Challenges and solutions 2017

MATHEON KALENDER

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Contents

1 A bug at the North Pole

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1.1 Challenge

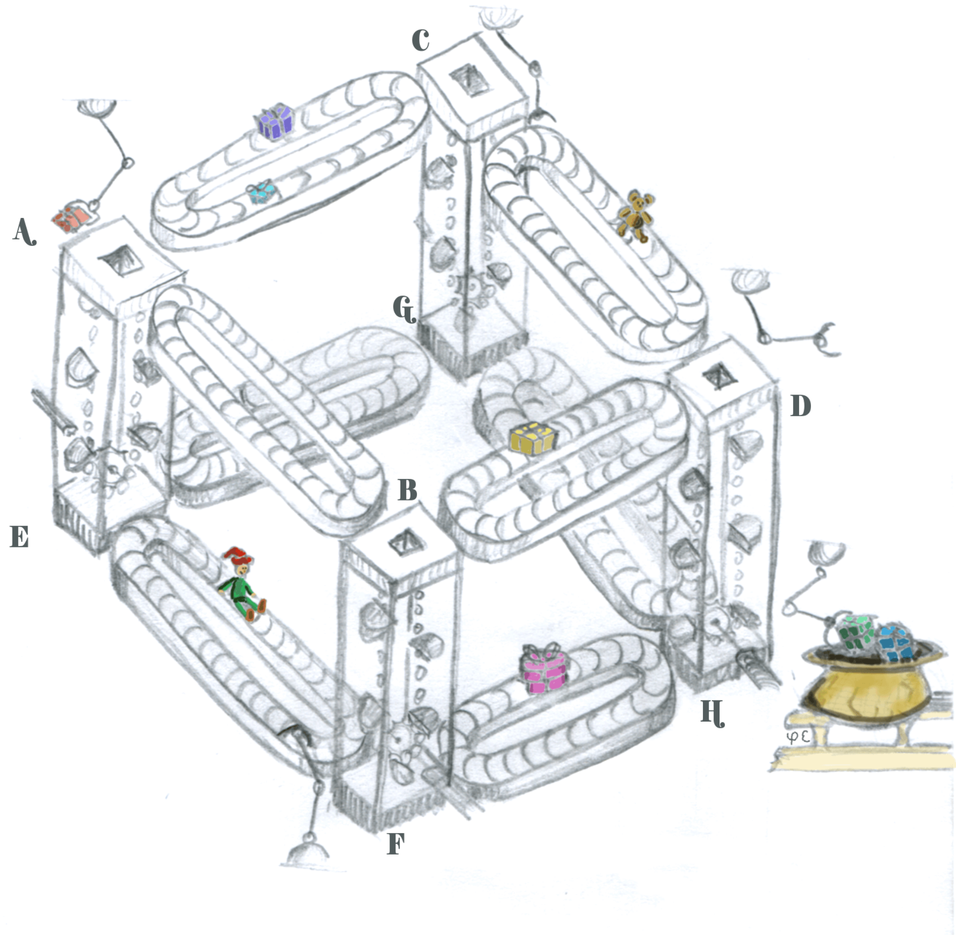
"Well, what is going on here?!" Gnome Willi is supervising the gifts' production, packaging and delivery (up to the sleigh) realizes that the conveyor system is not behaving as it should. "There must be a bug, a mistake in the software program."

A conveyor system allows the transportation of all the numerous gifts that Santa wants to give to the well-behaved children. Before ending up in Santa's bag on his sleigh, every gift moves on an ingenious—and normally perfectly working—system of conveyors. In this system, the gifts, produced by the busy gnomes, arrive at point A and are loaded onto the sleigh at point H. The conveyors connect the following points: A with B, A with C, B with D, C with D, A with E, B with F, C with G, D with H, E with F, E with G, F with H and G with H. (A sketch of the conveyor system is given below.)

To get from A to H, the gifts can drive on the conveyors from one point to the next. Every such drive from one point to another takes 1 minute. When they arrive at one point, the gifts are automatically and without any delay loaded onto the next conveyor and the trip continues. (In principle, the conveyors can be used in both directions, that is, one can get from A to B but also from B directly back to A.) Normally, the automatic transshipping should work such that every gift arrives at its destination at point H in the shortest time.

However, something is going completely wrong right now. Instead of sending the gifts through the conveyor system on an optimal route, the converters at the points A to H, which load the gifts from one conveyor onto another, do not work properly anymore: if a gift arrives at one of the points A to H, it is placed onto a *random* conveyor adjacent to this point and continues its journey from there. In fact, it is possible that a gift will be put on the conveyer it just has arrived on, because each adjacent conveyor is “chosen” equiprobably. And even at point H, the gift is not taken down and loaded onto the sleigh anymore, but drives on in the machinery as described above. Fortunately, Willi was able to detect the malfunction only 5 minutes after its start. He presses the red button, and the whole system stops. But where is the gift that began its journey at point A at the time when the malfunction started? More precisely, Willi wonders: What is the probability that this gift is at point H right now and can simply be loaded onto the sleigh?

Additional question (without rating): How would you expect the result to change if Willi had detected the malfunction one, two or four minutes later?



Possible answers:

1. 0
2. $\frac{1}{8}$
3. $\frac{1}{4}$
4. $\frac{1}{243}$
5. $\frac{2}{81}$

6. $\frac{1}{2}$
7. $\frac{2}{9}$
8. $\frac{20}{81}$
9. $\frac{32}{81}$
10. 1

1.2 Solution

The correct answer is: 8.

As a starting point, consider the 12 conveyors of the system as line segments in the 3-dimensional space. Then each conveyer lies in one of the following directions:

x-direction: A-B, C-D, E-F, G-H,
y-direction: A-C, B-D, E-G, F-H and
z-direction: A-E, B-F, C-G, D-H.

In each point, there is exactly one adjacent conveyor per direction (see the sketch of the conveyor system). Thus, every journey can be described not only by the sequence of points that are passed, but also by a sequence of directions in which the journey continues. Conversely, every sequence of such directions corresponds to a possible journey. Now one finds that, if one wants to get from A to H, one has to use every direction at least once—actually, exactly an odd number of times. So our problem corresponds to the question, how many possible sequences of length five exist such that every element is one of the three letters (= directions) x, y or z, and how many of them contain every letter an odd number of times. The first question is answered quickly: at every point, there exist three possible directions in which a gift can continue its journey. This makes $3^5 = 243$ possibilities. The second question is answered without difficulty too: obviously, a sequence of length five can contain every letter an odd number of times only if one letter is used three times and the others are both used exactly once. For the choice of the letter used three times, there are three possibilities (namely, x, y or z). The other two letters, used only once, are then determined. For the first one of these, there are five possible positions in the sequence. Then, there are four possible positions left for second one. All in all, there are $3 \cdot 5 \cdot 4 = 60$ possible routes. Thus, the possibility we were searching for is

$$\frac{60}{243} = \frac{20}{81}.$$

Remark: With the aid of this modelling, the additional question is answered easily too. After one additional minute, the gift cannot be at the point H. After two or four minutes additional minutes, the gift may have arrived and the possibilities for that can be determined in a similar way. Just try it!

2 Card magic

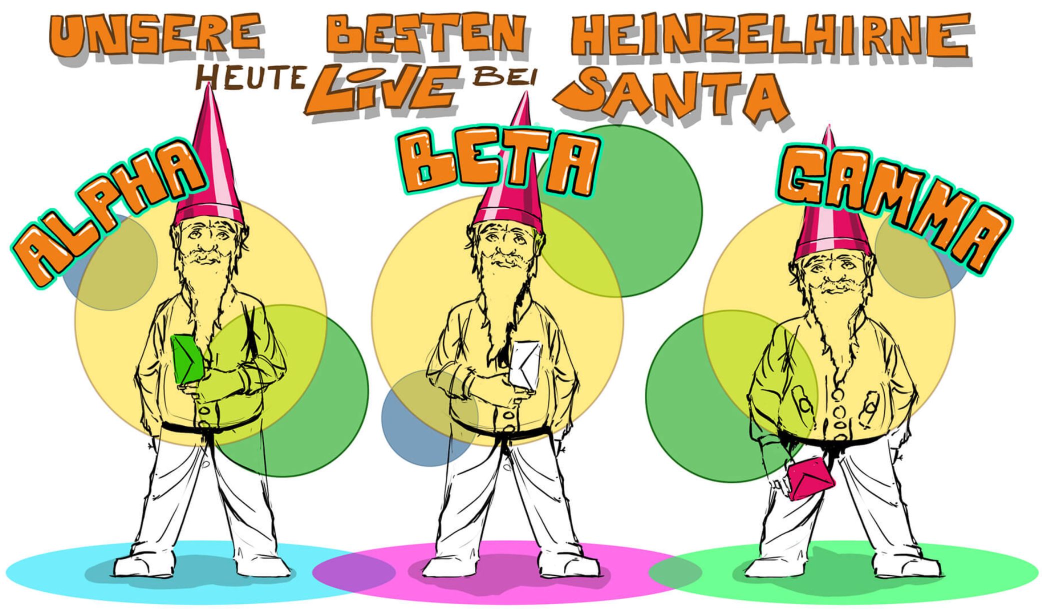
Author: Gerhard Woeginger (TU Eindhoven)

2.1 Challenge

Santa Claus hands an envelope to each of the three super-smart elves Alpha, Beta and Gamma. He tells them: *"In these three envelopes you will each find one of the three symbols from one of the cards K1, K2, . . . , K10 on the table. Please open your envelope, look at your symbol, but do not show it to your friends!"*

K1:	$\triangle$	$\oplus$	$\leftrightarrow$
K2:	$\bowtie$	$\clubsuit$	$\diamondsuit$
K3:	∞	$\bowtie$	$\times$
K4:	$\leftrightarrow$	$\heartsuit$	$\#$
K5:	$\spadesuit$	$\bowtie$	$\oplus$
K6:	$\equiv$	$\bigstar$	$\$$
K7:	$\bigstar$	$\square$	$\approx$
K8:	$\div$	$\triangle$	$\bowtie$
K9:	$\heartsuit$	∞	$\approx$
K10:	$\bigstar$	$\Rightarrow$	$\heartsuit$

Alpha, Beta and Gamma open their envelopes and each look at their symbol. Alpha says: *"I already know which of the ten cards carries our three symbols."* Then Beta says: *"I also know the card!"* And finally Gamma says: *"And now also I do know the card!"* And what about you? Are you able to tell us the card?



Possible answers:

1. It is card K1.
2. It is card K2.
3. It is card K3.
4. It is card K4.
5. It is card K5.
6. It is card K6.
7. It is card K7.
8. It is card K8.
9. It is card K9.
10. It is card K10.

2.2 Solution

The correct answer is: 4.

What we get out of Alpha's statement. As each of the eight symbols $\triangle$, $\oplus$, $\leftrightarrow$, $\bowtie$, ∞, $\heartsuit$, $\bigstar$, $\approx$ shows up on at least two different cards, Alpha would never be able to deduce the secret card from one of these symbols. Consequently Alpha's symbol must be one of the ten mavericks $\clubsuit$, $\diamondsuit$, $\times$, $\#$, $\spadesuit$, $\equiv$, $\$$, $\square$, $\div$, $\Rightarrow$, that each occur only on a single card. As the two cards K1 with $\boxed{\triangle \oplus \leftrightarrow}$ and K9 with $\boxed{\heartsuit \infty \approx}$ do not contain any maverick, neither of these two cards will be the secret card.

What we get out of Beta's statement. Beta knows that Alpha's symbol is a maverick. Beta also knows that neither K1 nor K9 is the secret card.
If Beta's symbol is also a maverick, the secret card must contain two distinct mavericks. There are only two such cards:

- K2 with $\boxed{\bowtie \clubsuit \diamondsuit}$, and Gamma has $\bowtie$
- K6 with $\boxed{\equiv \bigstar \$}$, and Gamma has $\bigstar$

If Beta's symbol is not a maverick, then Beta's symbol may only occur once on each of the eight surviving candidate cards K2, K3, K4, K5, K6, K7, K8, K10. There are only five possibilities for this:

- K3 with $\boxed{\infty \bowtie \times}$, and Beta has ∞, and Gamma has $\bowtie$
- K4 with $\boxed{\leftrightarrow \heartsuit \#}$, and Beta has $\leftrightarrow$, and Gamma has $\heartsuit$
- K5 with $\boxed{\spadesuit \bowtie \oplus}$, and Beta has $\oplus$, and Gamma has $\bowtie$
- K7 with $\boxed{\bigstar \square \approx}$, and Beta has $\approx$, and Gamma has $\bigstar$
- K8 with $\boxed{\div \triangle \bowtie}$, and Beta has $\triangle$, and Gamma has $\bowtie$

Consequently card K10 with $\boxed{\bigstar \Rightarrow \heartsuit}$ is no longer a candidate for the secret card.

