Spaxo's last statement, I did not know the file of the blue rook. Right before Spaxo's last statement the five squares a1, a4, a8, b1 and b8 were active. If the yellow rook were to stand on one of the files *a* or *b*, the blue rook would have to stand on the other file. However, since Yello could not deduce the file of the blue rook, the yellow rook cannot stand on file *a* or file *b*.

Reiko: Before Spaxo's last statement, I did not know the file of the blue rook. Since before Spaxo's last statement only the five squares a1, a4, a8, b1 and b8 were active, Reiko's statement eliminates rank 4 from consideration:



Yello: Eureka! Now, I know the square of the blue rook! The only remaining active squares after Reiko's claim were a1 and a8. Since Yello can eliminate one of the squares, the yellow rook has to stand either on rank 1 or rank 8. We already know that it stands neither on file *a* nor on file *b*. Therefore, the only square on which the yellow rook can stand is f1. We now finally know that the blue rook is positioned on a8:



10 Santa's Snowflake-powered Snowmobiles

Authors: Niels Lindner (Zuse Institute Berlin),
Fabian Löbel (Zuse Institute Berlin),
Berenike Masing (Zuse Institute Berlin)

Project: Electric Bus Scheduling (Research Campus MODAL)

Challenge

Each year, the preparation of Christmas gifts is a mammoth task for Santa and his elves. Not only the production of the presents, but also the logistics behind it pose a real challenge: there are different manufacturing steps and the needed components have to be transported to the correct locations on time. Since the task takes multiple days, Santa has been using the well-established system of reindeer sleds on the street network at the north pole up until now: each hour the same sled performed the same trip at the same departure and arrival times. The usual street network and the corresponding timetable can be found in Figure 6.

This year, however, everything is different: the reindeer do not want to do all this exhausting work anymore and have mutually decided to go on indefinite strike. Santa has to react immediately: the timetable for the sleds cannot be changed on such short notice, since the production process at each station is exactly adjusted to it. Luckily, Santa manages to procure environmentally friendly snowmobiles to replace the reindeers. They run as fast as the reindeers and can therefore keep the same travel times between stations as indicated in Figure 6. Just like the reindeer sleds, they can only use the indicated roads; their travel times are independent of the travel direction.

"This works out perfectly!", decides Santa. "It means that the timetable can be covered with six snowmobiles just as usual." The vehicles should travel through the network along circles, as the timetable is supposed to repeat itself after each hour. Each trip should be covered by exactly one circle, and no circle should contain a trip more than once. The vehicles can use the streets freely; in particular, they are allowed to return on the same road as they arrived. After arriving at a station, they can either wait or depart immediately. Concretely, Santa constructed the vehicle schedule depicted in Figure 7.

Just when he was about to order the six snowmobiles, Rudolf the reindeer knocks on Santa's door. On taking a look at the vehicle schedule, Rudolf starts laughing out loud. He explains to the confused Santa that he has forgotten about one crucial detail: the snowmobiles have a special engine, powered by snowflakes. Each vehicle is equipped with a bucket which can hold up to 500 snowflakes. However, the engine consumes 10 snowflakes per minute, which means that each snowmobile can run only for 50 minutes without breaks. The buckets can be refilled at each station, but this takes 5 minutes each time. As a result, the vehicles cannot be operated as planned, as some of the vehicle routes are scheduled to run for up to 110 minutes without breaks. However, if they simply refueled at some station within Santa's plan, they would miss their next departure and delay the entire process; the whole gift production would be disrupted! There is no getting around it—Santa has to re-plan the circles on which the snowmobiles have to operate. If worst comes to worst, Santa might have to buy even more than 6 snowmobiles in order to cover all trips...

Question: What is the minimum number of snowmobiles needed to realize all trips 1 to 8 in the schedule from Figure 6?

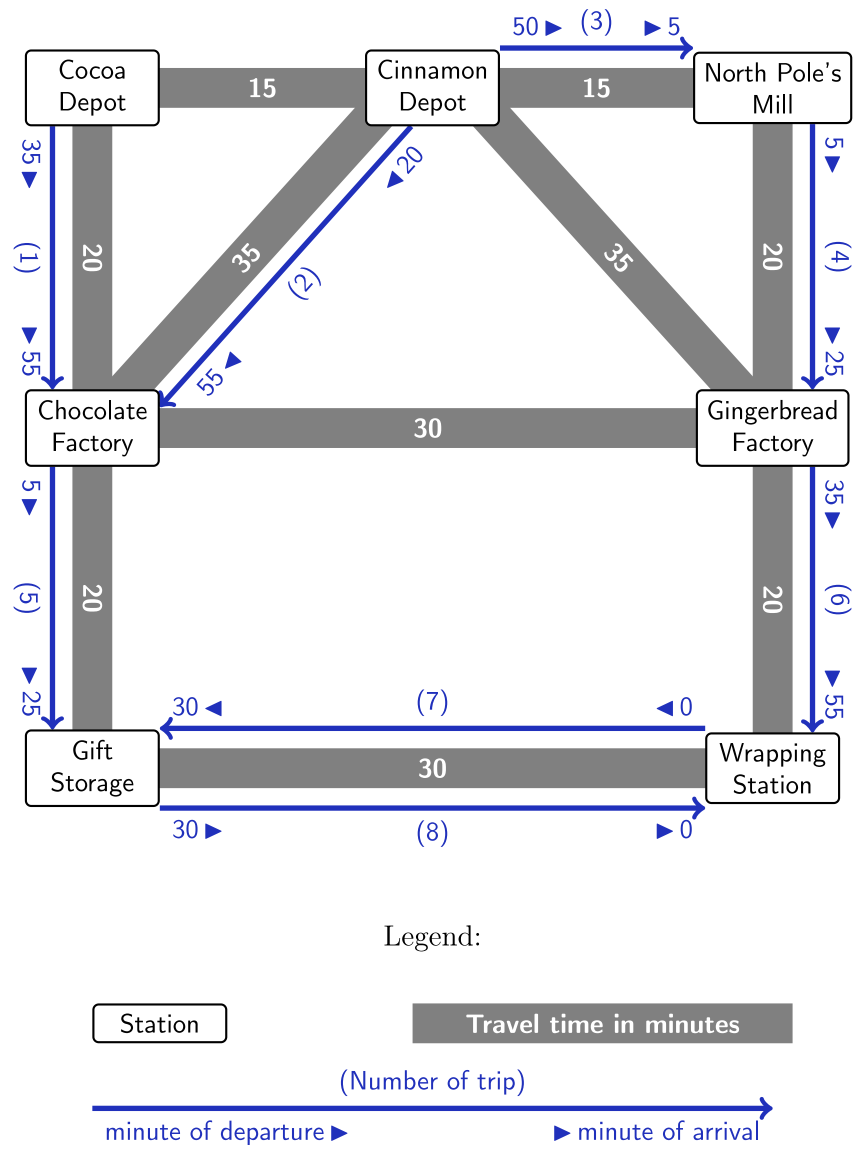


Рис. 6: Street network and timetable.

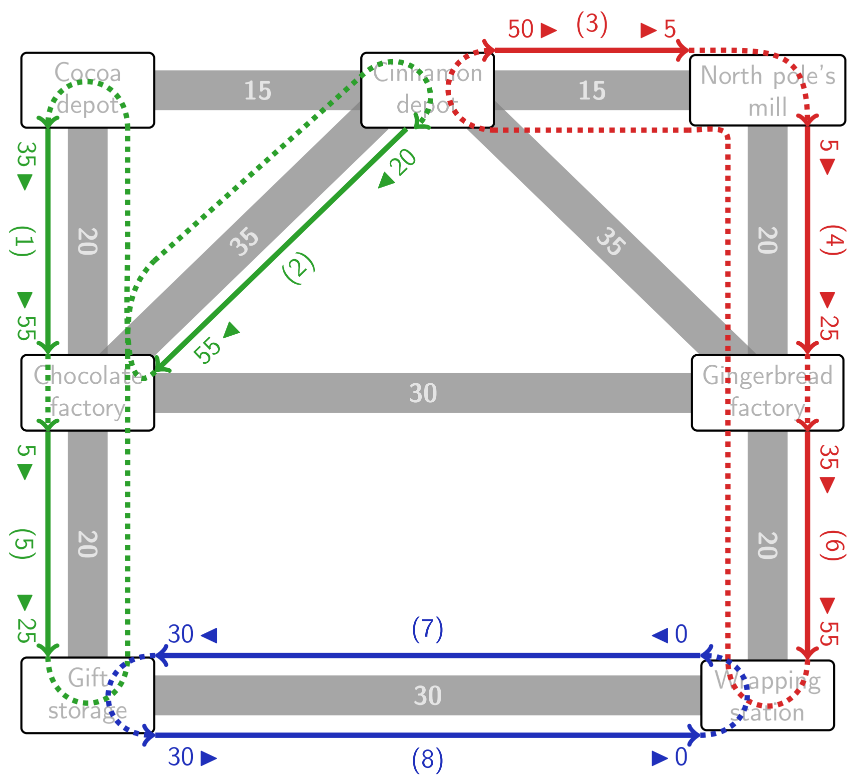


Рис. 7: Santa's Vehicle Schedule.

The blue circle serves Trips 7 and 8 in turns, one circulation takes 60 minutes. Consequently, the blue cycle needs exactly one vehicle in a periodic setting of 60 minutes. The corresponding vehicle departs immediately after each arrival without taking any breaks.

The green circle consists of Trip 1, followed by a wait of 10 minutes at the chocolate factory, Trip 5 (20 minutes), an empty run to the cinnamon depot (55 minutes), Trip 2 (35 minutes), an empty run to the cocoa depot and finally waiting for departure of Trip 1 (20 minutes). Consequently, travelling along the green circle takes 180 minutes in total, which means that 3 vehicles are needed.

On the red circle vehicles travel for exactly 120 minutes. For hourly departures, this corresponds to exactly 2 vehicles. Due to the pure travel time of exactly 55 minutes from the wrapping station to the cinnamon depot, a vehicle departing at the former at minute 55 manages to arrive at the latter at minute 50 exactly. Along this circle, the vehicles only have one break of 10 minutes at the gingerbread factory.

Artwork: Frauke Jansen

Possible Answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

Project reference:

The MobilityLab at the Zuse Institute Berlin researches how to improve operation, organisation, and passenger convenience of public transport with the help of Mathematics. As part of the Research Campus MODAL, the project *Electric Bus Scheduling* addresses the problem of scheduling routes for battery-powered buses. The greatest challenge when replacing diesel-fuelled buses by electric ones is that today's battery capacities are not sufficient for the typical distances that diesel buses accumulate over a day. Thus, the charging of the batteries has to be taken into account when planning vehicle schedules.

Solution

The correct answer is: 7.

Figure 8 gives an overview over the duration and the chronological arrangement of the trips, which is helpful for the following reasoning.

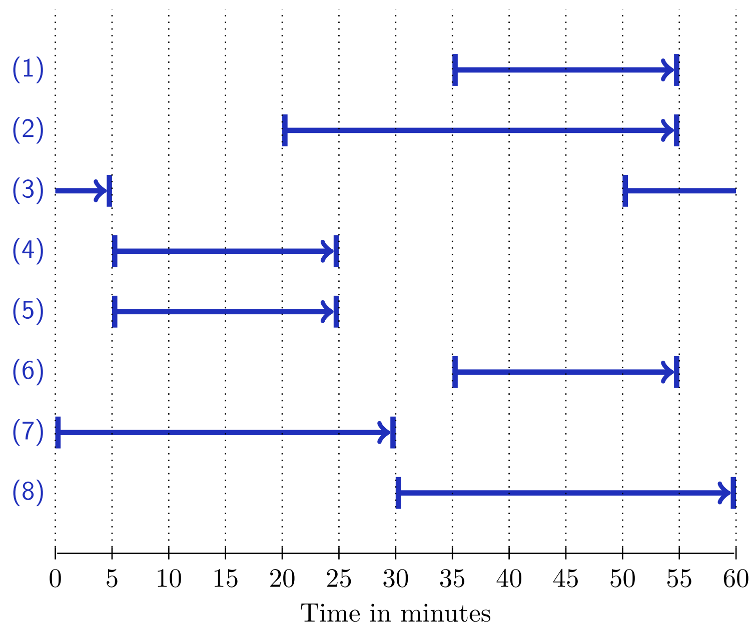


Рис. 8: Time table

≥ 5 snowmobiles: From the diagram, we can immediately deduct that during minute 50 and 55, there is one vehicle needed for each of the trips 1, 2, 3, 6, and 8. Consequently, at least 5 snowmobiles are needed.

≥ 6 snowmbobiles: The cinnamon depot is located quite far from the end points of the trips: from north pole's mill 15 minutes are needed, while 35 minutes or even more are needed to reach the cinnamon depot from any other end station. For the vehicle that starts Trip 2 at the cinnamon depot at minute 20, one of the following cases has to be true:

1. At minute 50 of the previous period, it is on an empty run, i. e. it is not on any of the Trips 1–8.
2. It served Trip 3 with arrival at the north pole's mill at minute 5 of the same period.

In the first case, at least 6 snowmobiles are needed at minute 50: the five vehicles on the Trips 1, 2, 3, 6, and 8, as well as on the empty run leading to Trip 2.
In the second case, the snowmobile travels along Trip 3 (15 min.), returns to the cinnamon depot (15 min.) and directly proceeds to the chocolate factory (35 min.). This amounts to 65 minutes of non-stop driving time, which would use 650 snowflakes. As the bucket holds only 500 and there is no time to refuel on the road, this case is not possible.

As a result, Santa needs at least 6 snowmobiles.

$\geq$ 7 snowmobiles: Let us now consider Trip 5. A vehicle that ends Trip 5 at minute 25 cannot serve any of the Trips 1,2,3 or 6 at minute 50 of the same period, since the time needed to reach any of the starting points of those trips from the gift storage is more than 25 minutes. Consequently, there are only three cases: at minute 50, this vehicle can be

1. on an empty run and then on Trip 2,
2. on an empty run and then not on Trip 2,
3. on Trip 8.

In the first case, we notice that Trip 5 ends at minute 25 while Trip 2 begins only at minute 20 of the next period. A vehicle would have enough time to reach the cinnamon depot from the gift storage within the 55 minutes. However, there is no time to take a break to refuel, leading to a shortage of snowflakes. As a consequence, it is only possible to serve Trip 2 after Trip 5, if Trip 2 is not served in the next but in the period thereafter. This results in 7 required vehicles, since at minute 50, apart from the 5 vehicles on Trips 1, 2, 3, 6, and 8, there are two more, which are on the empty runs between Trips 5 and 2.
In the second case, there are two distinct empty runs at minute 50, which again makes 7 vehicles necessary (see above).
In the third case, the snowmobiles of Trip 5 proceed to Trip 8. In particular, Trip 7 does not preceed Trip 8. Thus, the vehicle that served Trip 7 is on an empty run by necessity. Again, there are two options: either Trip 2 follows on this empty run or not. In the former case, the vehicle cannot enter Trip 2 at minute 20 of the next period, because of the empty run of 55 minutes towards the cinnamon depot. Consequently, it has to wait at the cinnamon depot for another 55 minutes for the period thereafter. If, at minute 50, Trip 2 is not served after Trip 7, there are two empty runs—one proceeding Trip 7 and one preceding Trip 2. Again, this results in at least 7 vehicles.

$\leq$ **7 snowmobiles:** It is possible, to operate the timetable with 7 snowmobiles, as is depicted in Figure 9.

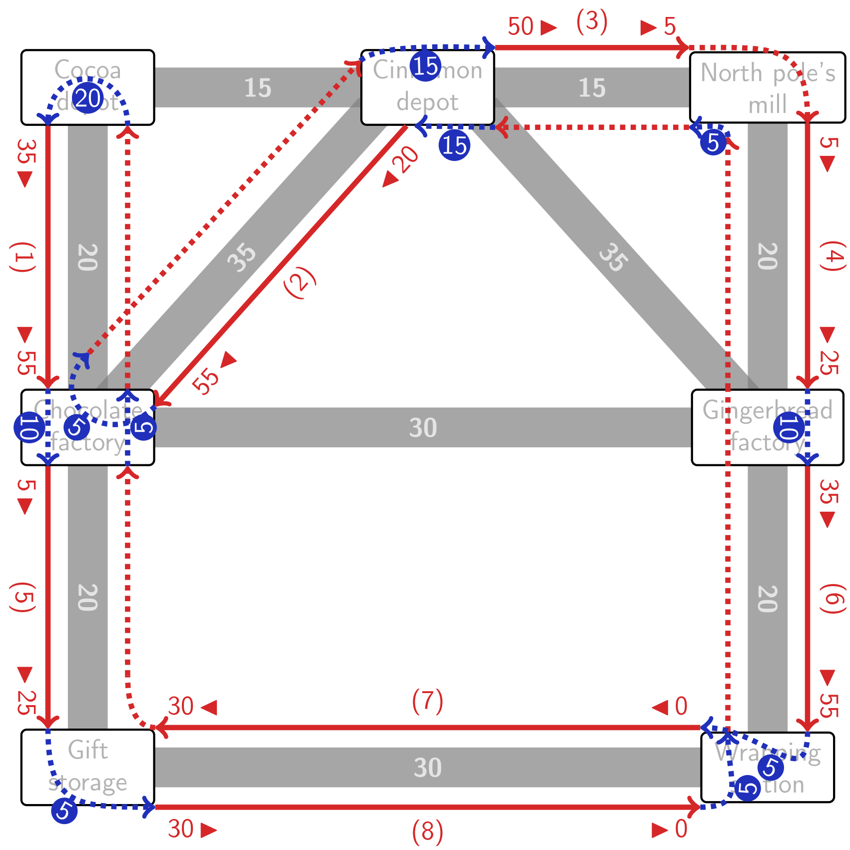


Рис. 9: Solution with 7 snowmobiles.

All 7 snowmobiles follow one large circle. The blue labels indicate how long a vehicle has to wait to recharge before continuing its journey. The total length of the circle is 420 minutes, which means that exactly $420/60 = 7$ snowmobiles are needed to operate this vehicle schedule.

