18   Hat Challenge 2021

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Challenge

Santa Claus tells the four super-smart elves Atto, Bilbo, Chico and Dondo, "For the sake of our annual tricky hat challenge, I would like to invite you to my place for coffee and cake."
"We are happy to accept!" answer the four elves.

Santa Claus continues, "Okay, then I am going to prepare several blue, yellow, and red hats. Tomorrow, I'll put one hat on each of your heads, so that none of you can see the hat color of your own hat. Each of you will be able to see the hat colors of the other three elves. However, you are not allowed to exchange any information with each other. Furthermore, each of you will receive seven cards with the following statements:

T1: My hat color is blue.
T2: My hat color is yellow.
T3: My hat color is red.
T4: My hat color is blue or yellow.
T5: My hat color is blue or red.
T6: My hat color is yellow or red.
T7: My hat color is blue or yellow or red.

Then, simultaneously each of you must guess the color of your own hat by publicly showing one of your seven cards. If at least one of you guesses wrong, the game is over, and you will have to leave. If all of your guesses are correct and if at least one of you guesses the correct color with one of the cards T1, T2, or T3, all of you will get a tasty piece of mozarttorte and a nice cup of coffee. If you all use T4, T5, T6, or T7 correctly, we will repeat the guessing round—but only if you don't all use T7. In this case, I will send you straight home."

Atto wants to know, "How are you going to choose our hat colors?"
"They are chosen randomly, so that each of the 81 color combinations arises with the same probability," answers Santa Claus.

Bilbo asks, "Are we going to get new hats, in case the guessing round is repeated?"
"No!" answers Santa Claus. "The colors will not be changed. But, of course, you are allowed to use different cards in every round."

Chico wants to know, "What is going to happen if in the second guessing round everybody guesses correctly again, and again only the cards T4, T5, T6, or T7 are used?"
"Then, the guessing round will be repeated. After that round, it may be repeated again, and again, and again, as long as you like," says Santa Clause.

The elves start to ponder. They discuss, and they think. They think, and they discuss. Then they discuss some more, and they think some more. Eventually, they manage to develop an amazing strategy that will guarantee them coffee and cake for M (of the 81) color combinations. This strategy is even optimal; that is, there is no strategy that guarantees them coffee and cake for $M + 1$ color combinations.

Which of the following statements is correct?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. $0 \leq M \leq 32$.

2. $33 \leq M \leq 37$.

3. $38 \leq M \leq 42$.

4. $43 \leq M \leq 47$.

5. $48 \leq M \leq 52$.

6. $53 \leq M \leq 57$.

7. $58 \leq M \leq 62$.

8. $63 \leq M \leq 67$.

9. $68 \leq M \leq 72$.

10. $73 \leq M \leq 81$.