What we get out of Gamma's statement. After Beta's statement, Gamma knows that only the seven cards K2, K3, K4, K5, K6, K7, K8 are surviving candidates for the secret card. Gamma cannot have the symbol $\bowtie$, as then the four cards K2, K3, K5 and K8 remain candidates. Gamma also cannot have the symbol $\bigstar$, as then the two cards K6 and K7 remain candidates. Hence Gamma must have the symbol $\heartsuit$.

All in all, the secret card is K4 with $\boxed{\leftrightarrow\ \heartsuit\ \#}$, and Alpha has $\#$,

3 Mirror Lake

Authors: Hennie ter Morsche (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)
Translation: Georg Prokert (TU Eindhoven)

3.1 Challenge

The five points A,B,C,D,E lie clockwise in that order at the edge of the circular Mirror Lake, with B and E diametrically opposed to each other. Since last week, the lake is frozen over completely. Yesterday, the Christmas elf Penelope went on a grand ice skating tour.

She started in point A, skated straight ahead 1922 meters to point B, then 1798 meters to point C, then 1798 meters to point D, then 2162 meters to point E, and finally back to the starting point A, all along straight lines.

Question: What is the second digit behind the decimal point in the decimal representation of the distance (in meters) from A to E?

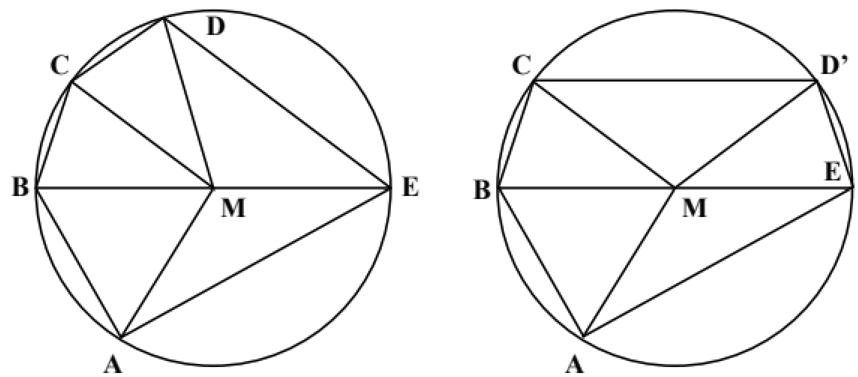
Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

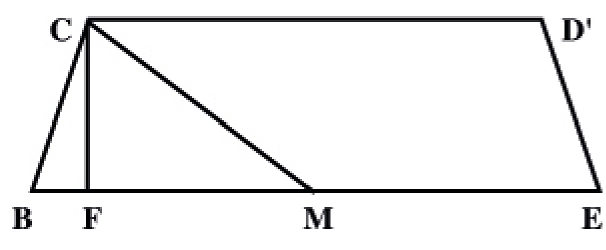
3.2 Solution

The correct answer is: 10.

The following figure shows the Mirror Lake with the five points A,B,C,D,E and the center M (left) and the Mirror Lake in which the pizza points CDM and DEM have been interchanged (right).



As the segments BC and CD in the left figure have the same length, the four points $BCD'E$ in the right figure form a trapezium with edge lengths $|BC| = |D'E| = 1798$ and $|CD'| = 2162$. Moreover, $|MB| = |MC| = |MD'| = |ME| = r$ are equal to the radius r of the circle.
Let us consider this trapezium $BCD'E$ a little closer. We introduce F as foot of the dropped perpendicular from C to the basis BE of the trapezium.



Then $|FM| = |CD'|/2 = 1081$. For the triangle ΔMCF, Pythagoras' theorem yields

$$|CF|^2 \;=\; |MC|^2 - |FM|^2 \;=\; r^2 - 1081^2.$$

For the triangle ΔBFC, Pythagoras' theorem yields

$$|CF|^2 \;=\; |BC|^2 - |BF|^2 \;=\; |BC|^2 - (|BM| - |FM|)^2 \;=\; 1798^2 - (r - 1081)^2.$$

Equating the right hand sides of both equations, we get

$$0 \;=\; 2r^2 - 2 \cdot 1081r - 1798^2 \;=\; (r + 841)(r - 1922).$$

Therefore, the radius of the circle is $r = 1922$.
Now we return to the original question. By Thales' theorem, the triangle ΔABE is rectangular. Its hypotenuse BE is a perimeter of the circle, so $|BE| = 2r$. According to the problem, the cathetus AB has length $|AB| = 1922 = r$. Pythagoras' theorem implies that for the other cathetus we have $|AE| = r \cdot \sqrt{3} \approx 3329.00165$. Therefore, answer **10** is correct.

Remark. Let $ABCD$ a quadrangle with edge lengths $|AB| = a$, $|BC| = b$ and $|CD| = c$. If the quadrangle has a circumcircle with radius r and the edge DA is a perimeter of the circumcircle then $4r^3 - (a^2 + b^2 + c^2)r - abc = 0$. With $a = b = 1798$ and $c = 2162$ this equation immediately gives the radius $r = 1922$.

4 Hyperinflation on Christmas eve

Author: Ernst van de Kerkhof

4.1 Challenge

Each year, Santa and his staff do the best they can to ensure an on-time delivery of Christmas gifts to children all over the world. As you can probably imagine, this is a tremendous logistical and financial operation. Apart from being the big-hearted children's friend we all know him to be, Santa is also a shrewd businessman, who will not miss out on a possible tax deduction. On his behalf an accountant office takes care of these matters. The expenses for the Christmas gifts are tax-deductible; so the office processes all purchases for submission to the Greenlandic Tax Office. Given the giant amount of items (presents), they are distributed over nine separate departments.

- Department D_1 handles all gifts whose selling price starts with the digit 1,
- Department D_2 handles all gifts whose selling price starts with the digit 2,
- etc.

For instance, a gift costing 122 Greenlandic Krones is handled by department D_1, whereas a gift costing 389 Krones is handled by D_3 and another costing 95 Krones by D_9. By doing so, the office hopes to distribute the work among the nine departments more or less evenly in a simple and transparent way. (By the way, there is no need for a department D_0, because all gifts cost at least 1 Greenlandic Krone.)

One week before Christmas Eve, the final lists of items are handed over to the nine departments. Now that each department knows the magnitude of its tasks ahead, proper capacity can be assigned and leaves of absence can be withdrawn in case of an unexpectedly high workload in some of the departments.

But then, a few days later... what a stroke of fate! Due to a severe stock market crash, the Greenlandic government decides in favour of an emergency hyperinflation: the Greenlandic Krone is devaluated to half of its original value. And since Santa Claus buys all of the presents abroad (most of them in well-known online stores...), all prices of the gifts, expressed in Krones, are instantly multiplied by 2.

As you can imagine, this causes a colossal panic in the accountant office. All of the lists are utterly mixed up and every single gift gets reassigned to a different department. Managers are frantically calling employees back from their holidays and everybody is preparing for the worst. Until, after the dust has cleared, a miraculous fact dawns among the accountants: though every item has been transferred to a different department, the *amount* of items stayed exactly constant for each department. A sigh of relief pervades the company. All workloads remain the same; no holidays need to be revoked.

Let N be the total amount of items (presents) and $n_1, n_2, \ldots, n_9$ the amount of items processed by the departments $D_1, D_2, \ldots, D_9$, respectively. (I.e., $N = \sum_{i=1}^{9} n_i$.

Given the fact that all of the n_i (i = 1,2, . . . , 9) have remained constant under the transformation described above **which of the following statement is guaranteed to be true**?

Possible answers:

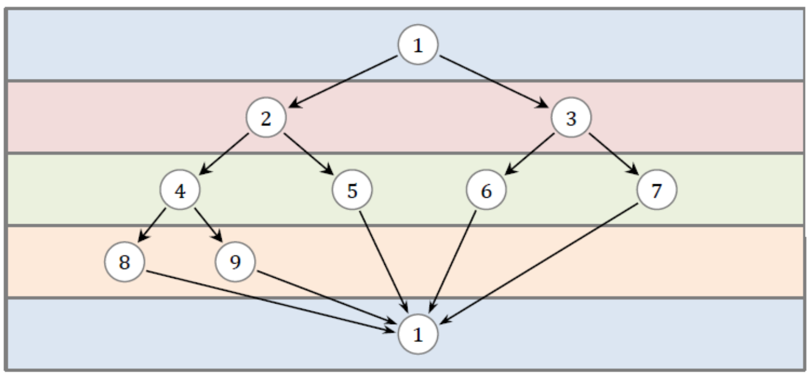
1. $n_1 + n_2 + n_3 > n_4 + n_5 + n_6 + n_7 + n_8 + n_9$
2. $n_2 \geq n_3$
3. $n_1 n_6 \geq n_2 n_3$
4. $n_3 \leq n_2 + n_4 + n_6 + n_8$
5. $\frac{1}{4}N \leq n_1 \leq \frac{1}{3}N$
6. $N \neq 3k + 2$ for all integer $k \geq 0$
7. $\lfloor \frac{1}{9}N \rfloor \leq n_5 \leq \lceil \frac{1}{9}N \rceil$
8. $n_8 \geq n_9$
9. $\frac{1}{3}N \leq n_1 + n_2 \leq \frac{1}{2}N$
10. $n_9 \leq \lceil \frac{1}{9}N \rceil$

Hint: The *floor function* $\lfloor \cdot \rfloor$ is the function mapping a real number x to the greatest integer $\lfloor x \rfloor$ that is less than or equal to x. Similarly, the *ceiling function* $\lceil \cdot \rceil$ maps a real number x to the smallest integer $\lceil x \rceil$ that is greater than or equal to x.

4.2 Solution

The correct answer is: 5.

The following graph shows the transition of the first digits of a number (≥ 1) in decimal notation, when multiplied by 2.



For a stable size of all first-digit subsets of a given set of numbers under this transformation (i.e., all of the n_i ($i = 1,2,\ldots,9$) are to remain unchanged), the following equations can be read off the graph above:

$$n_1 = n_2 + n_3 \tag{1}$$
$$n_2 = n_4 + n_5 \tag{2}$$
$$n_3 = n_6 + n_7 \tag{3}$$
$$n_4 = n_8 + n_9 \tag{4}$$
$$n_1 = n_5 + n_6 + n_7 + n_8 + n_9 \tag{5}$$

(Equation (5) can be read off the graph, but is also implied by the preceding four equations.)

Adding equations (1) and (5) yields

$$2n_1 = n_2 + n_3 + n_5 + n_6 + n_7 + n_8 + n_9 = N - (n_1 + n_4) \quad \Leftarrow \quad n_1 = \frac{N - n_4}{3}. \tag{6}$$

As a consequence, $n_1 \leq \frac{1}{3}N$, where the equal-sign corresponds to the case $n_4 = 0$.

Furthermore, inserting equations (2) and (3) into equation (1) gives

$$n_1 = n_4 + n_5 + n_6 + n_7.$$

If $n_5 = n_6 = n_7 = 0$, then $n_1 = n_4$ and equation (6) implies that $n_1 = \frac{1}{4}N$. Therefore, n_1 is bounded by

$$\frac{1}{4}N \leq n_1 \leq \frac{1}{3}N.$$

Thus, statement **5** is correct.

Finally, we give a short explication to the other statements:

1. $n_1 + n_2 + n_3 > n_4 + n_5 + n_6 + n_7 + n_8 + n_9$:
   The statement is true unless $n_5 = n_6 = n_7 = 0$, because in this case, one has $n_1 + n_2 + n_3 = n_4 + n_8 + n_9$.

2. $n_2 \geq n_3$:
   The statement is false. One only knows that $n_1 = n_2 + n_3$, and therefore that $n_1 \geq n_2$ and $n_1 \geq n_3$. One cannot deduce an ordering between n_2 and n_3.