11 Christmas Bauble Mashup

Authors: Christian Hercher (Europa-Universität Flensburg),
Michael Schmitz (Europa-Universität Flensburg)

Challenge

This year, elf Willi is operating the Christmas bauble machine in the Christmas workshop. The machine produces packs of coloured Christmas baubles. You can set the package size and the possible bauble colours. However, the machine randomly selects one of the selected colours for each bauble that goes into a package. Thus, it can happen, for example, that only baubles of one colour end up in a package, although more than one colour was set. At the moment, the machine is set to 5-packs with the colours red and silver. After Willi has opened three packs, he claims, "All 5-packs with colors red and silver are different from each other." Here we call two packs *the same* if they contain the same number of balls per colour, and *different* if they are not the same.
Elf Selma replies, "That can't be true, since there are only six different packs of this kind."

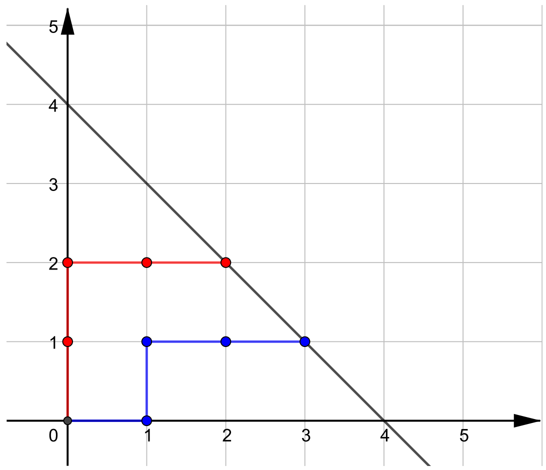
(a) How many different 10-packs of red, silver, and green baubles are there?

(b) Now, the machine is set to 4-packs with the colours red and silver. What is the probability that there are exactly 3 red balls in a pack?

(c) The machine is still set to 4-packs with the colours red and silver. What is the probability that two packages produced directly one after the other are the same?

(d) Now, the machine is set to 9-packs with the colors red and silver. What is the probability that two packages produced directly one after the other are the same?

To answer the questions in parts (b) to (d), the smart Selma visualizes the filling process for two possible colours, e. g. red and silver, using a path in a coordinate system. If a silver bauble enters the pack, the path goes one step to the right; if a red bauble enters the pack, it goes one step up. In the figure below, the blue path represents the filling of a 4-pack with three silver baubles and one red bauble. The red path represents filling a 4-pack with two silver baubles and two red baubles. Maybe this illustration will also help you?


5
4
3
2
1
0
1
2
3
4
5

Artwork: Julia Nurit Schönnagel

Possible answers:

1. (a) 59049, (c) $\frac{35}{128}$.
2. (a) 11, (d) $\frac{1}{10}$.
3. (b) $\frac{1}{5}$, (c) $\frac{1}{5}$.
4. (b) $\frac{1}{4}$, (d) $\frac{1}{10}$.
5. (c) $\frac{35}{128}$, (d) $\frac{12055}{65536}$.
6. (a) 66, (c) $\frac{1}{5}$.
7. (a) 59049, (d) $\frac{12155}{65536}$.
8. (b) $\frac{1}{4}$, (c) $\frac{35}{128}$.
9. (b) $\frac{1}{5}$, (d) $\frac{12155}{65536}$.
10. (c) $\frac{1}{5}$, (d) $\frac{12055}{65536}$.

Solution

The correct answer is: 8.

Only answer 8 is correct, all others contain at least one wrong value as we will see when calculating the individual solutions. Especially in combinatorics and stochastics, it is often the case thate there are many different possibilities and considerations that lead to the goal. We present two possible solutions here:

(a) We can represent each pack of 10 red, silver, and green baubles as a 12-tuple with exactly 10 zeros and 2 ones. Here, the numbers of zeros before the first one, between the two ones,and after the second one indicate the numbers of red, silver, and green baubles in the pack. E. g., the tuple $(0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0)$ represents a pack of 2 red, 3 silver, and 5 green baubles. Note that packs in which not all colours occur can also be represented in this way. For example, the tuple $(1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ represents a pack with only green baubles. Since there are $\binom{12}{2} = 66$ possibilities to choose the positions of the two ones from the 12 possible positions, there are as many tuples of this kind, that is packs of 10 with the coloured baubles in red, silver and, green.

Remark: This is the urn problem "drawing with placing back paying no regard to the order". For drawing k times from an urn with n balls, there are $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ possible outcomes. In our case, the machine draws $k = 10$ times from an urn with $n = 3$ baubles (in the colours red, silver, and green), i. e. there are $\binom{3+10-1}{n-1} = \binom{12}{2} = 66$ possible outcomes.

Alternative solution: Here we will proceed step by step and first look at Selma's statement in the case of packs with baubles with at most two colours.

Let $A_2(n)$ be the number of different packages with n baubles in red and silver. If the number r of red balls in such a package is known, we find that the number s of silver balls is already unambiguously determined: $s = n - r$; all balls in such a package that are not red are silver. Accordingly, for every number r of red balls, there is (up to equality) such a pack. Furthermore, for the number r of red baubles, any integer between inclusive zero and the total number of all baubles of such a pack n is admissible, we can state that there are exactly

$$A_2(n) = n + 1$$

distinct such packs in two available colours. In particular, $A_2(5) = 6$, and Selma's statement is correct.

Let us now turn to the task. We have to consider packages in three available colours. As before, we define the number of different such packs of size n with $A_3(n)$ and choose one colour: let g be the number of green baubles in such a package. Then, there is a total of $n - g$ baubles in red and silver. But we already know that there are $A_2(n - g)$ possibilities for this setting. Since the number of green baubles g must be between zero and n, we can conclude (exactly as above) that

$$\begin{aligned}A_3(n) &= A_2(n-0) + A_2(n-1) + \cdots + A_2(n-n)\\&= (n+1) + n + \cdots + 1\\&= \frac{(n+1)(n+2)}{2}.\end{aligned}$$

In particular, the number of 10-packs we are looking for is $A_3(10) = \frac{11 \cdot 12}{2} = 66$.

(b) We use Selma's illustration to solve this task. Each path from the coordinate origin with an endpoint on the straight line $x + y = 4$ represents a 4-pack with in red and silver. In each step, the path goes to the right with probability $\frac{1}{2}$ and up with probability $\frac{1}{2}$. Since there are four independent steps, each path has a probability of $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. All paths with endpoint $(1,3)$ represent a 4-pack with exactly three red baubles. Since there are exactly four such paths, the wanted probability is $\frac{4}{16} = \frac{1}{4}$.

Remark: Note that, in order to determine the probability, it is not productive to determine the number of different 4-packs win red and silver according to the approach from task part (a). Although is correct that there are $\binom{5}{1} = 5$ different such packs and that exactly one of them contains three red and one silver bauble, the wanted probability is not equal to $\frac{1}{5}$, because the different packs are not equally probable. This can be seen easily with Selma's path model: for example, only one path leads to the endpoint $(4,0)$. Hence, a pack of four red baubles is only produced with probability $\frac{1}{16}$.

Alternative solution: In order to make the probability calculation easier, we introduce a virtual order of the balls in the pack; so for us there is now a 1^{st} bauble, a 2^{nd} bauble and so on. This allows us to represent such a pack as a tuple, i. e., a sequence of entries whose order is relevant. If r stands for a red and s for a silver bauble, then (r, s, r, r) describes a 4-pack where the 1^{st}, 3^{rd} and 4^{th} baubles are red while the 2^{nd} is silver.

We introduced the order, because these tuples are all equally probable given a fixed pack of size n and a fixed number of colours. (This is called an *Laplace* experiment.) How many such tuples are there? For each entry, we have two possibilities: either the bauble is red or silver. For each new bauble, we can decide which colour it should have. Thus, the number of such tuples doubles with each further bauble that is packed, and we get $2 \cdot 2 \cdot 2 = 2^4 = 16$ such tuples. Hence, each tuple has a probability of $\frac{1}{16}$ when drawn at random.

And how many of these tuples contain exactly three red baubles? To answer this question, we have to choose the three of the four positions where the red baubles are to be placed. (The bauble in the remaining position is silver.) In this situation, there are $\binom{4}{3} = 4$ possibilities, so that the wanted probability is $\frac{4}{16} = \frac{1}{4}$.

(c) We use Selma's path model again and first consider, for any one $k \in \{0, 1, 2, 3, 4\}$, the probability that a path of length 4 ends at the point $(k, 4-k)$ (such a path represents a pack of k silver and $4-k$ red baubles). Because exactly k of the 4 steps must go to the right, there are $\binom{4}{k}$ different paths to the point $(k, 4-k)$. Because each of these paths has probability $\frac{1}{16}$, the probability that a path of length 4 ends at point $(k, 4-k)$ is $\binom{4}{k} \cdot \frac{1}{16}$. This is also the probability that a 4-pack with the colours red and silver contains exactly k red baubles. If two such packs are chosen at random and independently of each other, the probability that both contain exactly k red baubles is

$$\binom{4}{k} \cdot \frac{1}{16} \cdot \binom{4}{k} \cdot \frac{1}{16} = \binom{4}{k}^2 \cdot \frac{1}{256}.$$

Two randomly chosen packs are exactly identical if they both contain exactly 0 or 1 or 2 or 3 or 4 silver baubles. Consequently, the wanted probability is

$$\begin{aligned} \frac{1}{256} \cdot \sum_{k=0}^{4} \binom{4}{k}^2 &= \frac{1}{256} \cdot \left(\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 \right) \\ &= \frac{70}{256} = \frac{35}{128}. \end{aligned}$$

Alternative solution: Similar to the alternative solution to task part (b), we want to model the situation in such a way that we obtain a Laplace experiment again. Since we are now considering two 4-packs, we want to represent the baubles contained in them by a $2 \cdot 4 = 8$-tuple. (The fifth entry of this 8-tuple then represents the first ball of the second pack, the sixth of the tuple for the second ball of the second pack and so on).

However, we want to alter the modelling so that we can determine the wanted probability more easily: if we consider two 4-packs that are equal in the sense of the task, the number of red baubles in the first pack and the number of silver baubles in the second pack must always add up to 4 (or, in general, to the size of the pack). This is clear because the number of silver baubles in the second pack is equal to the number of silver baubles in the first pack. Of course, for the same reason, the reverse is also true: Whenever the number of red baubles in the first package is equal to the number of silver baubles in the second package, we have two packages that are equal in the sense of the above sense.

We now use this to adjust our modeling: We consider 8-tuples with entries 0 and 1, respectively. A 1 in the first 4 positions indicates that the corresponding bauble of the first pack is red, whereas a 0 indicates a silver bauble. For the second of the entries of the 8-tuple, however, the representation is exactly reversed: a 1 now indicates that the corresponding bauble of the second package is silver, and a 0 indicates a red bauble.

For example, the 8-tuple $(1, 1, 0, 0, 1, 1, 0, 0)$ encodes that the 1^{st} and 2^{nd} bauble of the first pack and the 3^{rd} and 4^{th} bauble of the second pack are red and the others are silver.

How many such tuples exist? As before, for each entry of the 8-tuple, we have two possibilities that can be chosen independently.Hence, the total number of such tuples is $2^8 = 256$.

And how many of these tuples encode two packages that are identical in the sense of the task? According to our observation, the number of red baubles in the first package and the number of silver baubles of the second package must add up to 4. We can read off this number directly from the constructed 8-tuples: it corresponds exactly to the number of 1s! Thus, we look for the number of those 8-tuples which contain exactly four 1s. These are $\binom{8}{4} = 70$, because we have to select the four places from the eight where the ones should be.

Therefore, the wanted probability is $\frac{70}{256} = \frac{35}{128}$.

(d) As in part (c), we obtain

$$\sum_{k=0}^{9} \binom{9}{k}^2 \cdot \left(\frac{1}{2^9}\right)^2 = \frac{12155}{65536}$$

as the wanted probability.

Remark: One can simplify the calculation of probabilities by acknowledging that the expression $\sum_{k=0}^{n} \binom{n}{k}^2$ is always equal to $\binom{2n}{n}$. This identity can be explained with Selma's path model: the expression $\binom{2n}{n}$ corresponds to the number of all paths of length $2n$ with endpoint (n, n), since exactly n of the $2n$ steps must go to the right. However, one can determine the number of these paths in another way: each such path must cross the straight line $x + y = n$ at some point $(l, n - l)$. Then, exactly l steps to the right were executed up to the point $(l, n - l)$. There are $\binom{n}{l}$ possibilities for this. To get to the point (n, n), exactly $n - l$ steps to the right must be executed afterwards. For this, there are $\binom{n}{n-l} = \binom{n}{l}$ possibilities. In total there are $\binom{n}{l} \cdot \binom{n}{l} = \binom{n}{l}^2$ possibilities to get to the point (n, n) via the point $(l, n - l)$. Because one can get to (n, n) via any of the points $(l, n - l)$ with $l \in \{0, 1, \ldots, n\}$, there are consequently $\sum_{l=0}^{n} \binom{n}{l}^2$ different paths of length $2n$ to the point (n, n).

This is a special case of Vandermonde's identity $\binom{m_1+m_2}{n} = \sum_{k=0}^{n} \binom{m_1}{k} \cdot \binom{m_2}{n-k}$ with $m_1 = m_2 = n$.

Alternative solution: According to the alternative solution to task (c), we derive the following probability for n-packs with baubles in red and silver to be the same (in the sense of the task):

$$p_n = \frac{\binom{2n}{n}}{2^{2n}}.$$

In particular, for $n = 9$, we have

$$p_9 = \frac{\binom{18}{9}}{2^{18}} = \frac{46820}{262144} = \frac{12155}{65536}.$$

Further information

For a general consideration (packs of n baubles in $d \leq n$ different colours) of the problem considered in the task, we refer to the following article:

J. Prochno, M. Schmitz. *A probabilistic way to discover the rainbow.*. The Mathematical Gazette, 106, 565, März 2022, S. 103–115.

arXiv:2011.01030, https://arxiv.org/abs/2011.01030 (2020).

12 Black Ice

Author: Cor Hurkens (TU Eindhoven)
Project: 4TU.AMI

Challenge

Last year's sporting highlight was certainly the *Iron Gnome 2021*, the traditional annual Iron Gnome competition. In the ninth part of the Iron Gnome competition, Ruprecht had to run across a snowfield from A to B. Figure 10 shows us the two points A and B in the snowfield together with a blue icy surface in the lower left corner, which was smooth as glass and extended hundreds of kilometers to the south and to the west. The northern edge and the eastern edge of the ice surface each form a perfectly straight line.

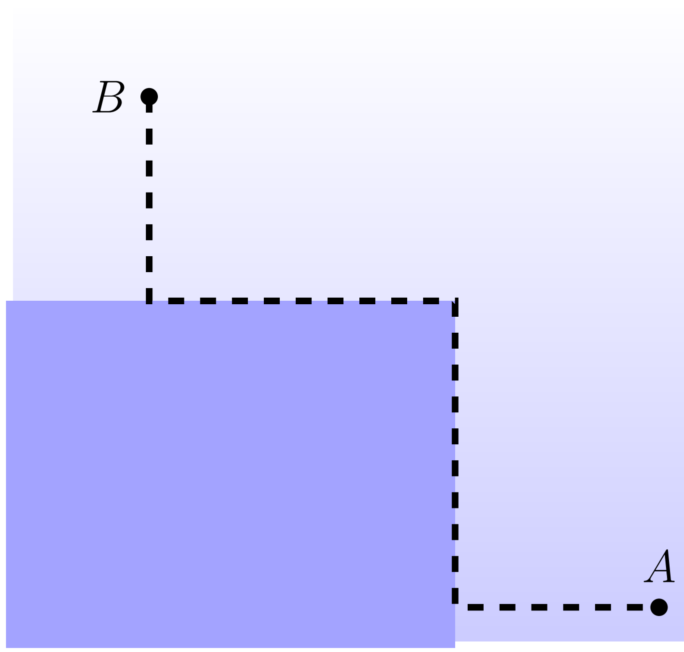


Рис. 10: The snowfield, the icy surface, and a possible path from A to B.

Figure 10 also shows a possible path that Ruprecht can take from A to B. The path goes from A first 6 km straight west until it reaches the eastern edge of the ice surface. Then, 9 km along the eastern edge to the north. Then, 9 km along the northern edge to the west. And finally, 6 km straight north to the point B.

On the icy surface, Ruprecht slides at a velocity of 10 km per hour. Walking across the snowfield is more arduous, and Ruprecht only manages to advance at $\sqrt{40}$ km per hour.
How much time does Ruprecht need for the *fastest possible* way from A to B?

Artwork: Friederike Hofmann

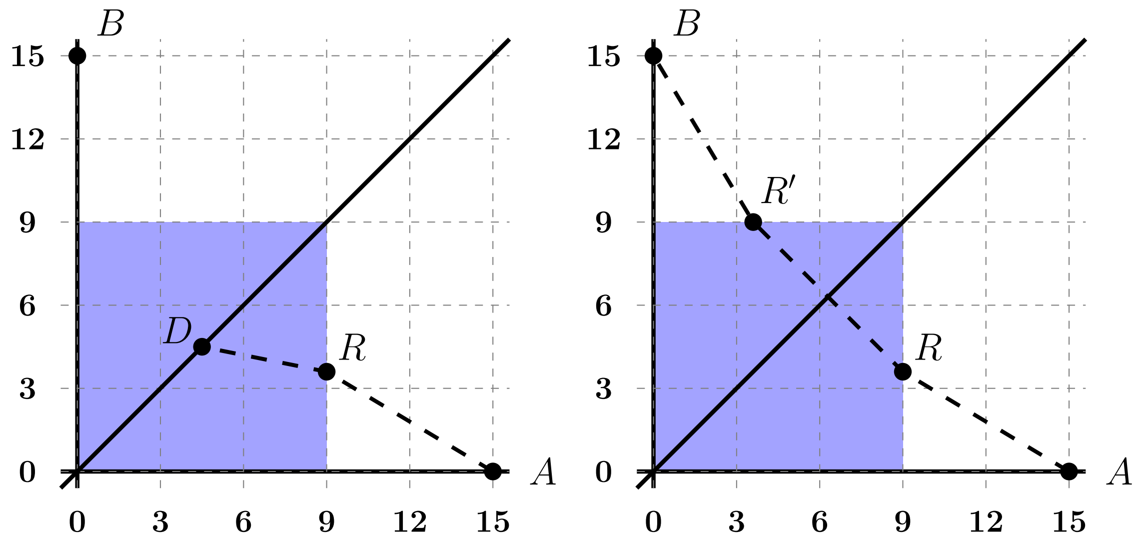
Possible answers:

1. Between 140 and 145 minutes.
2. Between 145 and 150 minutes.
3. Between 150 and 155 minutes.
4. Between 155 and 160 minutes.
5. Between 160 and 165 minutes.
6. Between 165 and 170 minutes.
7. Between 170 and 175 minutes.
8. Between 175 and 180 minutes.
9. Between 180 and 185 minutes.
10. Between 185 and 190 minutes.