3. $n_1 n_6 \geq n_2 n_3$:
   The statement is false, since n_6 can be zero while n_2 and n_3 are not.

4. $n_3 \leq n_2 + n_4 + n_6 + n_8$:
   The statement is false: Consider the case $n_2 = n_4 = n_8 = 0$ and $n_7 \neq 0$. Then, $n_3 \geq n_6$.

5. $\frac{1}{4}N \leq n_1 \leq \frac{1}{3}N$:
   The statement is always true, see above.

6. $N \neq 3k + 2$ for all integer $k \geq 0$:
   The statement is true for $k = 0$ and $k = 1$, but false for all $k \geq 2$.

7. $\lfloor \frac{1}{9}N \rfloor \leq n_5 \leq \lceil \frac{1}{9}N \rceil$:
   Drawing from the assumption that the median might have an average relative frequency, which is in generally false.

8. $n_8 \geq n_9$:
The statement is false. One only knows that $n_4 = n_8 + n_9$, and therefore that $n_4 \geq n_8$ and $n_4 \geq n_9$. One cannot deduce an ordering between n_8 and n_9.

9. $\frac{1}{3}N \leq n_1 + n_2 \leq \frac{1}{2}N$:
n_1 and n_2 can both be equal $\frac{1}{3}N$, in case $n_3 = n_4 = 0$.

10. $n_9 \leq \lceil \frac{1}{9}N \rceil$:
The statement is false: Consider the case $n_5 = n_6 = n_7 = n_8$. Then, $n_9 = \frac{1}{4}N$, which may be deduced from equations (1) and (6).

Relevance:

The question is an illustration of *Benford's Law*, an observation about the relative frequency of each of the digits 1 to 9 to appear as the first digit of a randomly chosen number in the decimal notation. (Randomness is in this exercise loosely represented by the stability under a given scale transformation, i.e. the multiplication by 2.)
According to Benford's Law, the theoretical fraction of random numbers starting with the digit 1 equals 0.301, which is in good agreement with the interval $\left[\frac{1}{4}, \frac{1}{3}\right]$ derived in this exercise.

5 The Christmas plane

Authors: Jo Andrea Brueggemann (WIAS), Jonas Holley (WIAS), Tobias Keil (WIAS)
Project: SE5 – Optimal design and control of optofluidic solar steerers and concentrators
Translation: Clara Jansen (MATHEON)

5.1 Challenge

“Only 24 days left until Christmas!”, Santa thinks, and nothing is settled. Nothing at all! “This year everything goes wrong”, he mumbles to himself. He cannot rely on the reindeers anymore, since chief reindeer Rudolf is being troubled with arthritis in his knees.

So it comes in handy for desperate Santa that shortly before Christmas the airline *Air Berolina* had to file for insolvency—especially considering the recent modernisation of its aircraft fleet for the purpose of sustainability. Santa senses an opportunity, since one can produce excellent organic kerosene with the *Fischer-Tropsch-Method* using his and his gnomes in-house biological waste.

Santa estimates shortly, talks to the packing-gnome and afterwards with *Air Berolina*: He has exactly 22 days left to buy his plane “Santa Air”. Then, he has two days left to load the gifts into the cargo hold and start to fly.

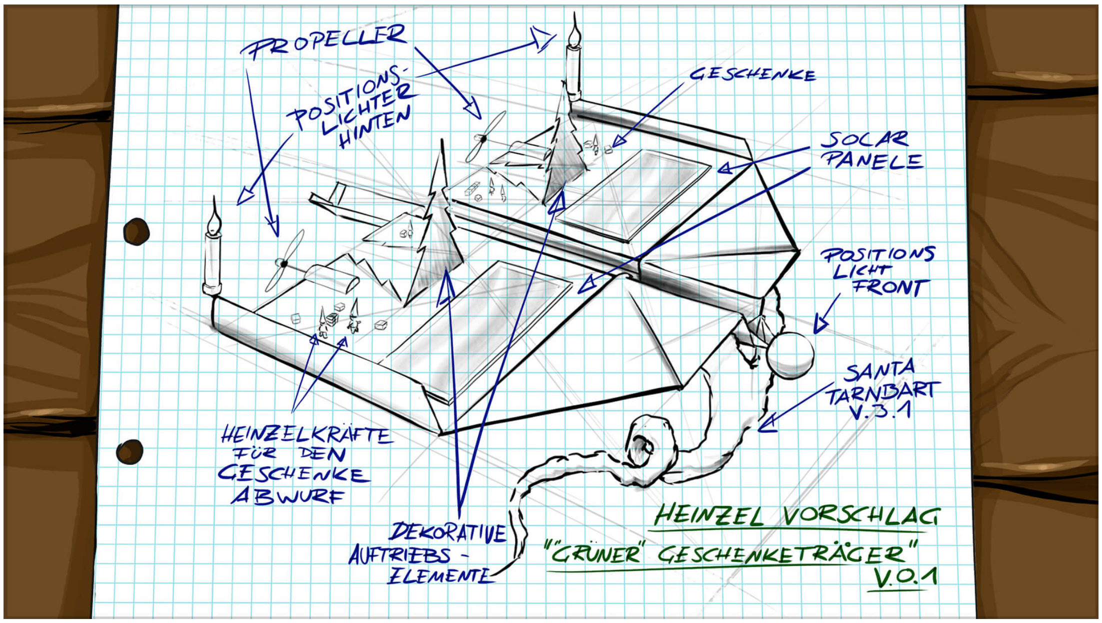
At the moment, *Air Berolina* has exactly 22 freight planes, everyone of them operating with organic kerosene. The planes only differ in the size of their cargo volume. Well, Santa being a very special client, *Air Berolina* has a very unique proposal: every day from the 1st until the 22nd of December, Santa will receive an offer to purchase one plane of the fleet—always at the same prize, but each one with a different cargo volume. However, *Air Berolina* will not reveal the maximum cargo volume of the planes in its fleet. Furthermore, the offer is valid only on the respective day; it expires at the following day.

Since Santa can only afford to buy one plane, he wants to choose the best offer—that is, the plane with the largest cargo volume—to be sure to find enough space to carry all the gifts.

Santa does not know what to do, but the experienced chief reindeer Rudolf, a smart strategist, suggests to wait for the first 8 days and to memorize the plane with the biggest cargo volume. After the eighth day, Rudolf advises that Santa should just take the first plane coming up with an even bigger cargo space (so the plane he chooses has a bigger cargo space than any plane from the first 8 days).

Santa is confused and asks: "Suppose that I apply your strategy as you suggested: Meaning, I actually get 22 independent offer, from which I reject the first 8 ones, and choose next best offer made afterwards. What is the probability that I actually get my hands on the plane with the largest cargo space?"

Which of the following possible answers is the best approximation of this probality?



Possible answers:

1. 100%
2. 9%
3. 51%
4. $\frac{1400}{22}\%$
5. $33{,}\bar{3}\%$
6. $\frac{1}{22}\%$
7. $\frac{800}{22}\%$
8. 42%
9. 38%
10. $\frac{100}{22}\%$

Reference to project:

Similar problems of optimal strategy occur in game theory, for example with equilibrium problems. In addition, they are also used for problems of optimization and controlling of physical and biomedical processes.

5.2 Solution

The correct answer is: 9.

To compute the probability of success of Rudolf's strategy, we consider the plane with the maximum cargo capacity. The day on which this plane is offered is denoted by a. Since Santa does not get any information about the day the best plane is offered, every number (= day) between 1 and 22 is equally possible for a.
Since the first 8 offers are turned down, the Rudolf's strategy fails if the best plane is among the first 8 offered planes (i.e. $a \leq 8$). Therefore, we consider the case when the best plane is offered from the ninth day on ($a \geq 9$).
For the success of the strategy it is essential on which day the *second best* among the first a planes is offered. We have to consider the following two cases:

1. Assume that this plane is not among the first 8 offers. Since, by definition, the second best plane among the first a planes will be offered before the best plane, i.e. within the first $a - 1$ days, the strategy fails: The second best plane will be offered to Santa between the ninth day and day $a - 1$, and he will not pick the best plane at the ath day.

2. Otherwise, the second best plane among the first a planes is offered within the first 8 days. Then, Santa memorizes the best among the first $a-1$ days (on the first 8 days) and actually chooses to buy the ath plane. Since this is the overall best plane, the strategy is successful in this case.

Altogether there are 8 admissible days among $a - 1$ possible days on which the plane is offered. Under the assumption that the best plane will be offered on the a-th day, the possibility p_a that the second best plane is offered in the first 8 days is $p_a = \frac{8}{a-1}$. If one considers that the best plane (in the successful case) is offered between the ninth and 22nd day and every day is equally possible, for the possibility P of the success of Rudolf's strategy we compute:

$$
\begin{aligned}
P_8 = \sum_{i=9}^{22} \frac{1}{22} \cdot p_i =& \frac{1}{22} \cdot p_9 + \frac{1}{22} \cdot p_{10} + ... + \frac{1}{22} \cdot p_{22} \\
=& \frac{1}{22} \cdot \frac{8}{8} + \frac{1}{22} \cdot \frac{8}{9} + ... + \frac{1}{22} \cdot \frac{8}{21} \\
=& \frac{8}{22} \cdot \left(\frac{1}{8} + \frac{1}{9} + ... + \frac{1}{21} \right) \\
=& 0{,}3827.
\end{aligned}
$$

Thus, with a possibility of about 38 % Santa chooses the plane with the biggest cargo volume.

In fact, one can show that Rudolf obviously is really smart and advised Santa to take the optimal strategy—that is the one that leads to the best probability that Santa catches the best plane. Assume that Rudolf had not revealed how many days Santa should wait to memorize the best plane, and Santa probably would have chosen another number r of days. Then, we can compute the probability P_r that he chooses the best plane in the following way:

$$
P_r = \sum_{i=r+1}^{22} \frac{1}{22} \cdot p_i = \sum_{i=r+1}^{22} \frac{1}{22} \cdot \frac{r}{i-1} = \frac{r}{22} \cdot \sum_{i=r}^{21} \frac{1}{i}.
$$

One can understand the last sum on the right-hand side of the equation chain as a step function of the function $f(x) = \frac{1}{x}$. More accurately, with this sum yields the area of the step function with the equidistant step width 1 of the function $f(x) = \frac{1}{x}$ on the interval $[r{,}22]$. So the value of $\sum_{i=r}^{21} \frac{1}{i}$ can be approximated with the integral of $f(x) = \frac{1}{x}$ over the interval $[r{,}22]$. This can be computed as follows:

$$
\int_r^{22} \frac{1}{x} dx = [\ln(x)]_r^{22} = \ln(22) - \ln(r) = \ln \left(\frac{22}{r} \right).
$$

Thus, we get for the probability P_r approximately:

$$
P_r = \frac{r}{22} \cdot \sum_{i=r}^{21} \frac{1}{i} \approx \frac{r}{22} \cdot \ln \left(\frac{22}{r} \right).
$$

If we want to determine the r which maximizes the probability P_r and with which we would most likely choose the plane with the biggest cargo space, we have to determine the maximum of the function $g(r) = \frac{r}{22} \cdot \ln\left(\frac{22}{r}\right)$. Since a function can only take its maximum value at a point where its derivative is equal to 0 (or on the boundary of the interval), we set:

$$g'(r) = \frac{1}{22} \cdot \ln\left(\frac{22}{r}\right) + \frac{r}{22} \cdot \left(-\frac{1}{r}\right) = 0.$$

Using an equivalent transformation, we get the following equation:

$$\ln\left(\frac{22}{r}\right) = 1 \Leftrightarrow r = \frac{22}{e} \approx 8.$$

That is, the function g has a local maximum near 8 and it is indeed the best choice to wait 8 days prior to choosing the next best plane!

6 Place cards

Author: Hennie ter Morsche (TU Eindhoven)

6.1 Challenge

At the Christmas party, five couples sit around a round table: Mr. and Mrs. Koriander, Mr. and Mrs. Fennel, Mr. and Mrs. Cinnamon, Mr. and Mrs. Ginger, and Mr. and Mrs. Rosemary. Every man sits next to his wife (either left or right). Mr. Koriander sits on the right of his wife. Furthermore, Mr. Koriander is the only man who does not sit next to another man.

The place cards of the couples are numbered clockwise from 1 to 10. During the dance, five mixed-gender pairs are formed with the place numbers 1–5, 2–4, 3–9, 6–8 and 7–10.

Question: What is the number of Mr. Koriander's place card?

Possible answers:

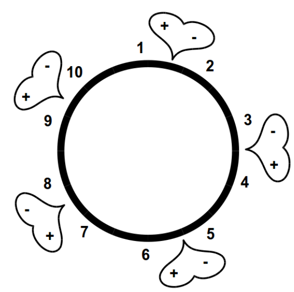
1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

6.2 Solution

The correct answer is: 2.

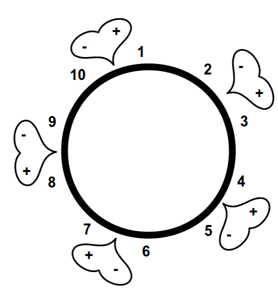
We know from the dancing couples that 1–5, 2–4, 3–9, 6–8 and 7–10 are man-woman couples. We denote the sex of 1 with $+$ (plus), and the sex of 5 with $-$ (minus).

Case 1:
The married couples are seated as couples 1–2, 3–4, 5–6, 7–8 and 9–10. Since each of the ten pairs 1–2, 2–4, 4–3, 3–9, 9–10, 10–7, 7–8, 8–6, 6–5, 5–1 contains a man and a woman, the sexes of 1,2,3,4,5,6,7,8,9,10 are then $+, -, -, +, -, +, +, -, +, -$. If $+$ designates a man, the three men 1,4,9 sit between two women; contradiction. If $-$ designates a man, the two men 5 and 8 sit between two women; contradiction. This case doesn't lead to any solution.



case 1

Case 2:
The married couples are seated as couples 1–10, 9–8, 7–6, 5–4 and 3–2. Since each of the ten pairs 1–10, 10–7, 7–6, 6–8, 8–9, 9–3, 3–2, 2–4, 4–5, 5–1 contains a man and a woman, the sexes of 1,2,3,4,5,6,7,8,9,10 are then $+, -, +, +, -, -, +, +, -, -$. If $+$ designates a man, only 1 sits between two women. Then Mr. Coriander would be 1 and would sit to the left of Mrs. Coriander 10; contradiction.



case 2

If $-$ designates a man, then 2 sits between two women. Then Mr. Coriander is 2 and sits to the right of Mrs. Coriander 3. Since this is the only remaining possibility, **answer #2** is correct.

7 Battlesleigh

Author: Falk Ebert (HU Berlin)
Translation: Ariane Beier (MATHEON)

7.1 Challenge

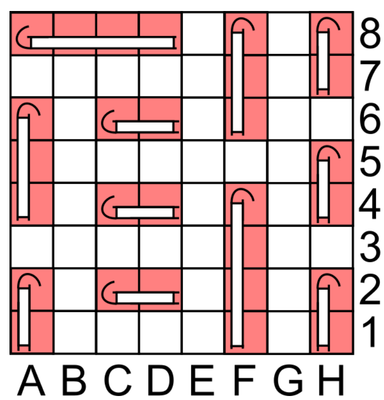
The gnomes Redda and Sledda play "Battlesleigh". In the past, they used to play this game with Santa's sleighs, which they parked on the different sides of a wall, throwing snowballs over the wall and trying to hit the opponent's sleighs. Once, Redda and Sledda had moved Santa's big sleigh with all the Christmas presents, so Santa Claus had to search for it all over the sleigh park to find his sleigh on Christmas eve. Since then, there are countries in which children receive their gifts not until December 25.

Since this incident, Redda and Sledda are not allowed to play with real sleighs anymore. Thus, they have to put up with a less bothering substitute of the game that is played with pencil and paper. Each gnome positions his or her sleighs on a grid of size 8×8 obeying the following rules:

- There exist sleighs of length $1, 2, 3, 4$ and 5. These are represented by a respective number of consecutive squares on the grid.
- The sleighs have to be positioned horizontally or vertically exactly inside the squares of the grid. In particular, there are not allowed to be put diagonally.

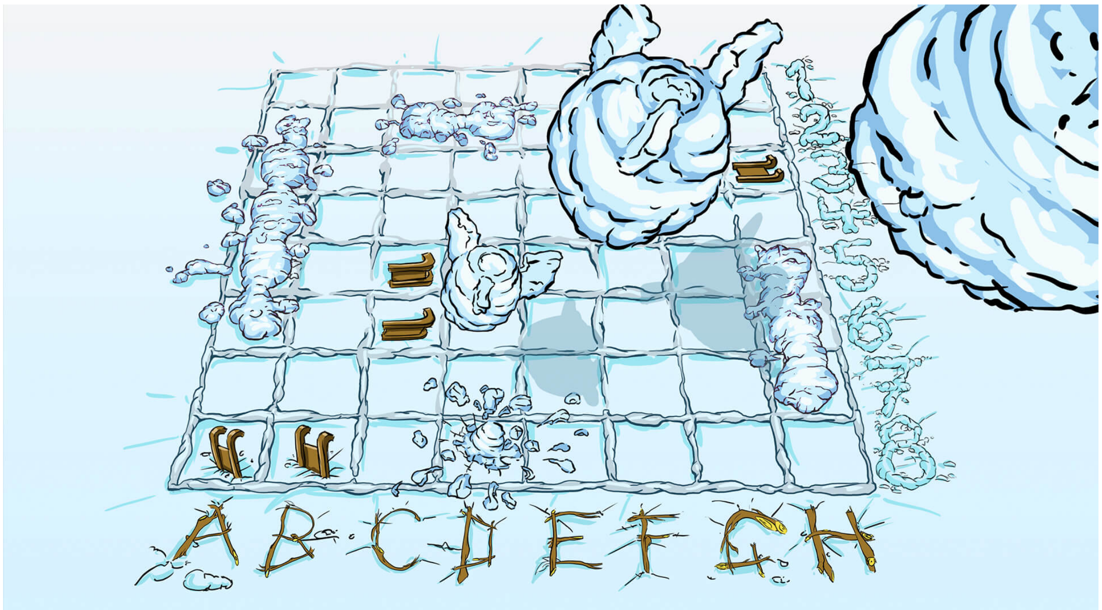
- Any two sleighs must not touch each other—not even at their vertices.

After positioning their sleighs, Redda and Sledda take turns calling out coordinates of a square on the grid and the opponent has to answer whether or not a sleigh was hit. (Maybe you recognise the game's concept.)
Today, they do not fancy to play the game in its classical variant. Instead, they aim to fill the grid as much as possible—i.e. maximizing the number of squares occupied by sleighs—obeying the above three rules. An example of a valid configuration is shown in the following figure:



Here, exactly 28 squares are occupied with sleighs.

What is the maximal number of legitimately occupied squares on the grid?



Possible answers:

1. 27
2. 28
3. 29
4. 30
5. 31
6. 32
7. 33
8. 34
9. 35
10. 42

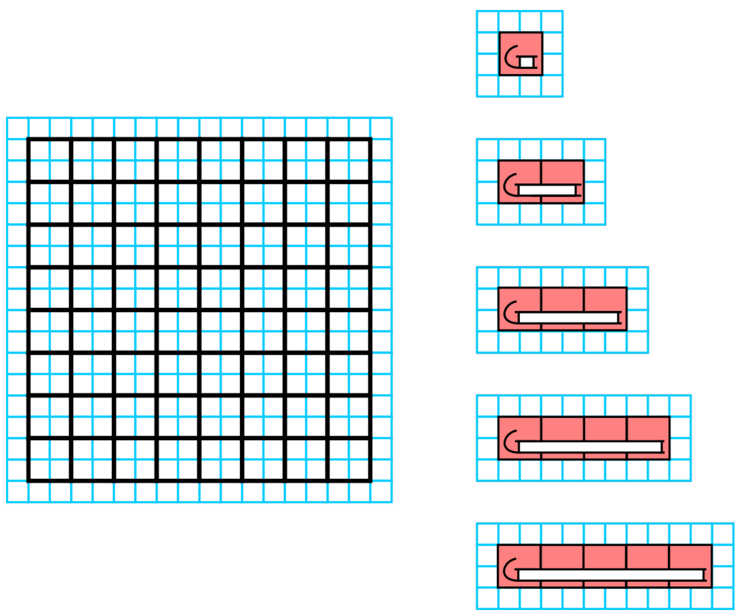
Please note:

This exercise is not directly related to industrial applications of Mathematics. However, there are smart ways to rephrase the constraints (i.e. the rules of the game) to yield a far easier problem. This is a common principle in Mathematical Optimization, because—even if many problems can be resolved by computers—a smart phrasing of a problem may reduce the computation time enormously.

7.2 Solution

The correct answer is: 6.

As long as the exercise is only about positioning the sleighs on the grid, it is fairly easy: the 64 squares of the grid are easily covered by horizontally and vertically placed sledges. The challenging part of the task is to incorporate the additional condition that any two sleighs must not touch each other. One possibility to implement this condition is to equip each sleigh with a boundary of the width of $\frac{1}{2}$ of a square. Thus, if any two of these "sleighs with boundary" are placed directly next to each other, there remains a space of exactly one square between these two sleighs. Since the sleighs are allowed to lie directly at the boundary of the grid, we also need to extend the grid with such a boundary.



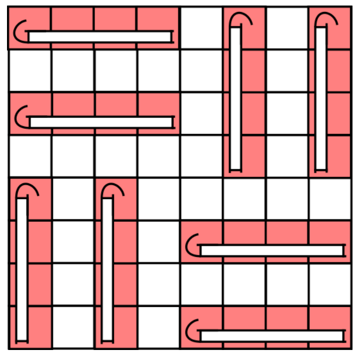
In the figure above, the grid and the sleighs are displayed with their boundaries. The modified task is now to fill the "grid with boundary" as much as possible with the "sleighs with boundary". As we will see in a moment, it is not possible to cover the new grid completely. Anyway, let us first consider the size of the extended sledges and the extended grid:

	originally	with boundary	ratio
1-sleigh	1 square	4 squares	$\frac{1}{4}$
2-sleigh	2 squares	6 squares	$\frac{2}{6} = \frac{1}{3}$
3-sleigh	3 squares	8 squares	$\frac{3}{8}$
4-sleigh	4 squares	10 squares	$\frac{4}{10} = \frac{2}{5}$
5-sleigh	5 squares	12 squares	$\frac{5}{12}$
grid	64 squares	81 squares	

From the table above, we deduce that the extended grid consists of 81 squares that have to be covered by sleighs that consists of 4, 6, 8, 10 and 12 squares, respectively. Obviously, this is not possible, since 81 is an odd integer. Thus, we are able to cover at most 80 squares. Furthermore, we recognize that the ratio of the actual sleigh inside the "sleigh with boundary" increases with the length of the sleigh. In conclusion, it is more efficient to use preferably long sleighs (which is not surprising). Using a clever trial and error routine, one notes that

$$\begin{aligned} 6 \cdot 12 + 1 \cdot 8 &= 80 \quad \text{and} \\ 5 \cdot 12 + 2 \cdot 10 &= 80 \end{aligned}$$

are the only partitions of 80 that lead to 33 covered squares on the grid ($6 \cdot 5 + 1 \cdot 3$ and $5 \cdot 5 + 2 \cdot 4$). However, the 5-sleigh does not exploit the space between its narrow edge and the boundary in a convenient way—no matter how it is positioned on the grid. Thus, both of the above partitions cannot be realised. In contrast, the partition $8 \cdot 10 = 80$ with 8 sleighs of length 4 is easily incorporated and covers 32 squares:



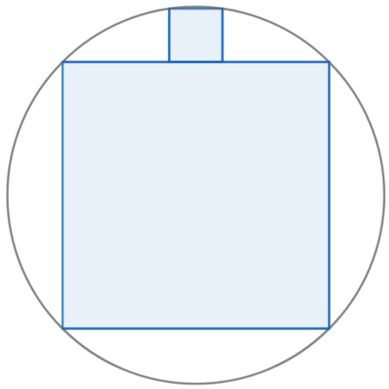
8 Look at all the wasted space!