Solution

The correct answer is: 8.

We introduce a coordinate system in which the unit length is 1 km: point A receives coordinates $(15, 0)$, point B receives coordinates $(0, 15)$, and the corner of the ice field is at point $(9, 9)$.



Now, we may assume that the fastest way from A to B is mirror-symmetric with respect to the diagonal $x = y$: the fastest path first leads Ruprecht from the starting point A to a point $R = (9, r)$ at the eastern edge of the ice field and then to some point $D = (d, d)$ on the diagonal. However, since the entire figure is symmetric to the diagonal, the remaining part of the fastest path from D to B can be chosen in such a way that it becomes the mirror image of the first part from D to A. In particular, this path will leave the ice field at a point $R' = (r, 9)$ such that the point R' is a mirror image of the point R. Hence, fastest path consists of three pieces:

- The straight line from $A = (15, 0)$ to $R = (9, r)$,
- the straight line from $R = (9, r)$ to $R' = (r, 9)$, and
- the straight line from $R' = (r, 9)$ to $B = (0, 15)$.

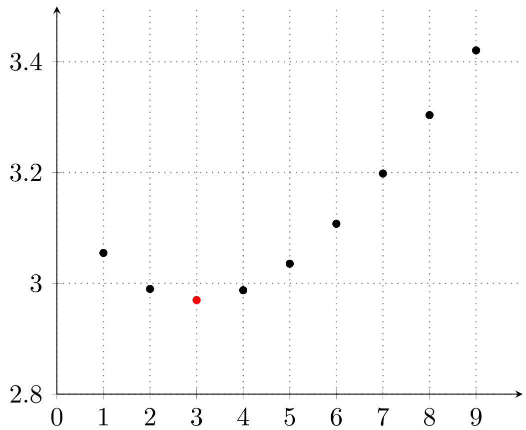
The total time (in hours) Ruprecht needs for this fastest path only depends on the coordinate r with $0 \leq r \leq 9$ and is given by

$$\begin{aligned} T(r) &= \frac{1}{\sqrt{40}} \cdot \sqrt{36 + r^2} + \frac{1}{10} \cdot \sqrt{2(9 - r)^2} + \frac{1}{\sqrt{40}} \cdot \sqrt{36 + r^2} \\ &= \sqrt{\frac{36 + r^2}{10}} + \frac{9 - r}{5\sqrt{2}}. \end{aligned}$$

We calculate some values of T in the range $0 \leq r \leq 9$ to get an impression of the behaviour of the function in this range:

r	1	2	3	4	5	6	7	8	9
$T(r) \approx$	3,05	2,99	2,97	2,99	3,04	3,11	3,20	3,30	3,42

If we draw these points in a coordinate system, we see that the function T in the range $0 \leq r \leq 9$ has a minimum point near $r = 3$ (see Fig. 11, left).



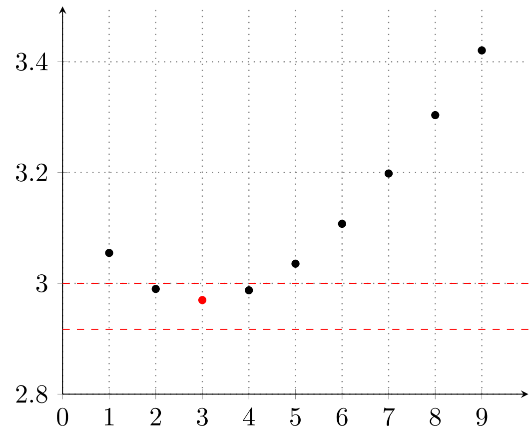


Рис. 11: Left: the calculated values of T in the coordinate system. Right: additionally plotted are the limits 175 and 180 minutes (red, dashed).

If one additionally draws the limits 175 minutes (corresponds to $\approx 2{,}92$ hours) and 180 minutes (corresponds to 3 hours) into the coordinate system (see Fig. 11, right), one recognises that the minimum value of the function T lies in this area.

Remark: If you are familiar with differential calculus, you may of course also consider the first derivative of the function T in order to find the minimum of T in the range $0 \leq r \leq 9$:

$$T'(r) = \frac{r}{\sqrt{10(36 + r^2)}} - \frac{1}{5\sqrt{2}}.$$

If one sets $T'(r) = 0$, one obtains the equation

$$\begin{aligned} & 0 = \frac{r}{\sqrt{10(36 + r^2)}} - \frac{1}{5\sqrt{2}} \\ \Rightarrow \quad & 5\sqrt{2}r = \sqrt{10(36 + r^2)} \\ \Rightarrow \quad & 50r^2 = 10(36 + r^2) \\ \Rightarrow \quad & r^2 = 9 \end{aligned}$$

and, furthermore, $r = 3$ in the range $0 \leq r \leq 9$. Now, one easily verifies that the function $T(r)$ has a minimum for the range $0 \leq r \leq 9$ at $r = 3$.

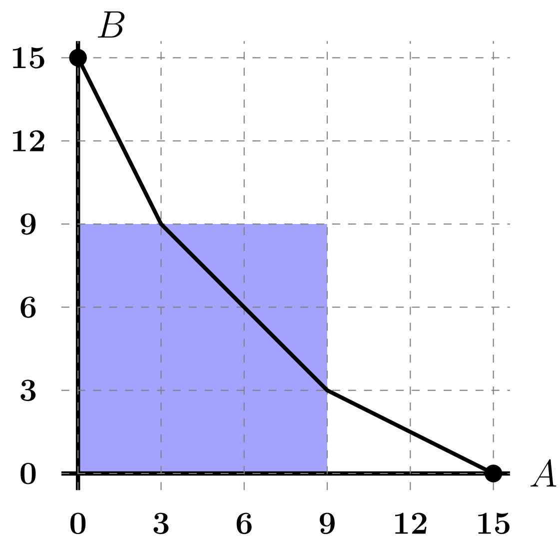


Рис. 12: The fastest pathway from A to B takes 178.2 minutes.

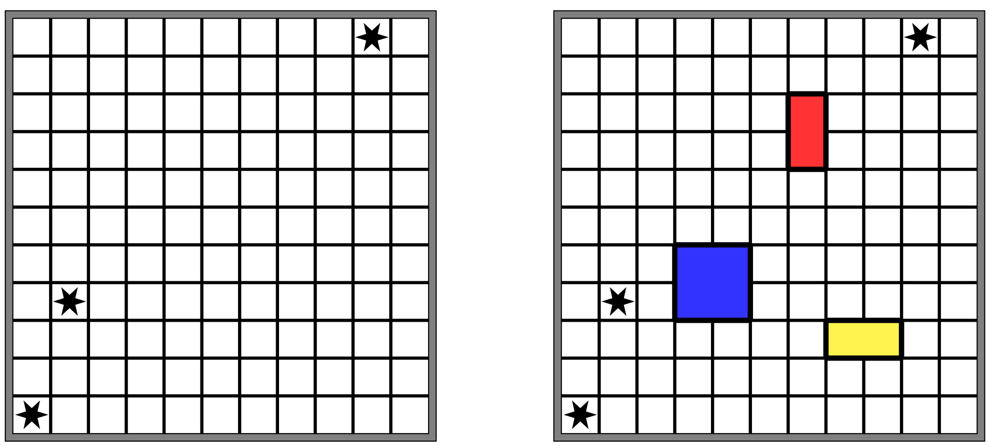
Hence, for the fastest path from A to B Knecht Ruprecht needs $T(3) = 21\sqrt{2}/10 \approx 2.97$ hours, and that is about 178.2 minutes (see Fig. 12).

13 Mondrian

Author: Cor Hurkens (TU Eindhoven)
Project: 4TU.AMI

Challenge

Mondrian, the painter-elf, has designed a square-shaped Christmas card and has divided it into 121 square-shaped cells in an 11×11 pattern (see Fig. 13a). Mondrian paints little stars into three of the cells *exactly* as shown in Figure 13a.



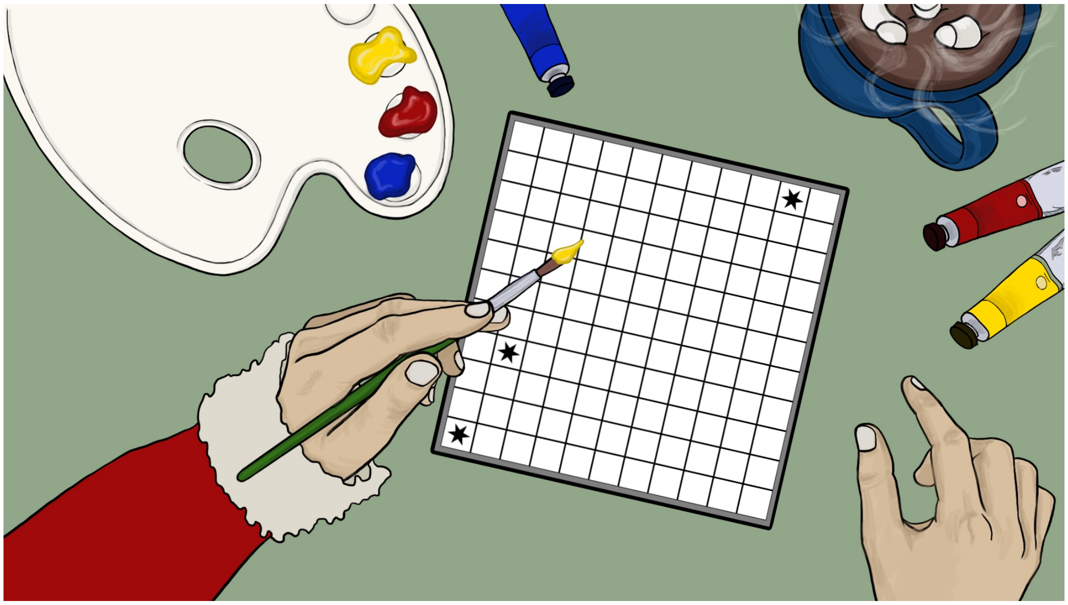
(a) Mondrian's Christmas card with the three little stars.

(b) Example: Mondrian's Christmas card with two rectangles and one square.

Рис. 13: Mondrian's Christmas card.

Then, Mondrian partitions the remaining grid of 118 cells into several 1×2 and 2×1 rectangles (each containing 2 cells), and 2×2 squares (each containing 4 cells) that he paints with bright colors (see Fig. 13b). In the end, *each* of the 118 cells belongs to *exactly one* such rectangle or square. The three cells with the little stars are not covered.

What is the largest possible number of 2×2 squares that Mondrian can paint onto his Christmas card?



Artwork: Frauke Jansen

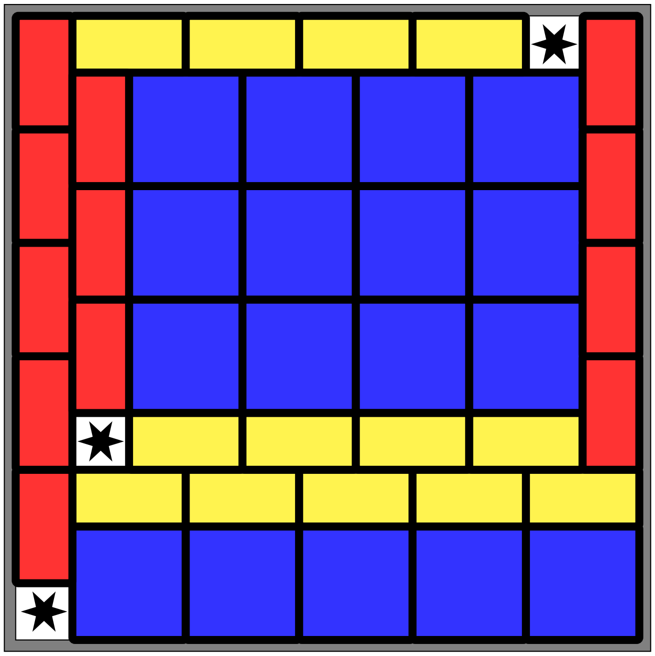
Possible answers:

1. The largest possible number of 2×2 squares is 14.
2. The largest possible number of 2×2 squares is 15.
3. The largest possible number of 2×2 squares is 16.
4. The largest possible number of 2×2 squares is 17.
5. The largest possible number of 2×2 squares is 18.
6. The largest possible number of 2×2 squares is 19.
7. The largest possible number of 2×2 squares is 20.
8. The largest possible number of 2×2 squares is 21.
9. The largest possible number of 2×2 squares is 22.
10. The largest possible number of 2×2 squares is 23.

Solution

The correct answer is: 4.

The largest possible number of 2×2 squares is 17. The following picture shows a possible solution with 17 blue squares, 13 yellow 2×1 rectangles and 12 red 1×2 rectangles:



Now, we show that there exists no solution with more than 17 squares. In the following figure, integers have been written into each of the 121 cells:

0	-1	2	-3	4	-5	6	-7	8	-9	10
1	0	-1	2	-3	4	-5	6	-7	8	-9
-2	1	0	-1	2	-3	4	-5	6	-7	8
3	-2	1	0	-1	2	-3	4	-5	6	-7
-4	3	-2	1	0	-1	2	-3	4	-5	6
5	-4	3	-2	1	0	-1	2	-3	4	-5
-6	5	-4	3	-2	1	0	-1	2	-3	4
7	-6	5	-4	3	-2	1	0	-1	2	-3
-8	7	-6	5	-4	3	-2	1	0	-1	2
9	-8	7	-6	5	-4	3	-2	1	0	-1
-10	9	-8	7	-6	5	-4	3	-2	1	0

The main diagonal of this array contains only zeros. Moreover, the array is *skew-symmetric* with respect to the main diagonals, i. e. if an entry below the main diagonal has the value w, the corresponding entry symmetrically located above the main diagonal has the value $-w$. It follows immediately that the sum of all 121 numbers is 0. Furthermore, the array also has the following two important properties:

- Each 2×2 square covers four numbers with sum 0.
- Each 2×1 or 1×2 rectangle covers two numbers with sum 1 or -1.

Now, consider an arbitrary partition of the 118 cells into rectangles and squares: since each 2×2 square covers four integers with sum 0 and since the three stars cover three cells with total sum $-10 - 6 - 9 = -25$, the 2×1 and 1×2 rectangles must cover a total sum of 25. Since each rectangle contributes at most 1 to this sum, there must be at least 25 rectangles in the partition. Thus, the total area of the rectangles is at least $25 \cdot 2 = 50$ cells; leaving at most $118 - 50 = 68$ cells for the total area of the squares. Therefore, there are at most $68/4 = 17$ squares in the Christmas card.

14 Auguste, the Christmas goose

Authors: Ariane Beier (MATH+ School Activities),
Mehran Seyed Hosseini (Universität Potsdam)

Challenge

Tuesday, December 14: a beautiful, cold winter day. Leopold returns from the farmer's market with full shopping bags and is sure: on Christmas Eve, his family will have the most delicious holiday feast. Fragrant pastries, the finest hors d'oeuvres, and, of course, a crispy Christmas goose. Leopold's mouth is already watering.
Attracted by the rustling and rumbling of the bags, his children, Elis, Gerda, and little Piotr, enter the kitchen with curious faces.
"Would you please get the goose from the cargo bike?!" Leopold asks his children.
"Dad! I thought we agreed on a vegetarian Christmas menu..." Gerda grumbles. But by then Elis and Piotr have already scurried downstairs. Shortly after, they come back escorted by a big chattering bird.
"Dad... it's still alive..." Gerda remarks the obvious.
"Yes, um... it was cheaper, and besides it's fresher then, isn't it?" Leopold stammers and scratches his half-bald head.
"We can't eat it!" protests Piotr.
"No way!" Elis and Gerda agree.
"But we can hardly keep it and it is already paid for..." Leopold says in desperation.

After a lengthy argument, Leopold, Elis, Gerda, and Piotr agree on the following: Leopold places five boxes side by side on an imaginary straight line and places Auguste (yes, the children have already grown fond of the animal and given it a name) in a box of his choice. Then, the children are allowed to look into exactly one box together. If Auguste is in this box, they have won: Auguste will not be eaten and is a free bird. However, if Auguste is not sitting in the chosen box, Leopold places her in an *directly adjacent* box on the following day, and the children may guess again. The "game" continues until the children have found Auguste or until 24 December, when the goose definitely goes into the oven.

In order to give Elis, Gerda, and Piotr time to think and Leopold to build appropriate boxes, the game does not begin until tomorrow—that is, 15 December. The children cannot see, hear, smell or feel from the outside which box the goose is in. And, of course, Auguste is fine in all five boxes.

Which of the ten statements is correct?