Authors: Luise Fehlinger (HU Berlin), Falk Ebert (HU Berlin)
Translation: Ariane Beier (MATHEON)

8.1 Challenge

Wrapping gifts is an art. However, for reasons of efficiency, gifts are mostly wrapped in boxes shaped like rectangular cuboids nowadays. The increase of efficiency produces rather strange effects: because of the aerodynamics of Santa's sleigh, its storage room for all the Christmas presents is shaped like a (non-solid) cylinder. The gnome wrapping station provides a large case for the presents shaped like a rectangular cuboid that fits exactly inside the cylindrical storage room. This case has a square profile and is the largest single rectangular cuboid that fits inside the cylinder.

For some years, wrapping gnome Wrapimir was getting annoyed at all the wasted space beside the large case inside the storage room. He aims to fit rectangular cuboids inside the four unoccupied segments of the cylinder. These are to be as long as the large cuboid and of square profile too. (By the way, Wrapimir ignores all of his colleagues' comments that square does not automatically translate to optimal.) Now, he is busy developing the respective cases.



In the example above, there is only one of the cylinder's empty segments filled with a rectangular cuboid of square profile.

If Wrapimir wants to fill all four of the unoccupied cylinder segments with rectangular cuboids of square profile and of maximal size, how much can the space available for presents be enlarged?

Possible answers:

1. 4%
2. 5%
3. 6%
4. 8%
5. 12%
6. 13%
7. 14%
8. 15%
9. 16%
10. 42%

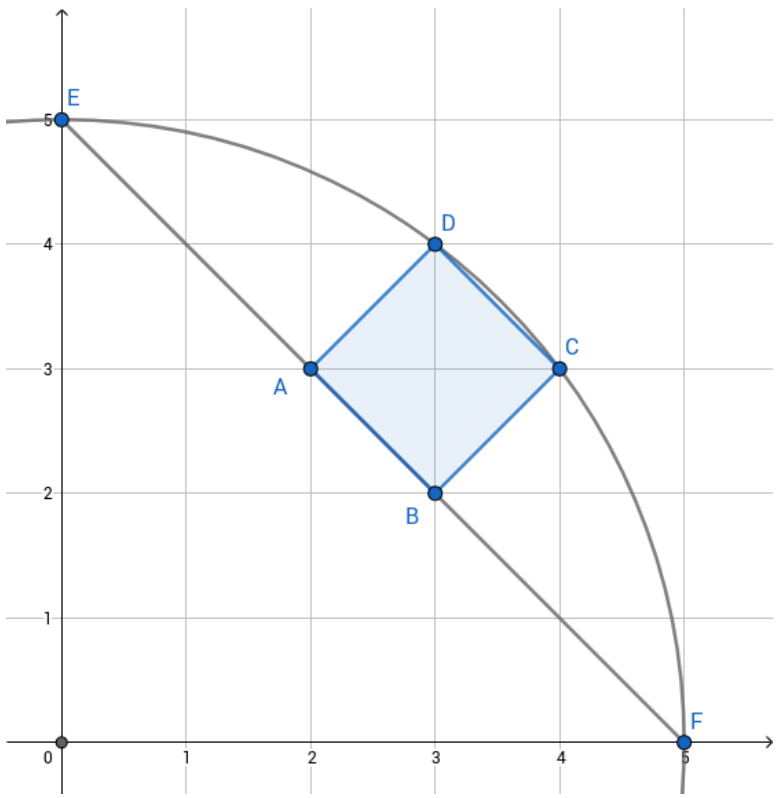
Project relevance:

Packing problems occur repeatedly in mathematical applications. But they are rarely as clearly determinate and exactly solvable as the one given above. In principle, the same happens when cutting out Christmas cookies. However, an optimal exploitation of the cookie dough succeeds only approximately—especially when there exists an additonal constraint given by nibbling children.

8.2 Solution

The correct answer is: 9.

There are various, more or less complicated ways to determine the area of one of the small squares to be fitted inside the cylinder as $\frac{1}{25}$ of the big square's area. The following method is particularly charming, since it needs just a few words:



The suitable choice of the coordinate system above demonstrates that $\square ABCD$ is in fact a square as required and that the length of its edges is $\frac{1}{5}$ of the side length of the big square. Consequently, the area of $\square ABCD$ is $\frac{1}{25}$ of the big square's area. Finally, since there are 4 of these small squares, the additional area amounts to $\frac{4}{25} = 16\%$ of the big square's.

9 Certified Christmas cookies from the 19th century

Authors: Rafael Arndt (HU Berlin), Caroline Löbhard (WIAS), Simon Rösel (HU Berlin)
Project: C-SE15

9.1 Challenge

Deeply worried, Santa Claus was sitting in the Christmas bakery thinking "oh man, this is really bad." Christmas was approaching fast, and the famous Christmas cookies still had to be produced. These cookies had been the Christmas eve delight for more than a century. This year, unfortunately, the Christmas elves had lost the traditional recipe from 1892 "somewhere on the way to the north pole while competing in a sleigh race." Luckily, Santa remembered that the very successful recipe contained a metric ton of flour (1 metric ton = 1000 kilogram), 4000 fresh eggs, 800 kilogram (kg) butter, 200kg of sugar, a pinch of cinnamon (very important!), as well as a certain amount of salt and/or ground almonds. Having been questioned by Santa, the elves said that they also did not know the optimal amount of almonds and salt. But they had written down the secret cookie-taste-function G, which had been obtained as a result of many years of experimentation (and after many complaints by frustrated kids).

In fact, the cookie taste is determined by the function

$$G(m,s) = -\frac{1}{8}\left(\frac{m}{20}-1\right)^2\left(\frac{m}{20}-2\right)\left(\frac{m}{20}-4\right)^2 - \left(\frac{s}{800}-1\right)\left(\frac{s}{800}-2\right)\left(\frac{s}{800}-7\right) + 22$$

where m is the amount of ground almonds (measured in kg) and s is the amount of salt (in gram (g)). The Christmas elves further mentioned that "the famous recipe from 1892 is determined by the property that the taste cannot possibly be improved, if, starting from the 1892 almond/salt combination, another certified recipe is employed, whose amounts of almonds and salt differ from the ones of the year 1892 by at most 40kg and 800g, respectively."

Of course, Santa Claus knew that the great cookie disaster of 1984 had led the elves to come up with a sophisticated security system: for future experiments, only those recipes were certified that had the property that the square root of the sum of the squares of the amounts of almond and salt (measured in kg and g, respectively) is less than or equal to the amount of almonds from the traditional 1892 recipe. Additionally, a recipe must contain almonds but not necessarily salt in order to get certified.

Santa was lost. At least, he knew for a fact that the recipe from 1892 contained at most 100kg of ground almonds and 5000g of salt, and that this traditional recipe was itself certified.

Now he relies on your assistance. Can you tell how much salt and ground almonds the 1892's recipe contains?

Possible answers:

1. 0kg of ground almonds, 0g of salt
2. 0kg of ground almonds, 4151g of salt
3. 20kg of ground almonds, 4151g of salt
4. 100kg of ground almonds, 4151g of salt
5. 32kg of ground almonds, 0g of salt
6. 60kg of ground almonds, 4151g of salt
7. 32kg of ground almonds, 4151g of salt
8. 80kg of ground almonds, 0g of salt
9. 80kg of ground almonds, 4151g of salt
10. 100kg of ground almonds, 0g of salt.

Project relevance:

The above task can be considered as a constraint optimization problem, where the feasible set varies with the solution. In a broad sense, the problem can be understood as a so-called *quasi-variational inequality*.

The investigation of quasi-variational inequalities and their control represents a current research topic in the field of mathematical optimization, which features a number of interesting aspects but also many open problems. This concerns, for instance, the solvability and the stability of problems of this type. Of course, we also want to develop efficient ways of solving these problems numerically.

Quasi-variational inequalities serve as mathematical model for many problems from various scientific fields. For example, this concerns the simulation of the behavior of bacteria, mechanical friction between two objects in contact or the growth of sand dunes. In financial mathematics, quasi-variational inequalities occur in game theory, where the interaction and strategies of different market actors are analyzed.

9.2 Solution

The correct answer is: 8.

Let us denote the 1892 combination of ground almonds and salt by $(\overline{m},\overline{s})$. According to Santa's memory, it holds $(\overline{m},\overline{s}) \in (0,100] \times [0,5000]$, the corresponding units understood. However, we do not need this additional information. According to the Christmas elves, $(\overline{m},\overline{s})$ is characterized by the optimality property

$$G(m,s) \leq G(\overline{m},\overline{s}), \quad \text{for all } (m,s) \in K(\overline{m},\overline{s}), \tag{7}$$

where the set

$$\begin{aligned} K(\overline{m},\overline{s}) = \{(m,s) \in \mathbb{R}^2 : m > 0, s \geq 0, \sqrt{m^2 + s^2} \leq \overline{m}, \\ \overline{m} - 40 \leq m \leq \overline{m} + 40, \overline{s} - 800 \leq s \leq \overline{s} + 800\} \end{aligned}$$

contains all almond/salt-pairs (m,s) that comply with the certification, and that differ from the 1892 recipe by at most 40kg and 800g, respectively. Note that the feasible set $K(\overline{m},\overline{s})$, over which we have to maximize, depends on the solution $(\overline{m},\overline{s})$.

First, we learn that the recipe of 1892 is itself certified. In particular, it holds $\overline{m} > 0$ and $\sqrt{\overline{m}^2 + \overline{s}^2} \leq \overline{m}$. On the other hand, it always holds $\sqrt{\overline{m}^2 + \overline{s}^2} \geq \overline{m}$, and thus, $\sqrt{\overline{m}^2 + \overline{s}^2} = \overline{m}$, which implies $\overline{s} = 0$. Thus, the amount of salt is already found. Therefore, problem (7) is equivalent to finding $\overline{m} \in (0,100]$, for which it holds that

$$G(m,s) \leq G(\overline{m},0), \quad \text{for all } (m,s) \in K(\overline{m},0), \tag{8}$$

which means that

$$\max_{(m,s) \in K(\overline{m},0)} G(m,s) \leq G(\overline{m},0). \tag{9}$$

Using the above definition of K, we may simplify the set $K(\overline{m},0)$ as follows:

$$\begin{aligned} K(\overline{m},0) = \{m > 0, s \in [0,800] : \\ \overline{m} - 40 \leq m \leq \overline{m}, \ \sqrt{m^2 + s^2} \leq \overline{m}\}. \end{aligned}$$

As one sees, we have used that the inequality $\sqrt{m^2+s^2} \leq \overline{m}$ implies $m \leq \overline{m}$. Now we examine the cookie-taste-function G more closely. Let us define the respective taste contributions of almonds and salt, respectively:

$$G(m,s) = G_1(m) + G_2(s) + 22,$$

where

$$\begin{aligned} G_1(m) &= -\frac{1}{8}\left(\frac{m}{20} - 1\right)^2 \left(\frac{m}{20} - 2\right) \left(\frac{m}{20} - 4\right)^2 \\ G_2(s) &= -\left(\frac{s}{800} - 1\right)\left(\frac{s}{800} - 2\right)\left(\frac{s}{800} - 7\right). \end{aligned}$$

In order to simplify the max-term in (9), we first observe that it holds that

$$\max_{(m,s)\in K(\overline{m},0)} G(m,s) \geq \max_{\{m>0:\, \overline{m}-40\leq m\leq\overline{m}\}} G_1(m) + G_2(0) + 22; \tag{10}$$

this follows by choosing $s = 0$. On the other hand, one may enlarge the set $K(\overline{m},0)$ by requiring $m \leq \overline{m}$ instead of $\sqrt{m^2+s^2} \leq \overline{m}$. This entails

$$\max_{(m,s)\in K(\overline{m},0)} G(m,s) \leq \max_{\{m>0:\, \overline{m}-40\leq m\leq\overline{m}\}} G_1(m) + \max_{s\in[0,800]} G_2(s) + 22 \tag{11}$$

Note that on the right hand side, the maximum of G_2 is sought to be taken on a set that does not depend on the solution any more! Since G_2 is a cubic polynomial with a negative leading coefficient and three different roots, that has the smallest root at $s = 800$, the maximum of G_2 on the interval $[0,800]$ is attained at $s = 0$. Hence, $\max_{0\leq s\leq 800} G_2(s) = G_2(0) = 14$. Alternatively, one may show that the first derivative of G_2 has no zeros in the interval $[0,800]$ and one may compare $G_2(0)$ and $G_2(800)$ (monotonicity!).

As a result, one obtains

$$\max_{(m,s)\in K(\overline{m},0)} G(m,s) \leq \max_{\{m>0:\, \overline{m}-40\leq m\leq\overline{m}\}} G_1(m) + G_2(0) + 22.$$

According to (10), this inequality is also valid in the opposite direction! Hence,

$$\max_{(m,s)\in K(\overline{m},0)} G(m,s) = \max_{\{m>0:\, \overline{m}-40\leq m\leq\overline{m}\}} G_1(m) + G_2(0) + 22.$$

Consequently, (9) has been reduced to finding an $\overline{m} \in (0,100]$ such that

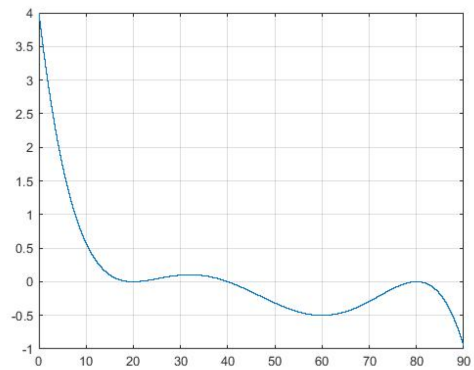


Figure 1: Graph von $G_1 = G_1(m)$

$$\max_{\{m>0:\, \overline{m}-40\leq m\leq \overline{m}\}} G_1(m) \leq G_1(\overline{m}). \tag{12}$$

In other words, $\overline{m}$ has to be determined in such a way that a decrease in the amount of almonds by at most 40kg cannot improve the taste. In doing so, the amount of almonds must be kept positive in order to comply with the certification. To find the solution of (12), we analyze the function G_1 by computing its first derivative. To reduce the work, we may use the reparametrized function $\tilde{G}_1(m) := G(20m)$, and one obtains

$$\tilde{G}_1'(m) = -\frac{5}{8}(m-1)(m-4)(m^2 - \frac{23}{5}m + \frac{24}{5}).$$

The zeros of $G_1'(m)$ are

$$m_1 = 20, m_2 = 32, m_3 = 60, m_4 = 80.$$

Since G_1 is strictly monotonically decreasing on the interval (0,20], there is no solution of (12) in this interval, as we may always halve the amount of almonds in order to improve the taste. On the interval [20,40], one may use the following estimate:

$$G_1(m) = \frac{1}{8}\left(\frac{m}{20} - 1\right)^2 \left(2 - \frac{m}{20}\right)\left(\frac{m}{20} - 4\right)^2 \leq \frac{1}{8} \cdot 1^2 \cdot 1 \cdot 3^2 = 9/8.$$

Since $G_1(0) = 4$, the solution of (12) is also not contained in [20,40]. As $G_1(80) = 0$ and $G_1(m) \leq 0$ for all $m \in [40{,}80]$, we find that $\overline{m} = 80$ is a solution of (12). Furthermore, there are no other solutions since G_1' is negative on the open interval $(80, +\infty)$ such that G_1 is strictly monotonically decreasing on $(80, +\infty)$. Again, one may reduce the amount of almonds in order to improve the taste of the cookies.

As a conclusion, Santa Claus should use 80kg of ground almonds and no salt in order to bake the Christmas cookies according to the recipe of 1892. **Answer 8** is correct.

10 Chaos at the cookie factory

Authors: Miriam Schlöter (TU Berlin), Leon Sering (TU Berlin)
Projects: SPP 1736 – *Algorithms for Big Data* (DFG), MI12 – *Dynamic Models and Algorithms for Equilibria in Traffic Networks* (MATHEON)
Translation: Clara Jansen (MATHEON), Ariane Beier (MATHEON)

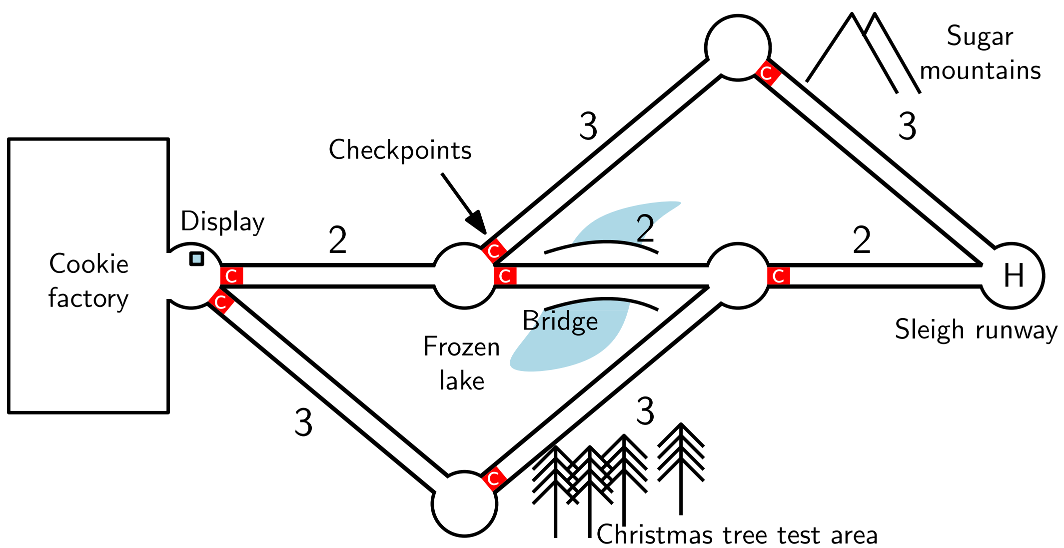
10.1 Challenge

How could this happen? Not only that elf Alvina—who is also a brilliant mathematician—and her stand-in Melvin BOTH have reported sick, but also the reindeers of the *Rudolph's Parcel Service (RPS)* threaten to lay down their work again. Rudolph and Santa could tear down the candles from the Christmas tree. At least the first round of negotiations with the *United Union of Reindeers (UNIRED)* went quite well. Meanwhile, the cookie factory drowns in chaos: thousands of assistant elves, who actually are supposed to transport the cookies from the factory to the sleigh runway, shout all in a tumble, run in circles, and throw crumbs and flour at each other.

"That's enough", Santa shouts and heavily bangs on the table such that Rudolph drops his almond biscuit he just wanted to dip into his cocoa. "You will make sure that the reindeers will deliver the cookies until the end of the day, and I will ensure that as many cookies as possible will be transported to the sleigh runway!"

One telephone call later:

Rudolph: Today at 5 pm the reindeer union meeting will vote on my proposal. If they accept, the last supply will be picked up at 11 pm. Otherwise, if they do not accept my proposal, the reindeers will stop all deliveries at that time. In this case, all cookies arriving at the sleigh starting point until 5 pm will be picked up and delivered by the reindeers. However, cookies arriving afterwards will not be delivered anymore.
Santa: I don't know how Alvina and Melvin got this chaotic crowd of assistant elves under control! Fortunately, I found a plan of the network showing the route from the cookie factory to the sleigh runway. It does not look that complicated, but I still do not understand it completely...



Rudolph: Look Santa, this is actually quite easy! Here, at the left, the assistant elves leave the cookie factory and are supposed to transport the cookies to the sleigh runway. The road network consists of single road sections. There are short sections of length 2, i.e., the assistant elves need 2 hours to cross this section, and there are longer sections of length 3. Since all sections are one-way roads leading from left to right, there are three routes leading from the cookie factory to the reindeer starting point.

There are thousands of assistant elves working in our factory. They are very small compared to the road sections. Therefore, you have to imagine the assistant elves moving through the road network like a liquid flowing through pipes. The elves are measured in elf-units (one elf-unit consists of many elves), and they leave the cookie factory at a rate of two elf-units per hour.

At the beginning of every road section there is a checkpoint (c), where the cookie deliveries are checked for completeness. This control process does not need any extra time, but the capacity of the checkpoints is bounded: at most one elf-unit per hour can be controlled. You can imagine the checkpoints to be a bottleneck of the road network at which the elves may pile up.

If the elves arrive at the checkpoint at a higher rate than the checkpoint is able to control, there is a waiting area next to the road on which the elves line up. The queue works like a fluid flowing through narrows: for example, if there are two elf-units in the queue, the elves at the end of the queue have to wait exactly two hours until they can pass the checkpoint (see Fig. 2).

Now, we need to decide on our strategy as soon as possible such that the assistant elves can start working at 8 am. In order that as many cookies as possible will be delivered—independent of the reindeers' vote at union meeting—the strategy should ensure that as many elf-units as possible are able to arrive at the sleigh runway until 5 pm **and** until 11 pm. That means, that there should not exist a better strategy with the property that more elf-units arrive at the sleigh runway until 5 pm and 11 pm.

Santa: I see! But this is very easy! For delivering the maximum number of cookies—independent of the vote of UNIRED—we should send the assistant elves at a rate of one elf-unit per hour on the shortest path from the cookie factory over the bridge to the sleigh runway.

Knecht Ruprecht, who just was involved in an egg fight with the assistant elves, grins and interferes:

Knecht Ruprecht: Oh, Santa! That is a very bad idea! However, if all elves only take the upper path, passing the sugar mountains, and the lower path, passing the Christmas tree test area, and this at a rate of one elf-unit per hour, then we can—at any time—send more elf-units on the road than possible in your scheme. Thus, my strategy needs to be the better one.

Santa frantically tears wisps out of his beard. Only Rudolph keeps a cool head. After a little while, he has an idea.

Rudolph: None of your ideas is the best, because in both strategies routes re-

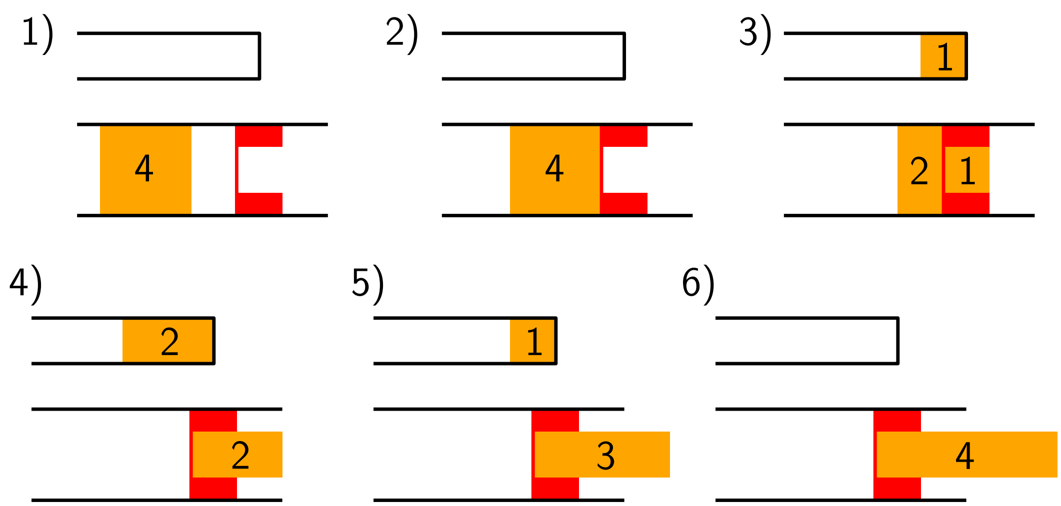


Figure 2: The functionality of a checkpoint when a rate of two elf-units per hour wants to pass the checkpoint. The situation is a continuous process, but it is shown at hourly intervals in the figure above.

main unused! But I know how we should send the elves through the network such that as many cookies as possible can be delivered by the reindeers, independent of the result of the negotiations with UNIRED. Using my optimal strategy, there already will be a complete elf-unit at the sleigh runway at 3 pm.