Artwork: Frauke Jansen

Possible answers:

1. There is no strategy with which the children can find Auguste.
2. Whether the children can find Auguste in time depends on which box Leopold places her in at the beginning.
3. With the best strategy, the children find Auguste in the worst case on 17 December.
4. With the best strategy, the children find Auguste in the worst case on 18 December.
5. With the best strategy, the children find Auguste in the worst case on 19 December.
6. With the best strategy, the children find Auguste in the worst case on 20 December.
7. With the best strategy, the children find Auguste in the worst case on 21 December.
8. With the best strategy, the children find Auguste in the worst case on 22 December.
9. With the best strategy, the children find Auguste in the worst case on 23 December.
10. With the best strategy, the children find Auguste in the worst case just in time on 24 December.

Solution

The correct answer is: 6.

Elis, Gerda, and Piotr can find Auguste in time before Christmas Eve— in the worst case even on 20 December. We derive a successful strategy below: to this end, we number the boxes from left to right with the numbers 1 to 5.

Case 1: Suppose Leopold puts Auguste in a box with an even number, i. e. 2 or 4.

On the first day, the children look in box no. 2. If Auguste is in this box, she is already saved. If Auguste is not in box no. 2, she *must* be in box 4. On the next day, she will be box no. 3 or 5. Thus, the children choose the third box on the second day. If Auguste is in this box, she is saved. If not, she *has to* sit in the fifth box. The following day Leopold *has to* put her in box no. 4, and the children will save her if they look in this box.

Conclusion 1: If, at the beginning, Auguste is in a box with an even number, she will be found on the third day at the latest if the children look into boxes no. 2, 3, and 4 one after the other.

Case 2: Suppose Leopold puts Auguste in a box with an odd number, i. e. 1, 3 or 5.

If Auguste is not found on the first day, she sits in box 2 or 4 on the second day. If she is not found then, she sits in 1, 3 or 5 on the third day. If she is not found then, she sits in 2 or 4.

Conclusion 2: If, at the beginning, Auguste sits in a box with an odd number, on the fourth day she will be in a box with an even number.

We can now combine the two results above into one strategy: Elis, Gerda, and Piotr look one after another in boxes no. 2, 3, and 4. If they have not found Auguste on the third day, then the children know that Auguste was in an odd-numbered box at the beginning, but is now sitting in box no. 2 or 4. Hence, they will find her after another three days if they look again in boxes no. 2, 3, and then 4. On the sixth day at the latest—that is, on 20 December—the goose will be saved from the oven.

Magic Ribbons

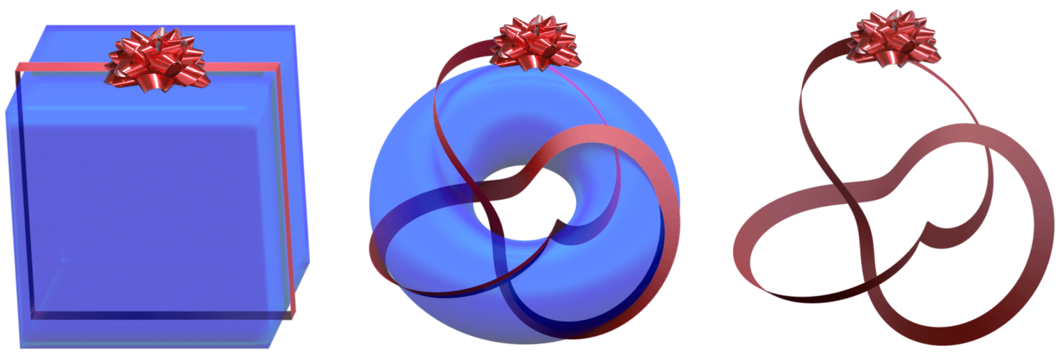
Authors: Myfanwy Evans (Uni Potsdam),
Frank Lutz (TU Berlin)

Project: Thematic Einstein Semester 2021
"Geometric and Topological Structure of Materials"

Challenge

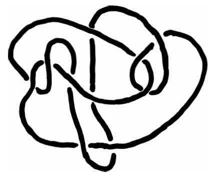
Far, far up in the north is the home of the elves. When winter is near, the elves help Santa Claire to wrap presents with their magic ribbons. Spheric sounds ring upon touching the strings.

In the past year, the elves faced a special challenge when Santa Claire asked them to wrap a chocolate torus. They cut open one of their ribbons, wound it around the chocolate torus and used magic to glue the ribbon back together. One of the elves realized that the ribbon on the torus became knotted, but soon forgot about it.

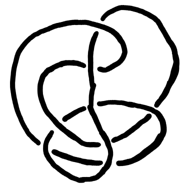


Every year, the magic ribbons are returned to the elves and stored to be reused for the next season. The special ribbon ended up in a bag with nine other ribbons. When the elves took out the ribbons in their preparations for this year's festivities, the ribbons appeared somewhat "entangled", and it took them quite some time to disentangle nine of them. Only then they remembered that one of the ten is special.

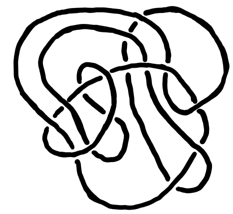
Which one is the special ribbon that was wrapped around the chocolate torus last year?



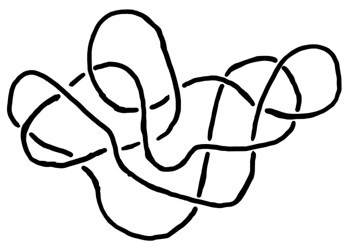
Ribbon no. 1.



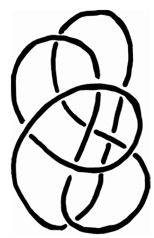
Ribbon no. 2.



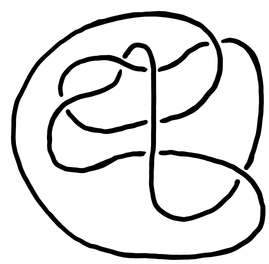
Ribbon no. 3.



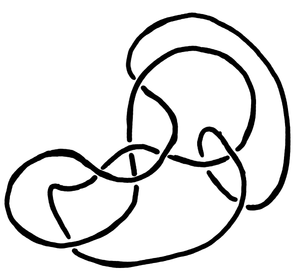
Ribbon no. 4.



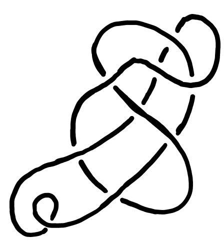
Ribbon no. 5.



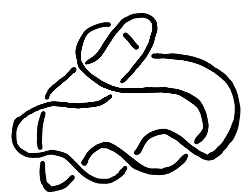
Ribbon no. 6.



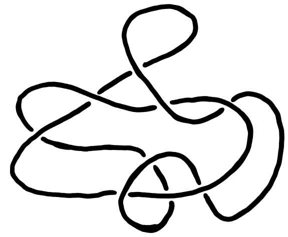
Ribbon no. 7.



Ribbon no. 8.



Ribbon no. 9.



Ribbon no. 10.

Artwork: Friederike Hofmann

Possible answers:

1. Ribbon no. 1.
2. Ribbon no. 2.
3. Ribbon no. 3.
4. Ribbon no. 4.
5. Ribbon no. 5.
6. Ribbon no. 6.
7. Ribbon no. 7.
8. Ribbon no. 8.
9. Ribbon no. 9.
10. Ribbon no. 10.

Project reference:

Physical properties of materials are governed to a large extent by their microstructure. Some materials are highly ordered like crystals, some are polycrystalline like rocks or metals, others are cellular like soap or metallic foams, are disordered like amorphous solids, and some even are entangled like DNA.

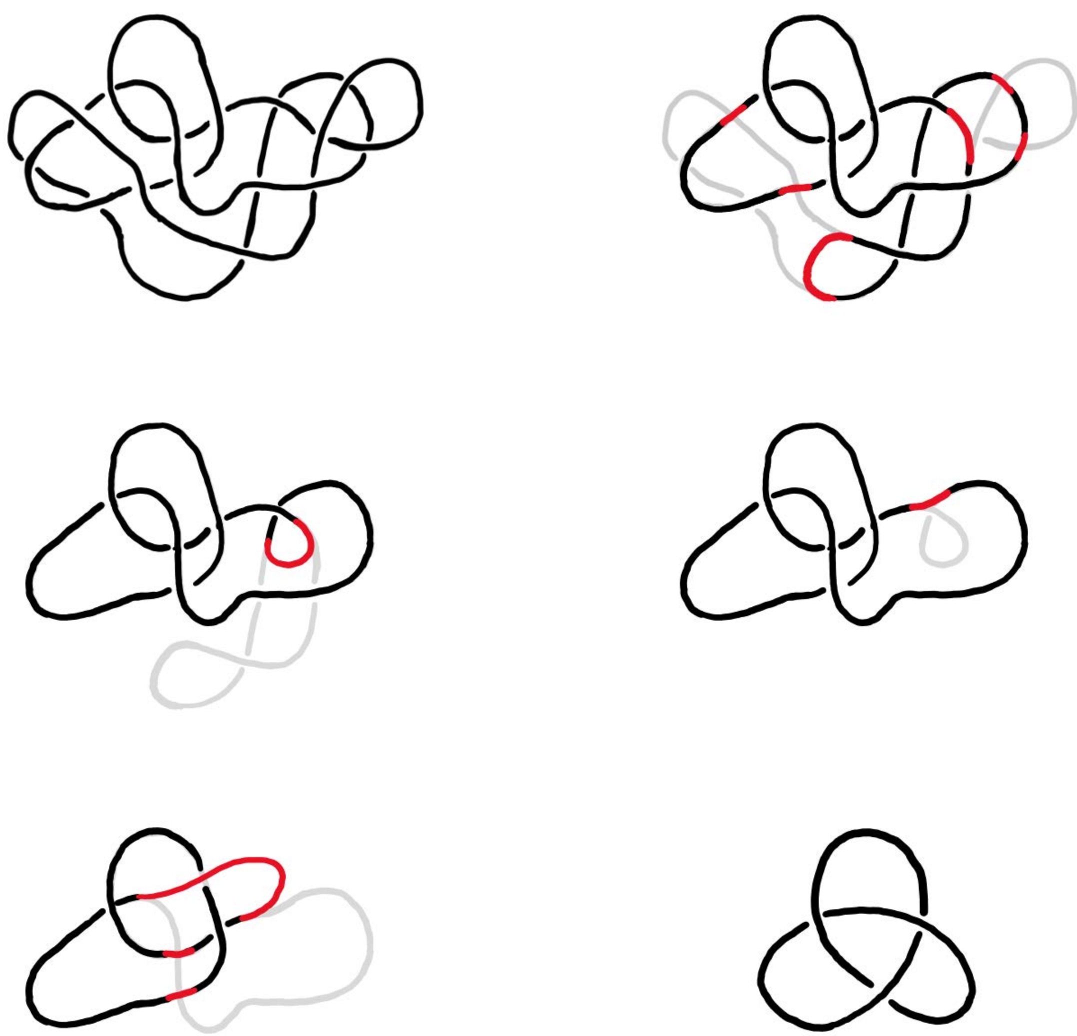
The Thematic Einstein Semester 2021 "Geometric and Topological Structure of Materials" was devoted to illuminate recent mathematical developments for a better understanding of materials by identifying or computing essential structural properties—eventually leading to improvements in production processes or to new designs of materials with controlled properties.

Solution

The correct answer is: 4.

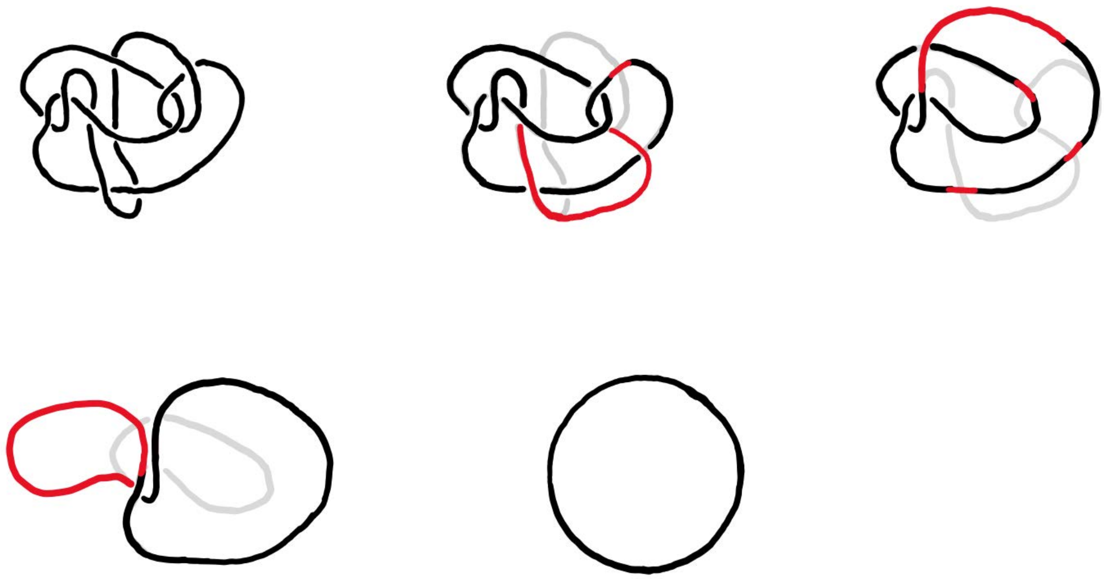
We will deform the fourth ribbon into the *trefoil knot*, which was wrapped around the chocolate torus. The trefoil knot is the simplest example of a non-trivial knot:

Ribbon no. 4:

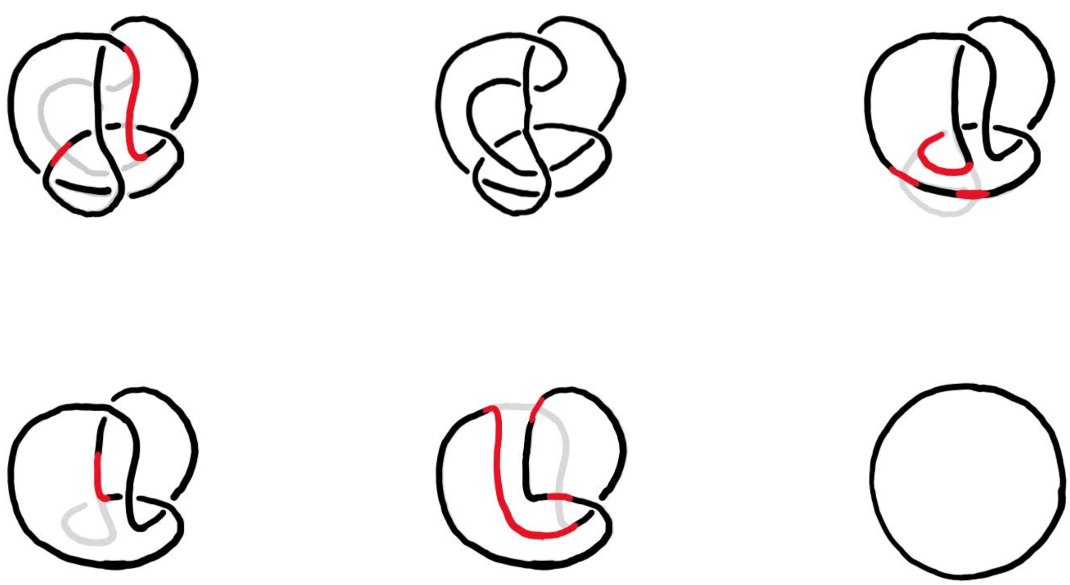


The other ribbons can indeed be transformed into the *unknot*, i. e. the trivial knot, which was wrapped around the other presents.