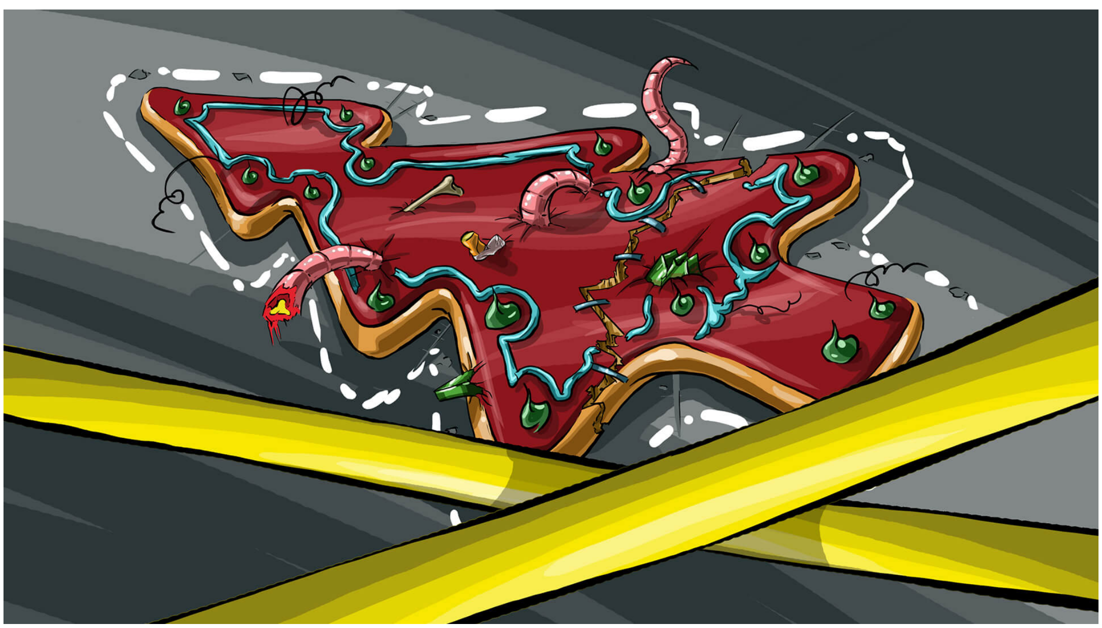
Rudolph explains his scheme to Santa. Santa is relieved: “That is fantastic! With this strategy, it is not important how UNIRED decides; we are not able to deliver more cookies. Christmas is saved!”

With the help of a megaphone, Rudolph and Santa try to communicate their scheme to the elves. But the elves only grumble and do not want to engage in it. For some reason, they think it is unfair, and Rudolph and Santa lack the authority to assert themselves. In the end, Santa looses his patience and resigns: “Okay, then we will try it another way: Every elf is allowed to decide for oneself which path to follow from the factory to the runway. Furthermore, everyone who has delivered his or her cookies to the sleigh runway is allowed to go home.”

To this proposal the elves agree. Of course, everybody wants to call it a day as soon as possible and decides to take the currently fastest route from the factory to the runway. At leaving the cookie factory, every elf knows exactly which route is the fastest, because they all read of a monitor at the exit of the cookie factory how many elf-units are currently in the network and how the queues at the checkpoints will evolve.

At 8 am sharp, the first elves come out of the factory and start their route. Until 10 am, all elves choose to take the direct path from the factory over the bridge to the runway. Thus, Santa doubts that it was a good idea letting the elves decide on their workplan. He is even considering to bribe them to use all the paths evenly, when—at exactly 10 am—the half of the elves are starting to move on the path along the Christmas tree test area, while the other half still decides to queue at the first checkpoint.

Which of the following statements is **false**?



Possible answers:

1. Knecht Ruprecht's strategy is optimal, if the reindeers will pick up the cookies until 11 pm.

2. According to the elves' workplan, the waiting time at the checkpoints amounts to at most two hours.

3. Subject to Rudolph's optimal strategy, there are elves arriving later at the sleigh runway than others, although they had to start their walk earlier.

4. If the reindeers decide to pick up the last cookie supply at 5 pm, then the elves' workplan is as good as Santa's strategy—more precisely, there will be the same amount of cookies arriving at the sleigh runway.

5. According to Santa's strategy, until 11 pm, 5 elf-units less will arrive at the sleigh runway than under Rudolph's optimal strategy.

6. Subject to the elves' workplan, some elves will use the upper path.

7. Santa's strategy is optimal, if the reindeers will start their last delivery at 5 pm.

8. According to the workplan of the elves, the sooner the elves leave the factory the sooner they will be at the sleigh runway.

9. Subject to Rudolph's optimal strategy, until 11 pm, at least 1.5-times as many elves will arrive at the sleigh runway as under the elves' workplan.

10. According to Rudolph's optimal strategy and the elves' workplan, in a particular time interval, the elves move as under Knecht Ruprecht's strategy.

Project relevance:

In the last years, the traffic density in big cities has increased dramatically. Since no end of this development is in sight, traffic planners and engineers constantly have to face new challenges. On the other side, new technologies such as navigation devices with access to more and more extensive and accurate real-time data enable, for example, car drivers to react spontaneously to emerging traffic jams.

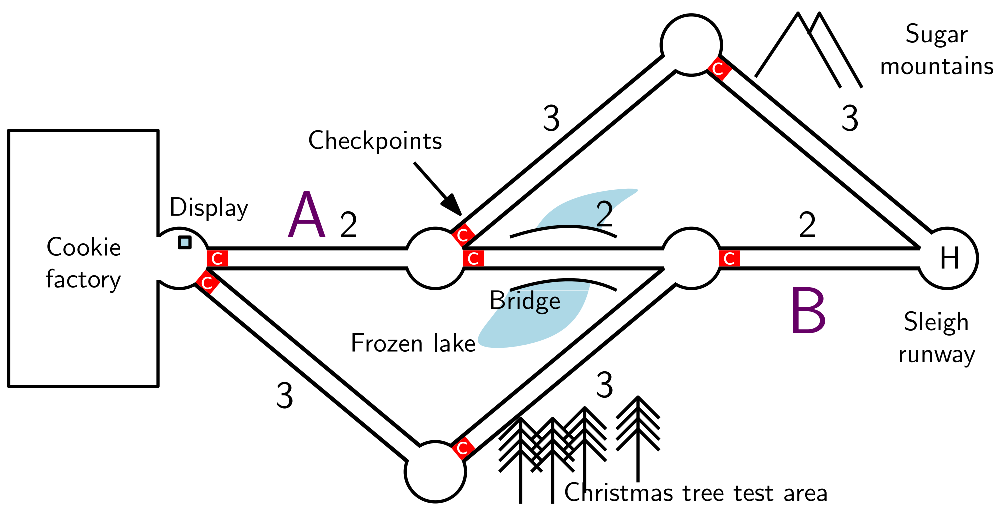
Our research project deals with dynamic network flow models for traffic streams. The traffic is modelled by a continuous flow moving through the road network modelled by a complicated system of pipes. Important new insights are provided by dynamic game theory, the so-called *Nash Flows*: Every particle of the flow is modelled by a single road user who wants to reach his/her destination as soon as possible—just as in the elves' workplan. To this end, it is advisable to avoid traffic jams and especially to have enough information about the behaviour of the other road users (not an unimaginable requirement in times of digitization and a big amount of real-time data). With this model we succeeded to compute a so-called *Nash equilibrium*, that is an equilibrium in which neither of the road users reaches his/her destination quicker with another route than the chosen one.

The dynamic flow models mentioned above are applied in other areas as well. One can also use this model to advance the fundamentals of optimized evacuation plans. Imagine the following scenario: A building, for example a concert hall crowded with people, finds itself confronted with a bomb threat and has to be evacuated as quick as possible. Nobody knows at which time the bomb will explode. Therefore the aim of a good evacuation strategy is to get as many people as possible out of the building, such that at every possible moment the maximal number of people would be saved. The optimal solution is called *Earliest Arrival Flow*. The optimal strategy of Rudolph corresponds to this Earliest Arrival Flow.

10.2 Solution

The correct answer is: 9.

There are three paths on which the elves can move from the cookie factory to the sleigh runway. We denote the upper path along the sugar mountains by P_u. This path has length 8. The straight path P_m in the middle has length 6, and the lower path P_l along the Christmas tree test area is of length 8.



According to the exercise, Santa's and Rudolph's goal is to send the elf-units through the road network such that as many elf-units as possible arrive at the sleigh runway both until 5 pm **and** 11 pm. There are different strategies described in the exercise:

- **Santa's strategy:** As many elf-units as possible are sent along the path P_m.
- **Ruprecht's strategy:** As many elf-units as possible are sent parallel along P_u and P_l.
- **Rudolph's optimal strategy:** The elf-units are sent in such a way that as many of them as possible reach the sleigh runway until 5 pm **and** 11 pm. Furthermore, we know that an elf-unit reaches the sleigh runway already at 3 pm.

In all of these strategies the elves are sent from 8 am and such that they arrive at the sleigh runway until 11 pm. In addition to the three strategies above, there is the **elves' workplan**.

In order to figure out the right answer to the question in the exercise, we need to make some calculations:
We already know how the elves cross the road system according to Santa's and Ruprecht's strategies. However, we lack a description of Rudolph's optimal strategy. To this end, we need to find out how the elves are sent in an optimal way. Similarly, we need to work out the elves' workplan. In particular, we need to know which path the elves will decide to follow in any time interval and how long the waiting times at the checkpoints will be at any time.
For all strategies and the elves' workplan, it is of particular interest how many elf units arrive at the sleigh runway at 5 pm and 11 pm, respectively. For Santa's and Ruprecht's strategies this question may be answered directly:

Santa's strategy:

In Santa's strategy, **as many elf-units as possible** are constantly sent along the path P_m such that they arrive at the sleigh runway until 11 pm. At first, we need to specify what "as many elf-units as possible" actually means: At the beginning of every road section there is a checkpoint, where at most one elf-unit per hour is controlled. Consequently, only one elf-unit per hour can be sent along the path P_m, and "as many elf-units as possible" means in this strategy that a **rate** of one elf-unit per hour is sent. The remaining elves need to wait in front of the first checkpoint.

The length of the path P_m is 6. Hence, elves leaving the cookie factory at 8 am arrive at the sleigh runway 6 hours later, that is at 2 pm. Accordingly, elves leaving at 9 am arrive at 3 pm, and elves leaving at 10:23 am arrive at 4:23 pm etc.

Now, we want to calculate how many elves reach the sleigh runway until 5 pm and 11 pm: Since the elves need 6 hours to cross the path P_m, only elves leaving the cookie factory by no later than 11 am arrive at the sleigh runway until 5 pm. Thus, the elves can be sent for 3 hours (from 8 am to 11 am) at a rate of one elf-unit per hour; in total, 3 elf-units in 3 hours. These 3 elf-units will arrive at the sleigh runway in time until 5 pm.

Analogously, elves leaving the cookie factory by no later than 5 pm arrive at the sleigh runway until 11 pm. In conclusion, elves being sent from 8 am to 5 pm will reach the sleigh runway until 11 pm. During these 9 hours, at most 9 elf-units can be sent.

Summarising, according to Santa's strategy **3 elf-units will arrive until 5 pm** and **9 elf-units will arrive until 11 pm**.

Ruprecht's strategy:

In Ruprecht's strategy, as many elf-units as possible are constantly sent along the paths P_u and P_l. As in the preceeding strategy "as many elf-units as possible" means that the elves are constantly sent with the maximal rate of one elf-unit per hour along the paths P_u and P_l. Both paths have length 8. Accordingly, elves need 8 hours to cross each of the paths. Since the paths P_u and P_l have no mutual crossings or sections, it suffices to calculate how many elf-units reach the sleigh runway until 5 pm and 11 pm, respectively, along one of them. Then, along both paths twice as many elf-units will reach the sleigh runway in the respecting times.

We will regard the path P_u. Since it has length 8, only elves leaving the factory by no later than 9 am will arrive at the sleigh runway until 5 pm. Consequently, the elves can be sent during the one hour between 8 and 9 am to ensure their arrival until 5 pm. Since the elves can only be sent at the rate of one elf-unit per hour, at most 1 elf-unit is able to cross the path P_u until 5 pm. Accordingly, only elves sent until 3 pm will cross P_u until 11 pm. From 8 am to 3 pm, seven hours pass in which elves can be sent on their way. Thus, at most 7 elf-units will reach the sleigh runway until 11 pm.

Summarising, according to Ruprecht's strategy **2 elf-units will arrive at the sleigh runway until 5 pm**, and **14 elf-units until 11 pm**.

Rudolph's optimal strategy:

In order to describe Rudolph's optimal strategy, we need to determine the maximal number of elf-units being able to reach the sleigh runway until 5 pm and 11 pm, respectively.