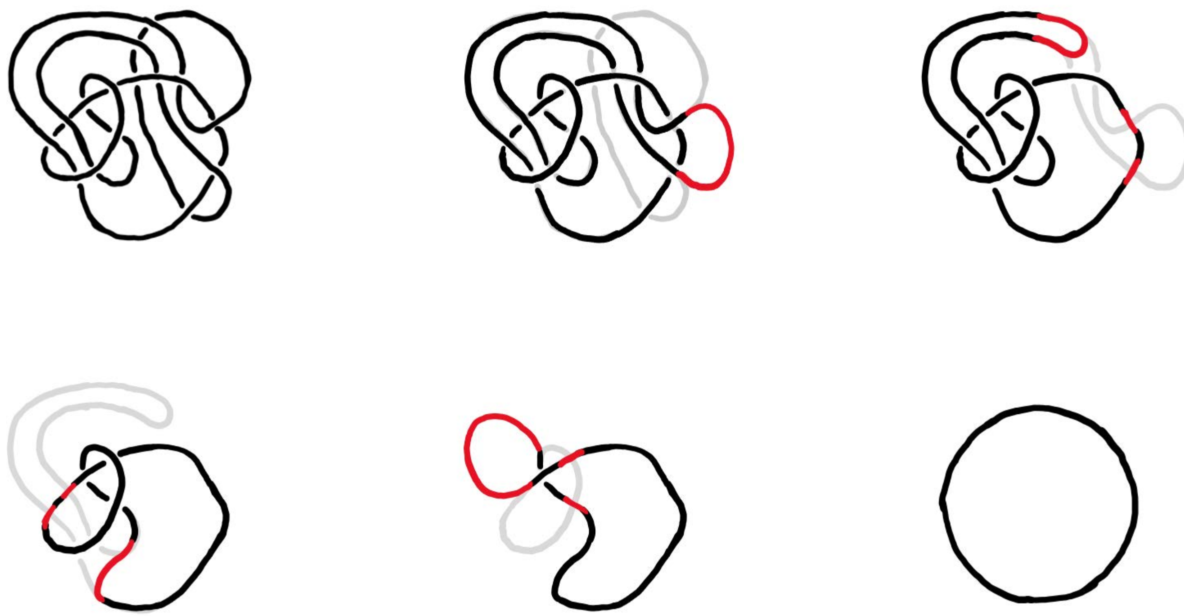
Ribbon no. 1:



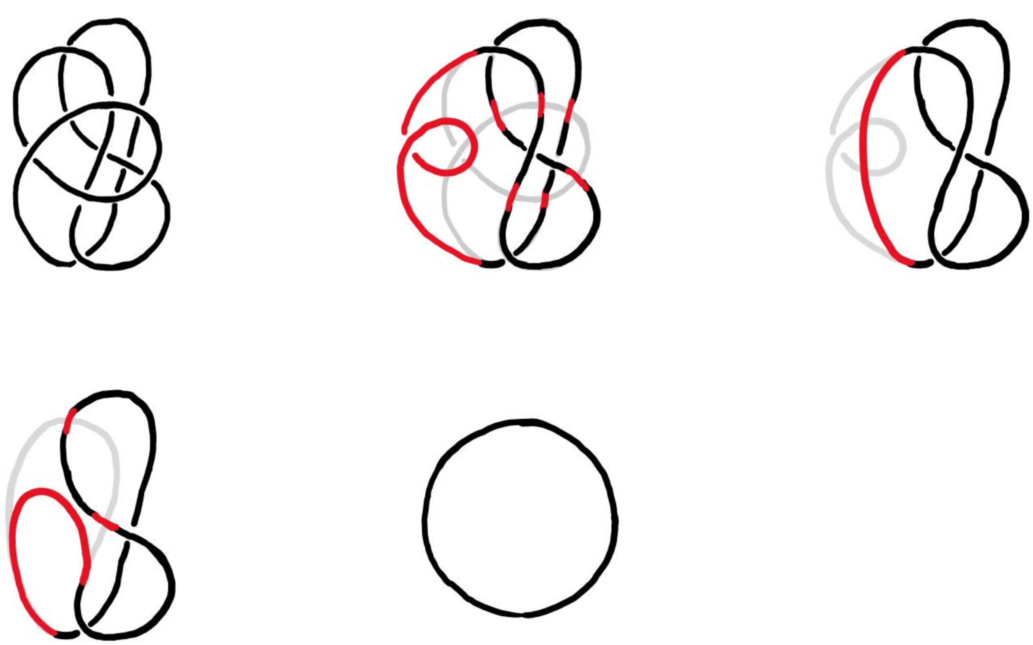
Ribbon no. 2:



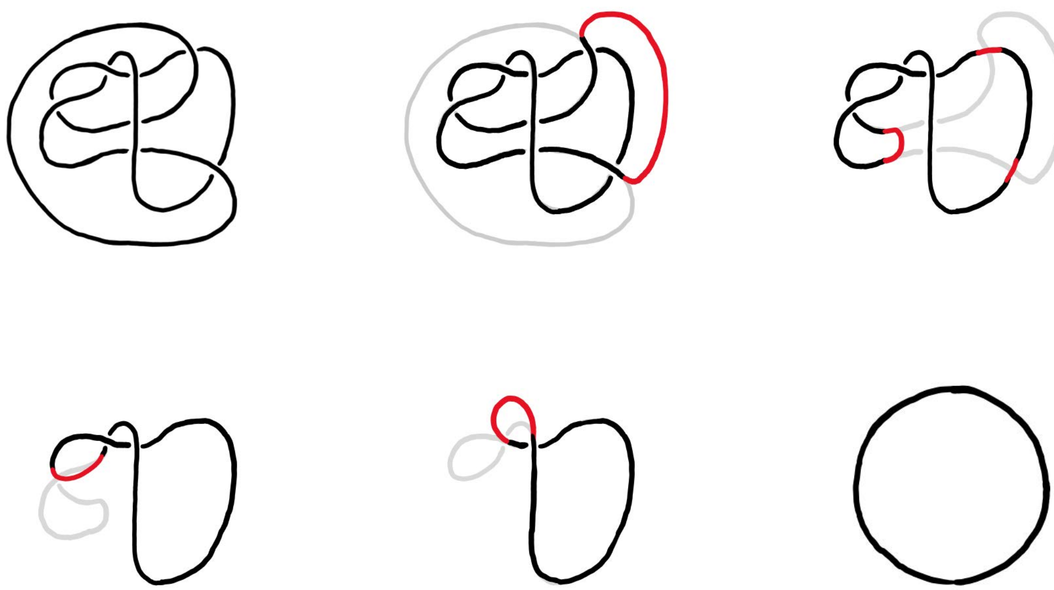
Ribbon no. 3:



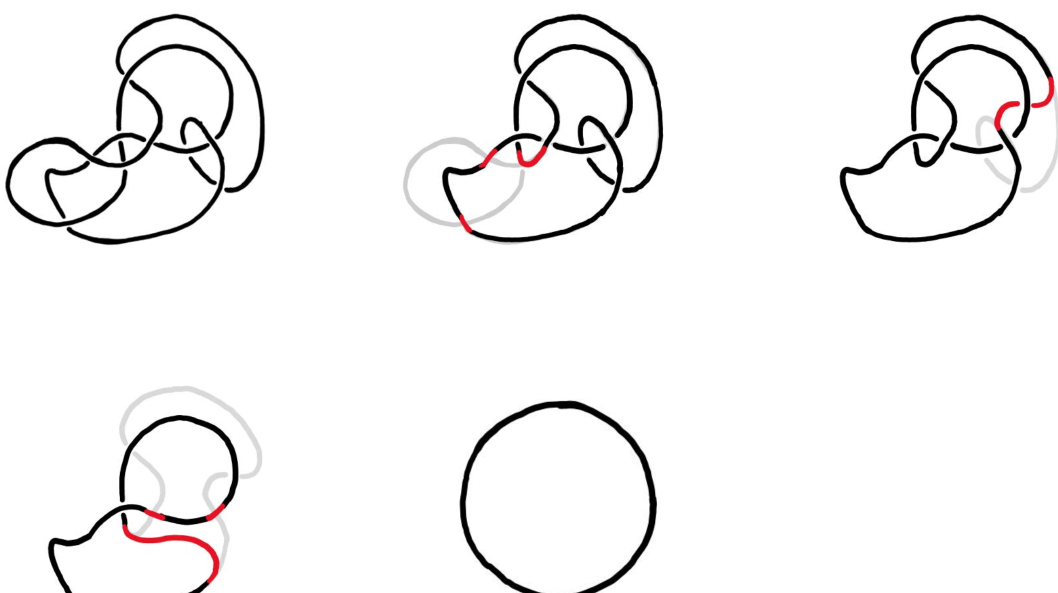
Ribbon no. 5:



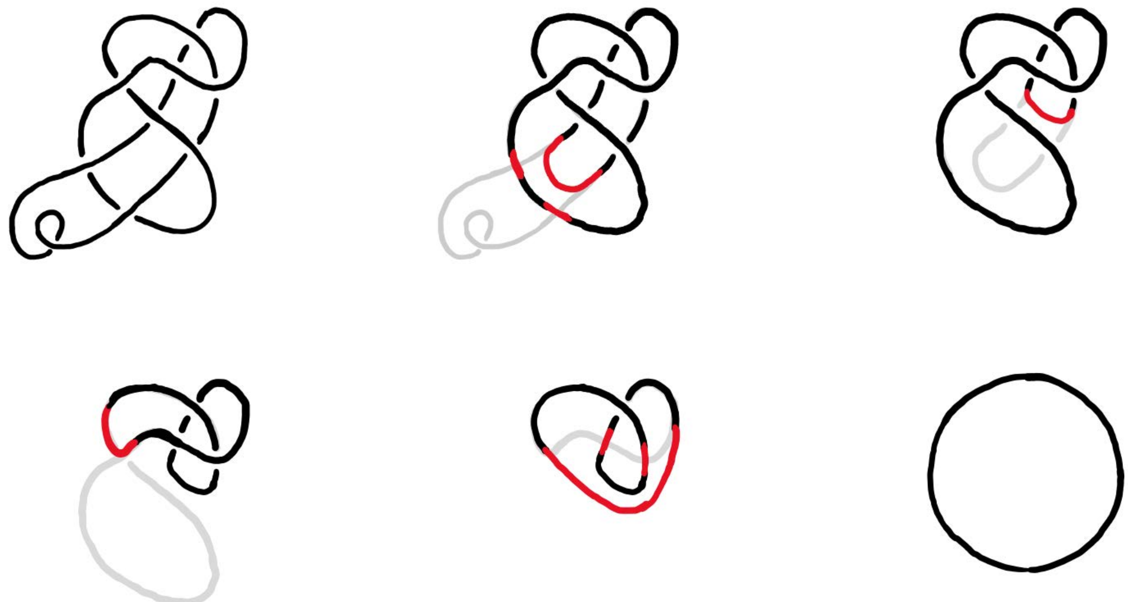
Ribbon no. 6:



Ribbon no. 7:

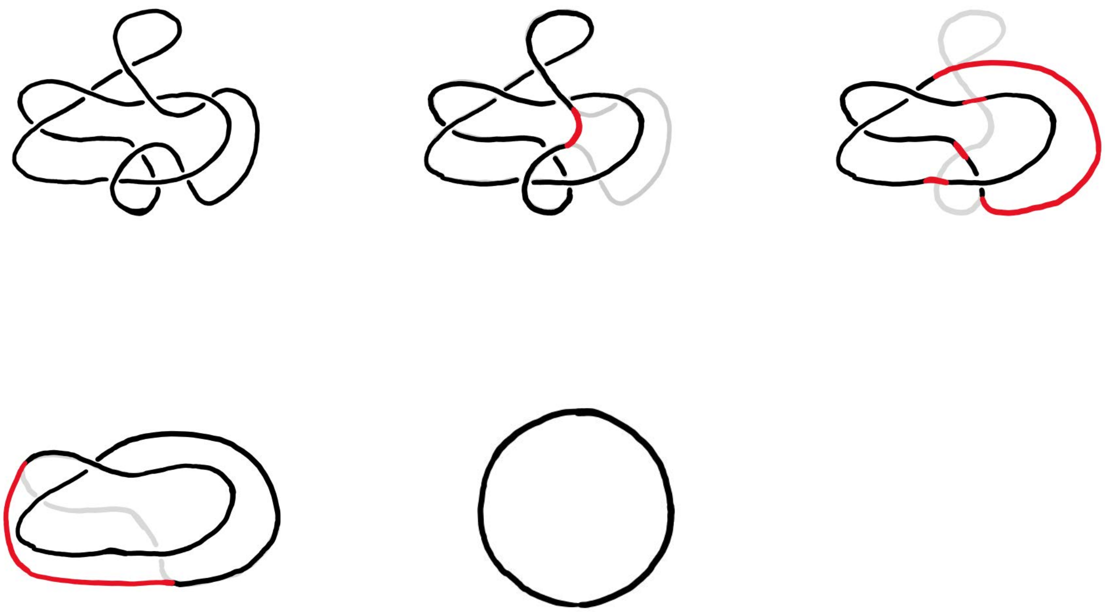


Ribbon no. 8:



Ribbon no. 9:

Ribbon no. 10:



The moves we performed on the ten ribbons are variants of the three *Reidemeister moves*, which transform a knot diagram into equivalent (or *isotopic*) knot diagrams.

15 Reindeer Election

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

Challenge

Once and for all, Santa has had enough of the quarrals among the five chief reindeer, Rudolph, Blitzen, Comet, Dasher, and Dancer, about who gets to lead Santa's sleigh on Christmas Eve this year. Therefore, he has decided to let the entire population of Elf Valley vote.

The election will have four rounds of voting: in the first round, Reindeer 1 competes against Reindeer 2. The winning reindeer from the first round will compete against reindeer 3 in the second round. In turn, the winning reindeer from this round competes against reindeer 4 in the third round. The reindeer that won the third round competes against reindeer 5 in the fourth (and final) round.

A round of voting is won by the reindeer that receives more than 50 % of the votes. The reindeer that wins the last is the winner of the election and will lead Santa's sleigh on Christmas Eve.

Santa is very pleased with his brilliant idea. Now, he just has to figure out how to determine the order of reindeer 1 through 5. Since he is quite sure that this will not affect the outcome of the election anyway, and Rudolph is secretly his favorite reindeer, Rudolph gets to decide the order.

Very pleased with Santa's choice, Rudolph takes a look at the predicted voting behavior of the people of Elf Valley—but quickly reckons that his chance of winning the election is pretty low. In fact, you can quite acurately tell how the population of Elf Valley will vote. According to this, the population is divided exactly into five equally sized groups with the following preferences:

Group 1: Dasher > Dancer > Blitzen > Rudolph > Comet

Group 2: Dasher > Dancer > Rudolph > Comet > Blitzen

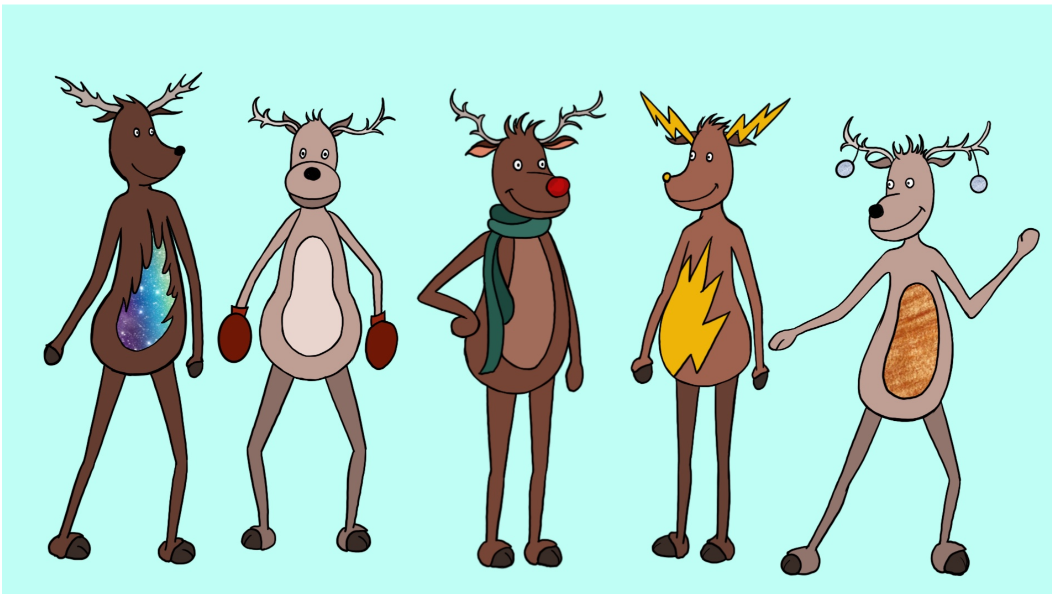
Group 3: Comet > Blitzen > Dasher > Dancer > Rudolph

Group 4: Dancer > Comet > Blitzen > Dasher > Rudolph

Group 5: Dancer > Blitzen > Dasher > Rudolph > Comet

That is, if a person from group 4 were to vote, they would vote for Comet in a round against Blitzen, Dasher, or Rudolph, but for Dancer in a round against Comet. We also know that the population of Elf Valley is very reliable: on election day, all residents of Elf Valley exercise their right to vote and vote exactly as predicted.

Which order of the reindeer should Rudolph choose to be allowed to lead the sleigh as the winner of the election process?



Artwork: Frauke Jansen

Possible answers:

1. Comet, Dasher, Dancer, Blitzen, Rudolph.
2. Dasher, Dancer, Blitzen, Rudolph, Comet.
3. Dasher, Dancer, Rudolph, Comet, Blitzen.
4. Dasher, Blitzen, Dancer, Comet, Rudolph.
5. Dasher, Blitzen, Dancer, Rudolph, Comet.
6. Dancer, Blitzen, Dasher, Comet, Rudolph.
7. Dancer, Dasher, Comet, Rudolph, Blitzen.
8. Dancer, Dasher, Blitzen, Comet, Rudolph.
9. Dancer, Dasher, Comet, Blitzen, Rudolph.
10. There is no order for which Rudolph is able to win the election.

Solution

The correct answer is: 8.

Rudolph wants to be the winner after the four rounds. Thus, he **has to** win the 4th round. Against which of the other four reindeers is this possible? We see that it cannot be Dasher or Dancer since Rudolph has less approval than them in all five groups. In a head-to-head election against either one of them, he would get 0 % of the votes and lose. Against Blitzen he would lose with only 20 % of the votes, since only the people in group 2 would choose him over Blitzen. In an election against Comet, however, Rudolph would win: he would get 60 % of the votes (from groups 1, 2 and 5). We know now that Rudolph would only win against Comet. Hence, we obtain:

- Reindeer 5: Rudolph,
- Reindeer 4: Comet.

Now, Comet **has to** win the third round. This is only possible in a duel against Blitzen, where Comet would win with 60 % of the votes (from groups 2, 3 and 4). However, Comet would lose against Dasher and Dancer with 40 % and 20 % of the votes respectively. We get:

- Reindeer 3: Blitzen.

Blitzen also **has to** win the second round. Against Dancer, Blitzen would lose the election with just 20 % of the votes. A direct duel against Dasher would be won by Blitzen with 60 % of the votes.
Therefore, reindeer 1 and 2 who take part in the first election are Dasher and Dancer. The round would be won by Dasher with 60 % of the votes.

Thus, an order that leads to Rudolph being the winner is given by:

1. Dasher, 2. Dancer, 3. Blitzen, 4. Comet, 5. Rudolph.

The only other possibility would be to switch the positions of Dasher and Dancer since round 1 is symmetrical.

16 The Eggnogg Chocolate Egg

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

Challenge

From the last Easter egg hunt, Christmas elves Annelie and Bernd still have 111 nougat chocolate eggs and one single eggnogg chocolate egg left. The two simply did not manage to eat up the treats yet, because they have been too busy with Christmas preparations since April. Now, the eggs are past their best-before date, and, besides, Santa will certainly will bring them new sweets for Christmas. So, they have to speed up their consumption. Annelie loves eggnogg, but Bernd would also like to eat the last egg of this kind. Therefore, he suggests the following game: the 111 nougat eggs are placed in a bowl and the two elves take turns grabbing one to eight nougat eggs. Before each turn, they can decide how many eggs they want to take from the bowl. Whoever takes the last nougat egg may also have the eggnogg egg.

Bernd leaves it up to Annelie to decide whether or not to start. Which of the following decisions should Annelie make if she wants to have the eggnogg egg?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. Annelie should make the first move, taking one nougat egg.
2. Annelie should make the first move, taking two nougat eggs.
3. Annelie should make the first move, taking three nougat eggs.
4. Annelie should make the first move, taking four nougat eggs.
5. Annelie should make the first move, taking five nougat eggs.
6. Annelie should make the first move, taking six nougat eggs.
7. Annelie should make the first move, taking seven nougat eggs.
8. Annelie should make the first move, taking eight nougat eggs.
9. Annelie should make the first move, but it does not matter how many eggs she takes.
10. Annelie only has a chance to win if she lets Bernd make the first move.

Solution

The correct answer is: 3.

We will solve the problem by working backwards from the game's end:

- If it is Bernd's turn and there are eight or less nougat eggs left, Bernd can just take them and get the eggnog egg.
- However, if there are nine nougat eggs in the bowl, there is at least one nougat egg and at most eight nougat eggs left after Bernd's turn. Annelie can then take them and win both the game and the eggnog egg.
- If there are ten to seventeen nougat eggs in the bowl, Bernd can make sure that—before it is Annelie's turn again—there are nine eggs left in the bowl. Of those eggs Annelie can take a maximum of eight eggs, thus making Bernd the winner in this case.

Taking all of this into account, Annelie should make sure that before Bernd's last turn there are exactly nine nougat eggs in the bowl. She can force that state if she leaves exactly eighteen nougat eggs to Bernd in the turn before. She achieves that by leaving 27 eggs to Bernd in the previous turn, and so on.

Hence, Annelie has to make sure that before Bernd's turns there are always $n \cdot 9$, where $n = 1, 2, 3, \ldots$, nougat eggs in the bowl. Since 111 is not a multiple of nine and $111 = 12 \cdot 9 + 3$ is true, Annelie should make the first turn and take three nougat eggs. Then, Bernd is left with $12 \cdot 9$ eggs in his first turn.

Note: Although Annelie makes sure to get the eggnog egg in the end when using this strategy, Bernd can get $12 \cdot 8 = 96$ nougat eggs if he always takes the maximum amount of nougat eggs possible. Since nougat eggs are Bernd's favourite, this deal works out quite well for him.

17 Hat Challenge 2021

Author Aart Blokhuis (TU Eindhoven)
Project: 4TU.AMI

Challenge

Santa Claus tells the four super-smart elves Atto, Bilbo, Chico and Dondo, "For the sake of our annual tricky hat challenge, I would like to invite you to my place for coffee and cake."
"We are happy to accept!" answer the four elves.

Santa Claus continues, "Okay, then I am going to prepare several blue, yellow, and red hats. Tomorrow, I'll put one hat on each of your heads, so that none of you can see the hat color of your own hat. Each of you will be able to see the hat colors of the other three elves. However, you are not allowed to exchange any information with each other. Furthermore, each of you will receive seven cards with the following statements:

T1: My hat color is blue.
T2: My hat color is yellow.
T3: My hat color is red.
T4: My hat color is blue or yellow.
T5: My hat color is blue or red.
T6: My hat color is yellow or red.
T7: My hat color is blue or yellow or red.

Then, simultaneously each of you must guess the color of your own hat by publicly showing one of your seven cards. If at least one of you guesses wrong, the game is over, and you will have to leave. If all of your guesses are correct and if at least one of you guesses the correct color with one of the cards T1, T2, or T3, all of you will get a tasty piece of mozarttorte and a nice cup of coffee. If you all use T4, T5, T6, or T7 correctly, we will repeat the guessing round—but only if you don't all use T7. In this case, I will send you straight home."

Atto wants to know, "How are you going to choose our hat colors?"
"They are chosen randomly, so that each of the 81 color combinations arises with the same probability," answers Santa Claus.

Bilbo asks, “Are we going to get new hats, in case the guessing round is repeated?”
“No!” answers Santa Claus. “The colors will not be changed. But, of course, you are allowed to use different cards in every round.”

Chico wants to know, “What is going to happen if in the second guessing round everybody guesses correctly again, and again only the cards T4, T5, T6, or T7 are used?”
“Then, the guessing round will be repeated. After that round, it may be repeated again, and again, and again, as long as you like,” says Santa Clause.

The elves start to ponder. They discuss, and they think. They think, and they discuss. Then they discuss some more, and they think some more. Eventually, they manage to develop an amazing strategy that will guarantee them coffee and cake for M (of the 81) color combinations. This strategy is even optimal; that is, there is no strategy that guarantees them coffee and cake for $M + 1$ color combinations.

Which of the following statements is correct?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. $0 \leq M \leq 32$.
2. $33 \leq M \leq 37$.
3. $38 \leq M \leq 42$.
4. $43 \leq M \leq 47$.
5. $48 \leq M \leq 52$.
6. $53 \leq M \leq 57$.

7. $58 \leq M \leq 62$.
8. $63 \leq M \leq 67$.
9. $68 \leq M \leq 72$.
10. $73 \leq M \leq 81$.

Solution

The correct answer is: 9.

We will show that $M = 72$. To this end, we represent the hat color combinations of Atto, Bilbo, Chico, Dondo as four-letter words with the letters B, Y, R (blue, yellow, red), where the four letters corresponding to the four elves. We say that two color combinations are *adjacent* if they differ in exactly one letter. Thus, two adjacent color combinations can be trandformed into one other by giving a single elf a different hat.

Why M=72 is possible. We describe a strategy that guarantees success in at least 72 of the 81 possible cases. This strategy is centered around the following nine *special* color combinations $F_1, \ldots, F_9$:

	Bxxx	Yxxx	Rxxx
xBxx	BBBB	YBRY	RBYR
xYxx	BYYY	YYBR	RYRB
xRxx	BRRR	YRYB	RRBY

Looking at two different special color combinations F_i and F_j with $i \neq j$, there is exactly one elf that has the same hat color in situations F_i *and* F_j. Or, in other words, if one wants to transform F_i into F_j, one has to change the hat color of exactly three elves. Furthermore, each non-special color combination is adjacent to exactly one special color combination. Consequently, the strategy of the elves in the first round is the following:

- Each elf looks at the hats of the other three elves. Since there are only three possibilities for its own hat color, B, Y, R, the elf knows that there are exactly three possible combinations K_B, K_Y, K_R.
- If one of the three combinations K_B, K_Y, K_R is a special color combination F_k, the elf guesses the two colors (using the cards T4, T5, T6) that do not result in F_k. If none of the three combinations equals F_k, then the elf uses cards T7.

Now let us consider what happens under this strategy in the first round of guessing. By choosing the four hats, Santa constructs a color combination F^*.

- If $F^* = F_k$ is a special color combination, then each of the four elves has the color combination F_k among his three possible combinations K_B, K_Y, K_R. Then each of the four elves guesses exactly the two colors (using the cards T4, T5, T6) that do not belong to his hat. All four elves guess wrong and lose the game in this case.
- However, if F^* is not a special color combination, then F^* is adjacent to exactly one special color combination F_k. We call the three elves that have the same hat color in F^* and F_k the *normal elves*, and the other elf is the *special elf*.

  Since F_k belongs to the three possible combinations K_B, K_Y, K_R of the special elf, the special elf will use one of the three cards T4, T5, T6 and guess correctly. Since F_k is not one of the three possible combinations for the normal elves, the normal elves will use cards T7. Therefore, the game goes to the second round.

In the 9 cases where F^* is a special color combination, the elves are sent home without coffee and cake. In the 72 cases where F^* is not a special color combination, the elves always reach the second round.

But what happens in the second round? The three normal elves have learned the special color combination F_k that is adjacent to F^* by the reaction of the special elf in the first round, since each normal elf knows the hat colors of the other two normal elves and these two colors determine F_k uniquely. Because the three normal elves in F^* and in F_k wear the same color, the normal elves know their hat color. In the second round, the normal elves guess their correct hat color using cards T1, T2, T3, and the special elf uses T7. Hence, the elves win the game and receive coffee and cake.

We illustrate our argument in the situation BYBB: for the special elf Bilbo the three possible color combinations are BBBB, BYBB, BRBB. Since BBBB is a special combination, Bilbo uses the card T6 (yellow or red) and guesses correctly. For the three normal elves Atto, Chico, Dondo, there is no special color combination that would be compatible with their information. Hence, the normal elves each use T7. In the second round, the normal elves know that BBBB is the special color combination adjacent to F^* and win the game.

In summary, the elves win the game with the given strategy in at least 72 of the 81 possible cases.

Why M=73 is impossible. We assume for the sake of contradiction that there exists a strategy by which the elves survive the first round under at least 73 color combinations. We consider a concrete color combination F^* under which the elves survive the first round. Then, at least one elf X under F^* must use one of the six cards T1–T6, guessing correctly. If we change the hat color of X to a color that does not match the chosen cards, we get a color combination F^{**} adjacent to F^*, under which the elves lose the game.
To summarize: If the elves survive the first round under color combination F^*, then there is a color combination F^{**} adjacent to F^* under which the elves are already sent home after the first round. But since there are at most $8 = 81 - 73$ color combinations under which the elves already lose in the first round, and since each color combination has only 8 neighbors, there are at most $8 \cdot 8 = 64$ color combinations under which the elves survive the first round. This contradiction implies the desired bound $M \leq 72$.

19 Casino

Author: Jaques Resing (TU Eindhoven)
Project: 4TU.AMI

Challenge

Ruprecht and the Grinch are playing the following game: Ruprecht starts with an initial capital of 189 €. The Grinch starts with six cards, three of which carry the word “DOUBLE”; whereas the other three cards carry the word “NOTHING”. The game is played over six rounds.

At the beginning of every round, Ruprecht announces his bet B for the current round, where B is an arbitrary non-negative real number that must not exceed Ruprecht's current capital. After hearing the bet B, the Grinch plays one of his cards; each of the six cards may be played only *once*.

- If the Grinch plays a card with the word NOTHING, Ruprecht loses B Euros to the Grinch.
- If the Grinch plays a card with the word DOUBLE, Ruprecht keeps his B Euros and receives another B Euros from the Grinch.

Of course, in every round, the Grinch and Ruprecht make the best possible decisions for themselves respectively.

What is the total amount of money Ruprecht possesses at the end of the game?

Artwork: Frauke Jansen

Possible answers:

1. Roughly 210 €.
2. Roughly 222 €.
3. Roughly 234 €.
4. Roughly 245 €.
5. Roughly 256 €.
6. Roughly 261 €.
7. Roughly 279 €.
8. Roughly 288 €.
9. Roughly 297 €.
10. Roughly 303 €.

Solution

The correct answer is: 8.

We want to analyse general situations $S(n,d)$ where the Grinch has n NOTHING cards and d DOUBLE cards. The seed capital of Ruprecht at the start of situation $S(n,d)$ will be denoted by K. Playing optimally, Ruprecht can raise K to $\tilde{K} = f(n,d) \cdot K$ in the remaining $n+d$ rounds.
First, we want to get a better understanding of the factor $f(n,d)$.

- For all $n \geq 0$, it is obvious that $f(n,0) = 1$: if there are no DOUBLE cards in the game, Ruprecht has nothing to win. Therefore, bets nothing and keeps his seed capital.
- Furthermore, we know that $f(0,d) = 2^d$ for $d \geq 0$: if there are no NOTHING cards in the game, Ruprecht will bet and double all of his money in every round. After d rounds, he will own $\tilde{K} = 2^d \cdot K$.

Now, we consider situations $S(n,d)$ with $n \geq 1$ and $d \geq 1$. For Ruprecht, the situation $S(n-1,d)$ is always better than $S(n,d-1)$: in both situations $n+d-1$ rounds are left to be played. However, in the first situation, the Grinch has one DOUBLE card more than in the second one. This yields

$$f(n,d-1) \leq f(n-1,d) \qquad \text{for } n \geq 1 \text{ and } d \geq 1. \tag{6}$$

In the situation $S(n,d)$, Ruprecht will bet a part W of his current capital K (with $0 \leq W \leq K$).

- If the Grinch uses a NOTHING card, Ruprecht loses and passes over to $S(n-1,d)$ with a capital of $K-W$. Hence, Ruprecht's capital at the end of the game will amount to $K' = f(n-1,d) \cdot (K-W)$.
- If the Grinch uses a DOUBLE card, Ruprecht will double its wager and passes over to $S(n,d-1)$ with a capital of $K+W$. Ruprecht's capital at the end of the game will be $K'' = f(n,d-1) \cdot (K+W)$.

Now, we claim that the following inequality holds for every possible wager W:

$$\min\left\{K',\, K''\right\} \leq \frac{2f(n-1,d)\, f(n,d-1)}{f(n-1,d) + f(n,d-1)}\, K. \tag{7}$$

Suppose that (7) is false. Then,

$$\begin{aligned}
K' := f(n-1,d)\cdot(K-W) &> \frac{2f(n-1,d)\,f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K\\
K-W &> \frac{2f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K\\
-W &> \frac{2f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K-K\\
-W &> \left(\frac{2f(n,d-1)}{f(n-1,d)+f(n,d-1)}-1\right)\cdot K\\
-W &> \frac{2f(n,d-1)-f(n-1,d)-f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K\\
-W &> \frac{f(n,d-1)-f(n-1,d)}{f(n-1,d)+f(n,d-1)}\,K\\
W &< \frac{f(n-1,d)-f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K \qquad (8)
\end{aligned}$$

would be true. Hence,

$$\begin{aligned}
K'' := f(n,d-1)\cdot(K+W) &> \frac{2f(n-1,d)\,f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K\\
K+W &> \frac{2f(n-1,d)}{f(n-1,d)+f(n,d-1)}\,K\\
W &> \frac{2f(n-1,d)}{f(n-1,d)+f(n,d-1)}\,K-K\\
W &> \left(\frac{2f(n-1,d)}{f(n-1,d)+f(n,d-1)}-1\right)\cdot K\\
W &> \frac{2f(n-1,d)-f(n-1,d)-f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K\\
W &> \frac{f(n-1,d)-f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K \qquad (9)
\end{aligned}$$

would hold.
Since the inequalities (8) and (9) contradict one another, (7) has to be true.
Now, we consider the case in which Ruprecht bets

$$W = \frac{f(n-1,d)-f(n,d-1)}{f(n-1,d)+f(n,d-1)}\,K. \qquad (10)$$

Because of (6), the numerator in (10) is non-negative, which yields $0 \leq W \leq K$. Furthermore,

$$\begin{aligned}
K' &:= f(n-1,d) \cdot (K-W) \\
&= f(n-1,d) \cdot \left(1 - \frac{f(n-1,d) - f(n,d-1)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= f(n-1,d) \cdot \left(\frac{f(n-1,d) + f(n,d-1) - f(n-1,d) + f(n,d-1)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= f(n-1,d) \cdot \left(\frac{2f(n,d-1)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= \frac{2f(n-1,d)f(n,d-1)}{f(n-1,d) + f(n,d-1)} K
\end{aligned}$$

and

$$\begin{aligned}
K'' &:= f(n,d-1) \cdot (K+W) \\
&= f(n,d-1) \cdot \left(1 + \frac{f(n-1,d) - f(n,d-1)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= f(n,d-1) \cdot \left(\frac{f(n-1,d) + f(n,d-1) + f(n-1,d) - f(n,d-1)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= f(n,d-1) \cdot \left(\frac{2f(n-1,d)}{f(n-1,d) + f(n,d-1)}\right) K \\
&= \frac{2f(n-1,d)f(n,d-1)}{f(n-1,d) + f(n,d-1)} K.
\end{aligned}$$

Therefore, if Ruprecht bets (10), the inequality (7) becomes an equation and the bet is ideal:

$$K' = K'' = \frac{2f(n-1,d)\,f(n,d-1)}{f(n-1,d) + f(n,d-1)} K. \tag{11}$$

Regardless of what the Grinch does in each round, Ruprecht can always bring his seed capital to the value in (11).

Thus, we proved that

$$f(n,d) = \frac{2f(n-1,d)\,f(n,d-1)}{f(n-1,d) + f(n,d-1)} \qquad \text{for } n \geq 1 \text{ and } d \geq 1. \tag{12}$$

Now, we can easily calculate the factors $f(n,d)$ with $0 \leq n \leq 3$ and $0 \leq d \leq 3$. The first column of the following table is calculated from $f(n,0) = 1$, and the first row is derived from $f(0,d) = 2^d$. The other entries $f(n,d)$ can be calculated recursively by means of (12).

n bzw. d	0	1	2	3
0	1	2	4	8
1	1	4/3	2	16/5
2	1	8/7	16/11	2
3	1	16/15	16/13	32/21

Since $f(3,3) = 32/21$ is true, Ruprecht can raise his seed capital of $K = 189\,e$ to

$$\tilde{K} = f(3,3) \cdot K = \frac{32}{21} \cdot 189\,e = 288\,e$$

if he plays optimally. In the first round, Ruprecht will bet

$$W_1 = \frac{f(2,3) - f(3,2)}{f(2,3) + f(3,2)} \cdot K = \frac{2 - 16/13}{2 + 16/13} \cdot K = \frac{10/13}{42/13} \cdot K = \frac{5}{21} \cdot K = \frac{5}{21} \cdot 189\,e = 45\,e$$

as can be deduced from (10). If the Grinch plays a DOUBLE card, Ruprecht finds himself in situation $S(3,2)$ with $234e$. If the Grinch plays a NOTHING card, Ruprecht passes over to $S(2,3)$ with $144e$, etc.3

20 Thieves in the Snow

Author: Maximilian Stahlberg (TU Berlin)
Project: Evolution Models for Historical Networks
(MATH+ Emerging Field Project EF5-6)

Challenge

The snowperson's face is gone! Their cucumber eyes as well as their carrot nose have been stolen this morning. The pixies suspect that the culprits are to be found among the animals living close to the field, as it would not be the first time that they fail to resist the temptation of vegetables not meant for them. Immediately, an investigation is launched to solve the crime and confront the ravenous thieves.

At a first hearing, the little rabbit, the fawn, and the beaver claim that the carrot was firmly stuck in the snowperson's face, but that the cucumber slices were already missing when they were going to school and work. The cat and the squirrel insist that they saw the snowperson only from a distance and that the carrot was in place, but that they were unsure about the cucumber eyes' condition.

After this initial collection of evidence, the pixies are certain that the animals were all on the move one after the other—at least that explains the lack of witness reports... Moreover, the first animal passing the snowperson this morning seized the early moment and stole the cucumber slices, whereas the last animal passing by the faceless victim must have taken the carrot. Presumeably, they overslept and were sure that no one would witness the crime. Just one thing stands in the way of solving the crime: the pixies do not know who was first and who was last, since the animals were smart enough not to state particularly early or late departure times.

But the scout pixie Masha has an idea: while the tracks in the deep snow are already buried under a thin layer of fresh snow, revealing no incriminating paw prints, the principal corridors of movement of all the animals are still easy to recognize. Moreover, Masha knows that the animals have a very hard time plowing through fresh snow and prefer paths that have already been cleared by someone else. More precisely, she estimates that the time it takes the animals to traverse a cleared path segment is proportional to the length of that segment. However, traversing path segments with fresh snow takes them twice as long. Masha is also certain that the animals would always take the fastest route to their destination. In addition, all the animals can overlook the entire field and therefore see, even from a distance, which paths have already been cleared. Convinced by Masha's theory, the pixies walk the crime scene and make a map:

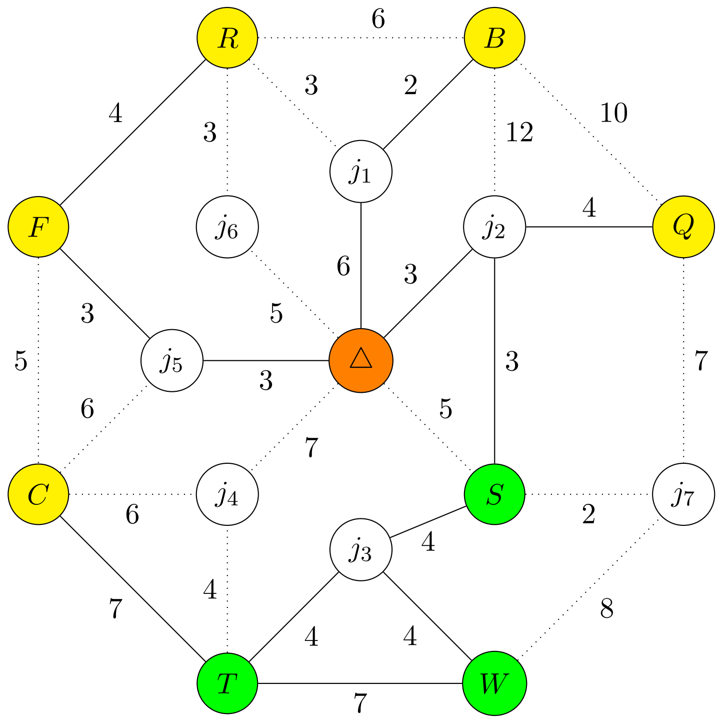


Рис. 15: **The map of the crime scene** shows the homes of the five animals (yellow), their daily destinations (green), the snowperson (orange), and the junction points (white). Solid lines mark cleared paths, dotted lines mark unused routes. Every path segment is labelled with its length in meters.

The pixies know that the little rabbit R and the fawn F still go to the school S, whereas the squirrel Q goes on a walk to the big tree T every morning. Both the cat C and the beaver B work in the wood workshop W. (The cat is very well known for her pretty scratch patterns.) The snowperson △ stands in the middle of the large field, where many animals could have passed them this morning.

After a thorough analysis of the snow path network, Masha is certain that she has determined the precise order in which the animals have traversed the field today. She also knows who has eaten from the snowperson's face.

Which of the following ten statements is correct?

Artwork: Friederike Hofmann

Possible answers:

1. The squirrel was on its way first and stole the cucumber slices. The beaver came last and ate the carrot.
2. The fawn was first, but the beaver ate the cucumber slices. The rabbit could not resist the temptation of the carrot.
3. The cat was first, but the fawn ate the cucumber slices. The beaver stole the carrot.
4. The cat came first and stole the cucumber slices. The beaver ate the carrot.
5. The squirrel was first, but the fawn stole the cucumber eyes. The rabbit ate the carrot. Everyone knows it's his favourite vegetable!
6. The cat was first, but the beaver stole the cucumber eyes. It should come to no surprise that the rabbit laid teeth on the carrot.
7. The squirrel was first, but the fawn ate the cucumber slices. The beaver stole the carrot nose.
8. The fawn was first and snatched the cucumber eyes. The beaver took the carrot.
9. The beaver was first and stole the cucumber slices. The rabbit is the carrot thief.
10. The fawn was first and snitched the cucumber slices. The rabbit came late to school and ate the carrot.

Project reference:

The task at hand can get extremely hard for networks with a larger number of origin and destination pairs, because the number of possible orderings explodes. Already with five animals, there are a total of $5! = 120$ possibilities to put their movement in a chronological order. In our research, we want to recover exactly such chronological orderings by analysing the network structure—except that, instead of cutting paths in the snow, we explore historical road networks, such as those built by the Romans. Also in this historical scenario it is often not clear which roads where constructed first. Furthermore, there may even be road segments that have not been discovered yet and are thus missing from our maps. But just as the animals preferred a path that has been cleared by someone else before them, later roads are expected to connect favourably to existing ones, since building new road segments was very expensive. By ordering assumed connections between ancient settlements, we try to reproduce the known road network and gain new insight into our ancient history.

Solution

The correct answer is: 3.

Masha makes the following observations:

- The cat C walked to the big tree T first, because no other snow trail starts from its sleeping place. All other animals that went either to the maple or to the workshop W must have passed the crossing point j_3. If one of these animals had been on its way before the cat, the cat would have taken the route via j_3 instead of the direct way from the tree to the workshop with length 7, because it would then only have needed $4 + 2 \cdot 4 = 12$ for this part of the way instead of $2 \cdot 7 = 14$ time units.Hence, the cat was on the move before the beaver and the squirrel, but did not pass the snowperson as claimed.

- The path of the rabbit R, fawn F, and beaver B clearly leads past the snowperson $\triangle$ and the school S. However, the direct path from $\triangle$ to S remained unused, although it is the shortest with only 5 length units (i. e. 10 time units for the first animal). This can only be explained by the fact that on the path from $\triangle$ via j_2 to S at least one section has already been cleared of snow, because then this can be passed in at most $3 + 2 \cdot 3 = 9$ instead of otherwise 12 time units. Only the squirrel Q can be responsible for this, whose walk to the big tree led via j_2 and S. Thus, the squirrel was on the move *before* the fawn, rabbit, and beaver, but *after* the cat. Moreover, the squirrel also tells the truth and did not pass directly by the snowperson.

- The rabbit's R way to school S provides the last indication: if it had been on the way before the fawn and the beaver, it would have taken the shortest way to the snowperson via j_6 and needed $2 \cdot 3 + 2 \cdot 5 = 16$ time units for it. However, this path is unused, so the path via F and j_5 must have been faster. Therefore, the fawn must have been on its way before the rabbit, and the rabbit needed only $2 \cdot 4 + 3 + 3 = 14$ time units for its way to school. If the beaver had been on the move before the rabbit (which, judging by the snow tracks, clearly ran via j_1 to $\triangle$), then the path via j_1 would also have been clearly advantageous for the rabbit, since the long stretch between j_1 and $\triangle$ would have been cleared of snow and the rabbit's way to school would then have taken only $2 \cdot 3 + 6 = 12$ time units. Thus, the rabbit was on its way before the beaver but after the fawn.

Therefore, Masha is able to reconstruct the exact order in which the animals were on the move in the morning: **the cat was out first**, then the squirrel. Both animals did not pass the snowperson. After the squirrel, the fawn followed. **The fawn was thus the first animal to pass the snowperson and snatched the cucumber slices.** After the fawn, the rabbit set off. When he passed the snowperson, it was already missing the cucumber eyes. Thus, he is also telling the truth, since **first after him—and thus last—the beaver was on his way and stole the carrot**. Hemce, answer 3 is correct.

When Masha confronts the thieving animals, they confess their crimes and show deep regret.

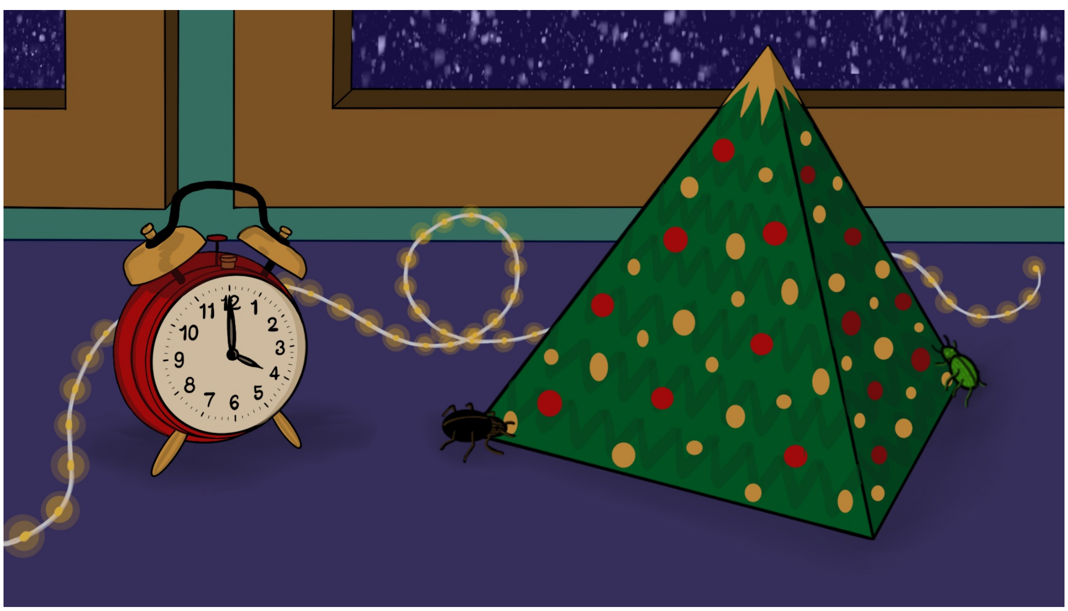
21 Tetrahedron

Author: Hennie ter Morsche (TU Eindhoven)
Project: 4TU.AMI

Challenge

A black and a green bug are sitting on a regular tetrahedron $ABCD$. The black bug starts its journey at 4 pm at vertex A, crawls with constant velocity along the edge AB, and reaches vertex B at 6 pm. The green bug starts its journey at 4 pm in vertex C, crawls with constant velocity along the edge CD, reaches vertex D at 5 pm, and then stays sitting in D.

We want to know from you: at which point T in time are the two bugs at minimum distance from each other?



Artwork: Frauke Jansen

Possible answers:

1. At time $T = 4{:}31$ pm.
2. At time $T = 4{:}32$ pm.
3. At time $T = 4{:}33$ pm.
4. At time $T = 4{:}34$ pm.
5. At time $T = 4{:}35$ pm.
6. At time $T = 4{:}36$ pm.
7. At time $T = 4{:}37$ pm.
8. At time $T = 4{:}38$ pm.
9. At time $T = 4{:}39$ pm.
10. At time $T = 4{:}40$ pm.

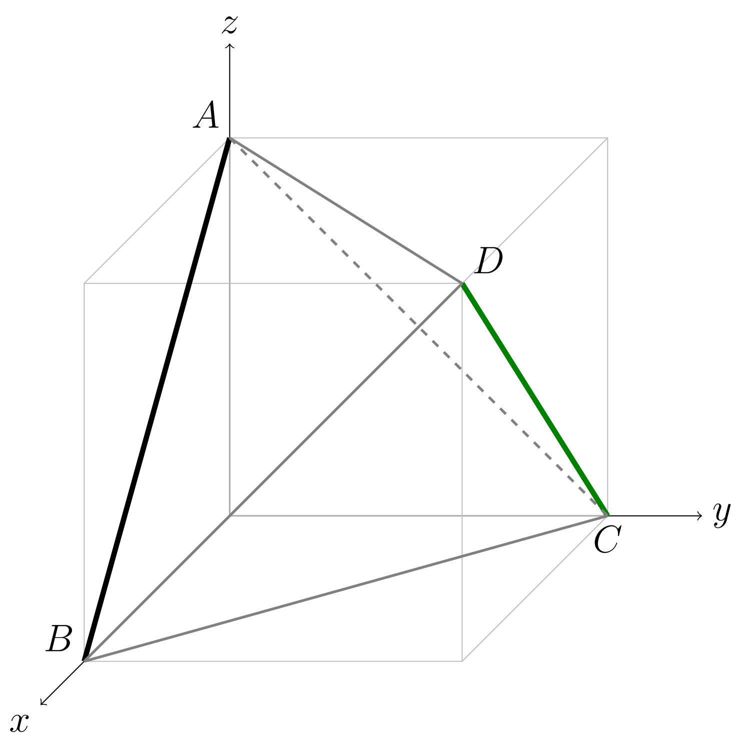
Solution

The correct answer is: 6.

We embed the tetrahedron in a cube of edge length 120 such that the edges AB and CD lie on opposite faces of this cube. We can do this, for example, by setting

$$A = (0, 0, 120), \quad B = (120, 0, 0), \quad C = (0, 120, 0), \quad D = (120, 120, 120)$$

.



Hence, m minutes after 4 pm, the black bug is located at

$$S(m) = (m, 0, 120 - m)$$

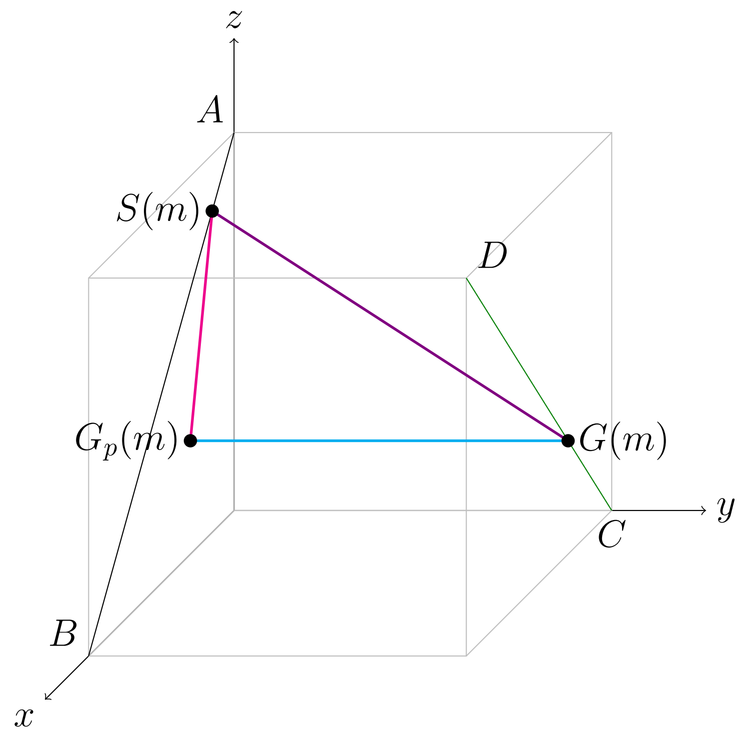
for all $0 \leq m \leq 120$.
For $0 \leq m \leq 60$, the green bug is at

$$G(m) = (2m, 120, 2m).$$

m minutes after 4 pm.
For $60 \leq m \leq 120$, the green bug sits in $D = (120, 120, 120)$.

The distance $d(S(m), G(m))$ of the two points $S(m)$ and $G(m)$ can be calculated using the Pythagorean theorem: we denote by $G_p(m)$ the *projection* of $G(m)$ onto the face of the cube on which AB and, in particular, $S(m)$ lies:

$$G_p(m) = (2m, 0, 2m).$$

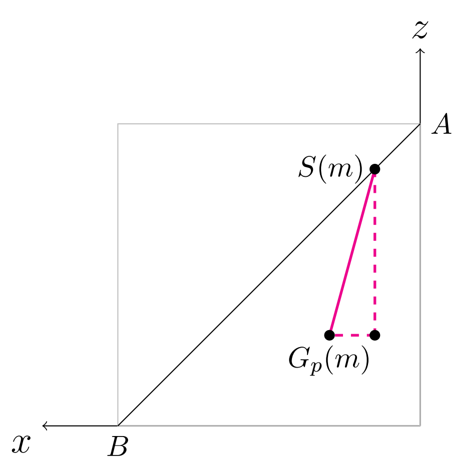


The distance $d(S(m), G(m))$ results from the distances $d(G(m), G_p(m))$ between $G(m)$ and $G_p(m)$ and $d(S(m), G_p(m))$ between $S(m)$ and $G_p(m)$:

$$d(S(m), G(m))^2 = d(G(m), G_p(m))^2 + d(S(m), G_p(m))^2.$$

The distance $d(G(m), G_p(m))$ from $G(m)$ to $G_p(m)$ is 120. We calculate the distance between $S(m)$ and $G_p(m)$ by calculating the distance of these points in the xz-plane:

$$\begin{aligned} d(S(m), G_p(m))^2 &= d((m, 120-m), (2m, 2m))^2 \\ &= (2m-m)^2 + (2m-120+m)^2 \\ &= m^2 + 9m^2 - 720m + 120^2 \\ &= 10m^2 - 720m + 120^2. \end{aligned}$$



This yields

$$\begin{aligned} d(S(m),G(m))^2 &= d(G(m),G_p(m))^2 + d(S(m),G_p(m))^2 \\ &= 120^2 + 10m^2 - 720m + 120^2 \\ &= 10m^2 - 720m + 2\cdot 120^2. \end{aligned}$$

Thus, for $0 \le m \le 60$, the square of the distance of the two bugs is a quadratic function in m:

$$\begin{aligned} f(m) &:= d(S(m),G(m))^2 \\ &= 10m^2 - 720m + 2\cdot 120^2 \\ &= 10\cdot\left(m^2 - 72m + \tfrac{1}{5}\cdot 120^2\right) \\ &= 10\cdot\left((m-36)^2 - 36^2 + \tfrac{1}{5}\cdot 120^2\right). \end{aligned}$$

Since the factor before the quadratic term of $f(m)$ is positive, the function has an absolute minimum at its vertex located at $m = 36$, as can easily be read off the vertex form of the quadratic function f.
For $60 \le m \le 120$, we define $\tilde{f}$ analogously as the square of the distance of the two bugs (where the green bug silently stays put in D):

$$\begin{aligned} d(S(m),D)^2 &= 120^2 + d((m,120-m),(120,120))^2 \\ &= 120^2 + (120-m)^2 + (120-120+m)^2 \\ &= 120^2 + 120^2 - 240m + m^2 + m^2 \\ &= 2m^2 - 240m + 2\cdot 120^2 \\ &= 2\cdot\left(m^2 - 120m + 120^2\right) \\ &= 2\cdot\left((m-60)^2 - 60^2 + 120^2\right). \end{aligned}$$

Thus, $\tilde{f}$ is also a quadratic function that has a absolute minimum. The vertex is at the point $m = 60$, which can directly be read off the vertex form.

Furthermore, according to the definition of f and $\tilde{f}$ at $m = 60$, we have

$$\tilde{f}(60) = f(60) > f(36),$$

since the function f has its minimum at $m = 36$. Therefore, the distance between the two bugs is minimal 36 minutes after 4 pm, i. e. at 4:36 pm.

22 Zelda in Distress

Authors: Luise Fehlinger (HU Berlin)
Florian Matthies (HU Berlin, Herder-Gymnasium Berlin)
Project: Research qroup "Mathematics and Education" (HU Berlin)

Challenge

Princess Zelda has been kidnapped again. This time, it is clear that courage, strength, and wisdom, which form the corners of the magic Triforce, will not be enough to save her. The fourth power, *patience*, is also crucial. Thus, the Triforce must be transformed into a magical tetrahedron. A new dimension has to be added. The equilateral triangle of the Triforce becomes a regular tetrahedron. As with the Triforce, the corners of the tetrahedron must be made up of smaller tetrahedra. These touch at their vertices, forming the scaffold of the magical tetrahedron (see Fig. 16). However, the interior must not remain empty. It must be perfectly filled by an inner solid K_1, which in turn consists of corners and another inner solid K_2 and so on.

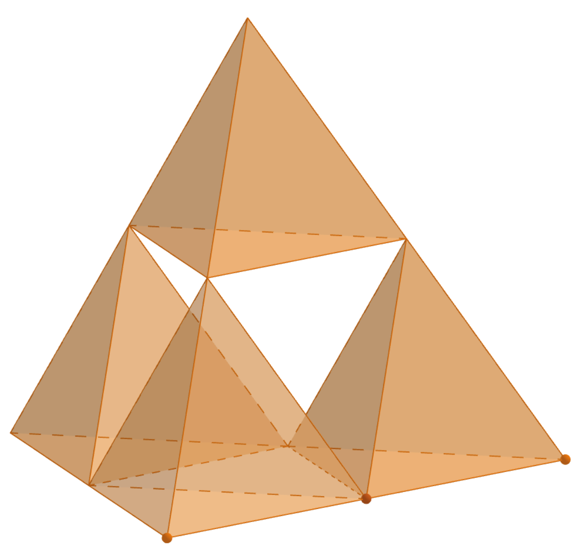


Рис. 16: The tetrahedron K_0. The solid K_1 is obtained from K_0 by cutting of the corners (in orange). Hence, K_1 is the (colourless) remainder in the center of the tetrahderon K_0.

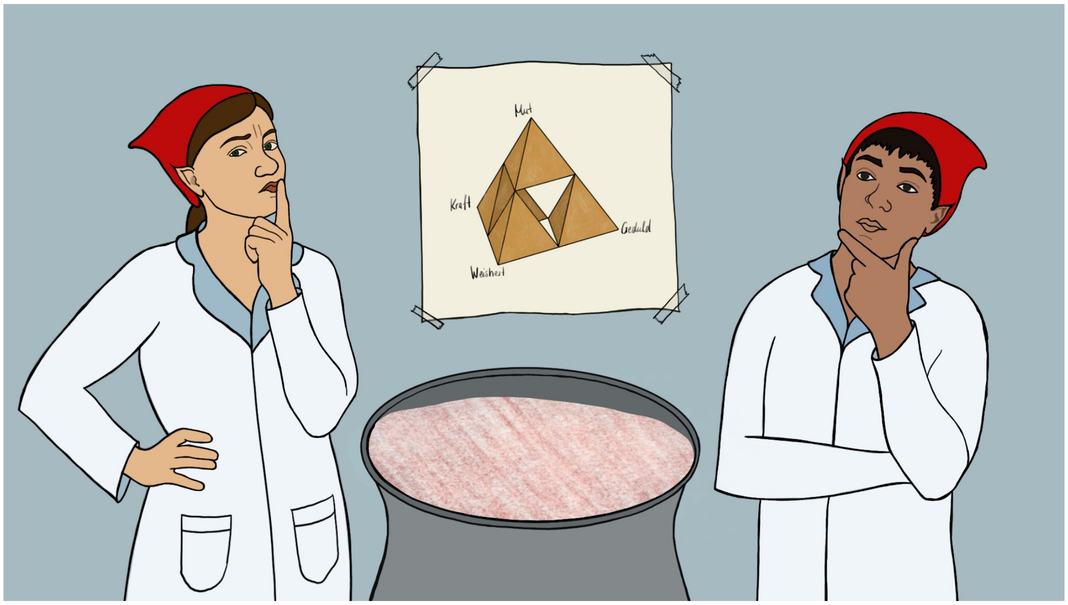
Hence, the Tetraforce is a regular tetrahedron that will be called K_0.

- K_1 is obtained from K_0 by cutting off the corners of K_0. The sectional surfaces are determined by the midpoints of the edges meeting in the respective vertex (see Fig. 16).
- K_2 is obtained from K_1 by cutting off the corners of K_1. Again, the sectional surfaces are determined by the midpoints of the edges meeting in the respective vertex.
- And so on: K_{n+1} is obtained from K_n by cutting off the corners of K_n. The sectional surfaces are determined by the midpoints of the edges meeting in the respective vertex.

  After an infinite number of cuts, the magical heart of the Tetraforce remains.

This heart will be manufactured by the elves—who else would be able to do this—from a magic crystal, which must be cast from Christmas elixir at -40°C at the magnetic pole of the Earth.

But what are the properties of the magical heart? The elves make various guesses. But one statement is wrong. Which one?



Artwork: Frauke Jansen

Possible answers:

1. K_1 is an octahedron.
2. The volume of K_1 is only half of the volume of K_0.
3. All of the K_n are convex.
4. K_0 is the only one of the solids where only three edges meet in each vertex. For all other K_n, four edges meet in each vertex.
5. Each of the K_n has faces that are equilateral triangles.
6. The faces of the K_n are triangles and/or quadrilaterals.
7. The centroids of the faces of K_n are part of all K_{n+k} ($k \in \mathbb{N}$).
8. In each step, the number of edges is doubled.
9. Each solid K_n besides K_0 has two faces more than it has vertices.
10. The magical heart is a ball.

Solution

Unfortunately, there is a serious error in the problem task. From the sixth iterations step on, the resulting quadrilaterals are no longer well-defined, i. e. they are not planar. Therefore, the problem cannot be solved. We are very sorry!

We would like to thank all dedicated participants who pointed out this error and for the vivid and enlightening discussion on the *Mathekalender Forum*.

The problem does not count for the MATH+ Advent Calendar 2021.

23 Traces in the Snow

Author: Christian Renau (HU Berlin, Heinrich-Hertz-Gymnasium Berlin)
Project: Research qroup “Mathematics and Education” (HU Berlin)

Challenge

As every morning, the little elves Juri and Sergei set off to school. First, they climb the Fairy Hill, then they walk along the Pixie Stream in the Elf Valley to the enchanted fir forest. Behind the fir forest, there is an extensive snow field the two have to cross.

When Juri and Sergei leave the fir forest, they cannot believe their eyes: in the snowfield, there are traces that look like a giant coordinate system in which, as far as one can see, the lines connecting the points (m^2, m) and $(n^2, -n)$ are plotted for all $m, n \in \mathbb{N}$ with $m, n > 1$. Have aliens been at work here?

Excited, Juri and Sergei run the rest of the way to school. There, they are met by their math teacher Emma. Immediately, they tell her about their mysterious discovery. Quite interested, Emma grabs her whole class, and together they trudge to the snow field. The little elves are keen to follow on this excursion, because one day before Christmas they can hardly sit still on their classroom chairs...

“However, we still need to do some math,” Emma thinks, and asks her class which numbers on the x-axis are “hit” by these connecting lines. But be cautious: in the midst of all that excitement, one little elf was not paying attention and made a mistake. Which one?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. Anh answers, "I think that 1.008.988.999 is hit."
2. Basima bets, "The number 2.517.849.199 is hit."
3. Charly checks their notes and says, "Also 3.497.348.737 is hit."
4. Dao deduces, "Certainly, 4.142.454.642 is hit."
5. Elif explains, "Naturally, 5.761.648.489 is hit."
6. Frieda figures, "The number 6.556.849.301 is hit as well."
7. Ghazal guarantees, "Most certainly, 7.481.075.262 is hit."
8. Hasina has the following idea, "I think 8.991.101.800 is hit."
9. Imani is perfectly sure that, "9.768.956.129 is hit."
10. João joins the discussion and says, "The number 10.567.846.453 is definitely hit."

Solution

The correct answer is: 2.

The intersections of the connecting lines between (m^2, m) and $(n^2, -n)$ on the x-axis are just the roots of the linear equations through (m^2, m) and $(n^2, -n)$.

Therefore, we first establish the linear equation of the line connecting (m^2, m) and $(n^2, -n)$ for $m, n \in \mathbb{N}$ with $m, n > 1$ and calculate their roots. The two-point form of this linear equation is

$$(y - m) \cdot (n^2 - m^2) = (x - m^2) \cdot (-n - m).$$

Equivalent transformations lead to the following linear equation

$$y = \frac{1}{n - m}(-x + mn),$$

from which one can immediately deduce the root $x_0 = mn$.

Thus, on the x-axis all numbers which are the product of two integers $n, m > 1$ are "hit". Below, we give the prime factorization of the ten given numbers:

1. $1,008,988,999 = 10,091 \cdot 99,989$
2. $2,517,849,199$ is a prime number and thus cannot be represented as the product of two integers $n, m > 1$, i. e. Basima made a mistake.
3. $3,497,348,737 = 54,577 \cdot 64,081$
4. $4,142,454,642 = 2 \cdot 3^3 \cdot 73 \cdot 1,050,851$
5. $5,761,648,489 = 74,597 \cdot 77,237$
6. $6,556,849,301 = 59 \cdot 111,133,039$
7. $7,481,075,262 = 2 \cdot 7 \cdot 23 \cdot 23,233,153$
8. $8,991,101,800 = 2^3 \cdot 5^2 \cdot 44,955,509$
9. $9,768,956,129 = 23 \cdot 193 \cdot 2,200,711$
10. $10,567,846,453 = 43 \cdot 103 \cdot 2,386,057$

note: In 1971, mathematician Yuri Vladimirovich Matiyasevich showed in the journal Kwant (Квант) that one can use the property of the normal parabola shown above to graphically multiply arbitrary (positive) numbers. Thus, the normal parabola is a so-called *nomogram*. The mathematician Sergei Borisovich Sztetchkin had the idea to construct a graphical prime number sieve using this property.

24 Tempting Vanilla Crescents

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

Challenge

The baking elves have outdone themselves again this year and baked the very best vanilla crescents. The crescents are delicately sweet, buttery, tenderly melt in the mouth, and leave a wonderful Christmas feeling. To prevent the recipe from being stolen and ending up as "Granny's vanilla crescents—the best!!1!" on elfrecipes.np, the elves have secured it against thieves in a special safe with seven switches.

Shortly before Christmas, however, the vanilla crescents have to be baked again, because the gluttonous elves have already eaten up all the supplies at their various Christmas parties. Hidden on the top shelf of the bakery, the baking elves must climb a ladder to get to the safe manual. It says:

> The safe opens only if all switches are set to "off". The switches can be operated as follows:
>
> - The rightmost switch can be turned on or off as desired.
> - Any other switch can be turned on and off only if the switch directly to its right is "on" and all the others to its right are "off".
> - Only one switch can be operated at a time.

At the moment, all the switches are turned "on". How many moves are necessary to open the safe? Here, toggling one switch counts as one move.

Artwork: Frauke Jansen

Possible answers:

1. 81
2. 82
3. 83
4. 84
5. 85
6. 86
7. 87
8. 88
9. 89
10. 90

Solution

The correct answer is: 2.

Let $n \in \mathbb{N}$ be the amount of switches. We number the switches *from left to right* with 1 to n and denote the states "on" and "off" by 1 and 0 respectively. Furthermore, let $M(n)$ denote the minimal amount of moves necessary to change n switches from "on" to "off".

To get an understanding of the process, we will first consider small n. For $n = 1$, one only has to change one switch to get from state 1 to state 0 ($M(1) = 1$). For $n = 2$, one needs two moves. Therefore, $M(2) = 2$. For $n = 3$, we need five moves and for $n = 4$ we need ten. Hence, giving us $M(3) = 5$ and $M(4) = 10$.

1
0

1	**1**
0	1
0	0

1	**1**	**1**
1	1	0
0	1	0
0	1	1
0	0	1
0	0	0

1	**1**	**1**	**1**
1	1	0	1
1	1	0	0
0	1	0	0
0	1	0	1
0	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1
0	0	0	1
0	0	0	0

Now, let $n \geq 3$. We can divide the $M(n)$ moves into four phases:

1. Before we can turn off the first of the n switches, the state $1100\ldots00$ has to be reached. Therefore, we have to turn off the $n-2$ switches to the right of switch number two. For this, we need at least $M(n-2)$ moves. (For $n = 4$, these moves are shown in blue in the following table.

2. From state $1100\ldots00$ to state $0100\ldots00$, we get in one move.

3. Before we can turn off the second switch, the state $0111\ldots11$ has to be reached. We have to make the same moves as in 1) in reverse order. This adds another $M(n-2)$ moves. (For $n = 4$, these moves are depicted in red in the following table.)

4. Omitting the first 0 on the left, the state $0111\ldots11$ equals the case $n-1$. We therefore need at least $M(n-1)$ moves to get to the state $00\ldots00$. (For $n = 4$, these moves are coloured in green in the following table.)

Over all, we get the following recursive formula for all $n \geq 3$:

$$M(n) = M(n-1) + 1 + M(n-1) + M(n-2) = 2M(n-2) + M(n-1) + 1. \tag{13}$$

$$\begin{array}{cccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}$$

With the help of (13), we can calculate $M(n)$ for higher n in the following way:

$$\begin{aligned}
n = 5: \quad & M(5) = 2M(3) + M(4) + 1 = 2 \cdot 5 + 10 + 1 = 21, \\
n = 6: \quad & M(6) = 2M(4) + M(5) + 1 = 2 \cdot 10 + 21 + 1 = 42, \\
n = 7: \quad & M(7) = 2M(5) + M(6) + 1 = 2 \cdot 21 + 42 + 1 = 85.
\end{aligned}$$

Hence, for seven switches, the baking elves need at least 85 moves.

Remark: From (13), we can also get the following formula for $M(n)$ and prove it by mathematical induction:

$$M(n) = \tfrac{2}{3}2^n - \tfrac{1}{6}(-1)^n - \tfrac{1}{2}. \tag{14}$$

Base case: For $n = 1$, (14) equals:

$$M(1) = \tfrac{2}{3}2^1 - \tfrac{1}{6}(-1)^1 - \tfrac{1}{2} = \tfrac{4}{3} + \tfrac{1}{6} - \tfrac{1}{2} = \tfrac{4}{3} - \tfrac{1}{3} = 1.$$

Similarly, for $n = 2$, we get:

$$M(2) = \tfrac{2}{3}2^2 - \tfrac{1}{6}(-1)^2 - \tfrac{1}{2} = \tfrac{8}{3} - \tfrac{1}{6} - \tfrac{1}{2} = \tfrac{8}{3} - \tfrac{2}{3} = 2.$$

Induction step: Let $k \geq 3$. Using the inductive hypothesis, (14) holds true for $n = k - 1$ and $n = k - 2$. It follows that

$$\begin{aligned}
M(k) &= 2M(k-2) + M(k-1) + 1 \\
&= 2 \cdot \left(\tfrac{2}{3}2^{k-2} - \tfrac{1}{6}(-1)^{k-2} - \tfrac{1}{2}\right) + \left(\tfrac{2}{3}2^{k-1} - \tfrac{1}{6}(-1)^{k-1} - \tfrac{1}{2}\right) + 1 \\
&= \tfrac{2}{3}2^{k-1} - \tfrac{1}{3}(-1)^{k-2} - 1 + \tfrac{2}{3}2^{k-1} - \tfrac{1}{6}(-1)^{k-1} - \tfrac{1}{2} + 1 \\
&= 2 \cdot \tfrac{2}{3}2^{k-1} - \left(\tfrac{1}{3}(-1)^k(-1)^{-2} + \tfrac{1}{6}(-1)^k(-1)^{-1}\right) - \tfrac{1}{2} \\
&= \tfrac{2}{3}2^k - \left(\tfrac{1}{3}(-1)^k - \tfrac{1}{6}(-1)^k\right) - \tfrac{1}{2} \\
&= \tfrac{2}{3}2^k - \tfrac{1}{6}(-1)^k - \tfrac{1}{2},
\end{aligned}$$

proving (14).