We start by calculating the maximal number of elves arrriving at sleigh runway until 5 pm. The number M of elf-units arriving at the sleigh runway until 5 pm is maximal if there is **no** strategy in which more than M elf-units will arrive until 5 pm. In the following, we will compute M: one elf-unit will reach the sleigh runway until 5 pm along the path P_u. This elf-unit needs to be sent from 8 am to 9 am along P_u to ensure its arrival until 5 pm.

The paths P_u and P_m share a mutual road section, denoted by A. From 8 am to 9 am, it is possible to sent elves along both ways, P_u and P_m, if one ensures to sent them with an overall rate of one elf-unit per hour onto these two paths. Thus, in every imaginable strategy, at most one elf-unit can be sent along the paths P_u and P_m from 8 am to 9 am. This elf-unit will reach the sleigh runway until 5 pm.

Similarly, the paths P_m and P_l share a mutual road section, denoted by B. As above, only elves sent from 8 am to 9 am along P_l will reach the sleigh runway until 5 pm. Elves starting x minutes past 8 am along P_l will reach the road section B at the same time as elves starting x minutes past 10 am along P_m. However, there is a strategy that sends elves from 8 am to 9 am along P_l and from 10 am to 11 am along P_m, ensuring that the rate of elves being sent from 8 am to 9 am along P_l and the rate of elves being sent from 10 am to 11 am along P_m sums up to at most one elf-unit per hour. In every possible strategy, the elves being sent along P_l from 8 am to 9 am and along P_m from 10 am to 11 am add up to at most one elf-unit. This elf-unit will reach the sleigh runway until 5 pm.

In order to arrive at the sleigh runway in time until 5 pm, elves leaving the cookie factory from 9 am to 10 am have to be sent along P_m. These elves will not collide with any elves using any other path. Thus, from 9 am to 10 am there is another elf-unit that can be sent along P_m and that will reach the sleigh runway until 5 pm.

Consequently, in any possible strategy, there are at most **3 elf-units reaching the sleigh runway until 5 pm**:

- One elf-unit being sent from 8 am to 9 am along P_u and P_m,
- one elf-unit being sent from 9 am tod 10 am along P_m,
- and (in total) one elf-unit being sent from 8 am to 9 am along P_l and from 10 am to 11 am along P_m.

Such a plan is, for example, realised by Santa's strategy.

Similarly, we can compute the maximal number of elf-units arriving at the sleigh runway until 11 pm:
According to Ruprecht's strategy, until 11 pm, at most 7 elf-units are able to reach the sleigh runway along the path P_u. These elves have to start their walk from 8 am to 3 pm.
The paths P_u and P_m share a mutual road section, denoted by A. Again, from 8 am to 3 pm, it is possible that elves can be sent on both paths, P_u and P_m, if one ensures to sent them with an overall rate of one elf-unit per hour. In any imaginable strategy, at most seven elf-units can be sent from 8 am to 3 pm along the road network onto the paths P_u and P_m. In order to maximise the amount of elves reaching the sleigh runway until 11 pm, one needs to send the elves constantly along the paths P_u and P_m at a rate of one elf-unit per hour. These altogether seven elf-units will arrive at the sleigh runway until 11 pm.
Similarly, the paths P_m and P_l share a mutual road section, denoted by B. Again, only elves sent from 8 am to 3 pm are able to reach the sleigh runway in time, that is, until 11 pm. Elves starting at time x along P_l will reach the road section B at the same time as elves starting two hours later along P_m. However, there is a strategy that sends elves from 8 am to 3 pm along P_l and from 10 am to 5 pm along P_m, ensuring that the rate of elves being sent from 8 am to 3 pm along P_l and the rate of elves being sent from 10 am to 5 pm along P_m sums up to at most one elf-unit per hour. In every possible strategy, the elves being sent along P_l from 8 am to 3 pm and along P_m from 10 am to 5 pm add up to at most seven elf-units.
Consequently, there is an optimal strategy that sends (in total) seven elf-units from 8 am to 3 pm along P_m and P_u, and (in total) seven elf-units from 8 am to 3 pm along P_l and from 10 am to 5 pm along P_m. Thus, at most 14 elf-units will reach the sleigh runway in time until 11 pm. This value is also maximal and is, for example, realised by Ruprecht's strategy.

Now, we are able to describe Rudolph's optimal strategy, that is, the strategy in which one elf-unit reaches the sleigh runway until 3 pm, three elf-units reach the sleigh runway until 5 pm, and 14 elf-units reach the sleigh runway until 11 pm.
Elves being sent along P_u and P_l reach the sleigh runway at 5 pm at the earliest. Thus, in order for one elf-unit to arrive at the sleigh runway until 3 pm, one needs to send one elf-unit from 8 am to 9 am along P_m.
In addition, 3 elf-units need to be at the sleigh runway until 5 pm and altogether 14 elf-units until 11 pm. For 3 elf-units to arrive until 5 pm, one could send the

elves along P_m as in Santa's strategy. But then, not enough elves will reach the sleigh runway until 11 pm. Thus, we need to come up with a better strategy.
As long as one sends elves at a rate of one elf-unit per hour along P_m, it is not possible to send along P_u. Anyhow, for elves to cross along P_u until 5 pm, they need to leave the factory until 9 am. But this is not possible since an elf-unit is sent along P_m from 8 am to 9 am. Thus, the 3 elf-units (crossing until 5 pm) need to be sent along P_m and P_l. If we do not want to send all 3 elf-units along P_m, the only remaining solution is to send from 8 am to 10 am (altogether) two elf-units along P_m and one elf-unit from 8 am to 9 am along P_l. These three elf-units will arrive at the sleigh runway until 5 pm. Afterwards, we send from 9 am to 3 pm another 6 elf-units along P_m and from 10 am to 3 pm another 5 elf-units along P_u.
Summarising, Rudolph's optimal strategy is the following:

- From 8 am to 10 am send (in total) 2 elf-units along the path P_m.
- From 8 am to 3 pm send (in total) 7 elf-units along the path P_l.
- From 10 am to 3 pm send (in total) 5 elf-units along the path P_u.

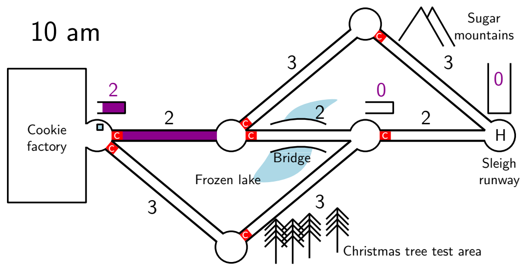
The elves' workplan:

From 8 am the elves leave the cookie factory at a rate of 2 elf-units per hour. During the first two hours, the elves want to take the shortest path (P_m), because they will finish their work—despite the waiting time—after less than 8 hours. (If they chose to walk along P_u or P_l, they would need at least 8 hours.) Thus, from 8 am to 10 am, there will be 2 elf-units per hour arriving at the checkpoint at the beginning of road section A. Since the processing rate of the checkpoint is one elf-unit per hour, the checkpoint's queue will increase by one elf-unit per hour. At 10 am there will be two elf-units waiting in the queue. Succeeding elves need 8 hours to cross the road system along P_m (including the waiting time of two hours) and also 8 hours along the paths P_u or P_l. Afterwards, a rate of one elf-unit per hour will join the queue and take the direct path P_m, whereas a rate of one elf-unit per hour will cross the road section along the lower path P_l. Thus, the queue at the checkpoint towards road section A will remain of length 2.
If the rate of elves taking the straight path P_m was smaller than 1, then the queue (and with it the waiting time) would decrease and elves would have taken the straight path P_m after all. On the other side, if the rate of elves was greater than

1, then the queue and the waiting time would increase and a crossing along P_m would take more than 8 hours, i. e. longer than a crossing along P_l.

At 4 pm, elves crossing along P_m and P_l will simultaneously arrive at the checkpoint at the beginning of road section B. Since a total rate of two elf-units per hour needs to pass this checkpoint, its queue will increase at a rate of one elf-unit per hour. At 6 pm, the length of the queue will be two elf-units, and the waiting time will be two hours. Consequently, it is convenient to additionally use the upper path P_u along the sugar mountains.
Because all elves monitor also future situations at the exit of the cookie factory, the elves leaving the factory at 12 pm know that, if they decide to follow P_m, they will arrive at the checkpoint at the beginning of road section B not until 6 pm.
Accordingly, from 12 pm, all elves will take the paths P_u and P_l each with a rate of one elf-unit per hour—just as in Ruprecht's strategy. Since the waiting times both along P_u at the checkpoint at the beginning of road section A and along P_l at the beginning of the road section B amount to two hours, the elves need 10 hours to finish the crossing along P_u or P_l. Thus, elves leaving the cookie factory at 1 pm will not reach the sleigh runway in time until 11 pm.

As depicted in the following figure, **3 elf-units will arrive at sleigh runway until 5 pm** and **10 elf-units until 11 pm**.


10 am
Cookie factory
2
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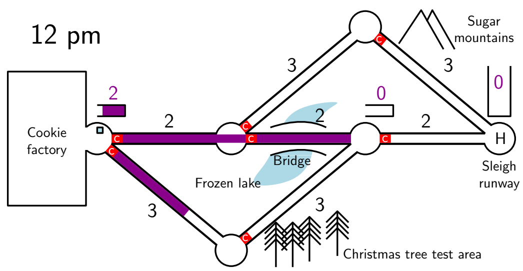
Sleigh runway
Bridge
Frozen lake
Christmas tree test area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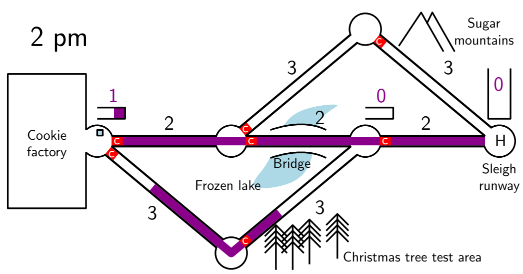
Sleigh runway
Bridge
Frozen lake
Christmas tree test area


2 pm
Cookie factory
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Sugar mountains
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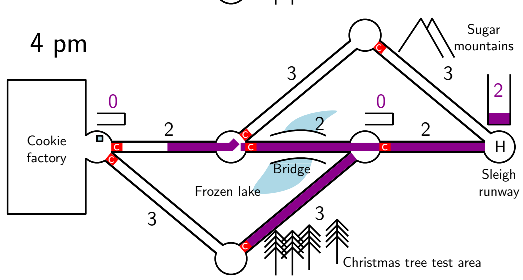
Sleigh runway
Bridge
Frozen lake
Christmas tree test area


4 pm
Cookie factory
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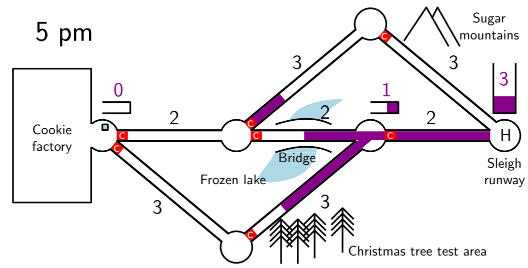
Frozen lake
Christmas tree test area


5 pm
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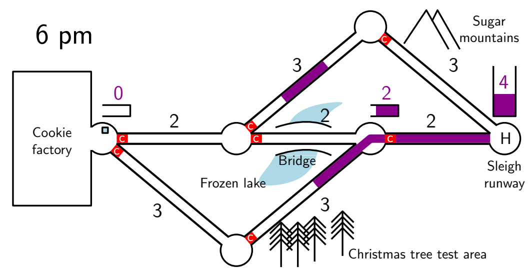
Frozen lake
Christmas tree test area


6 pm
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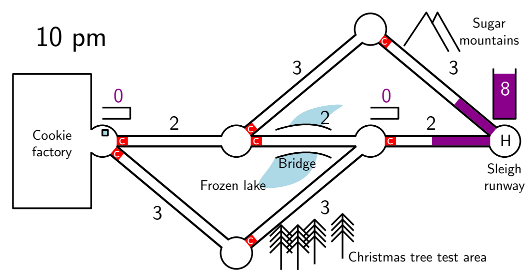
Christmas tree test area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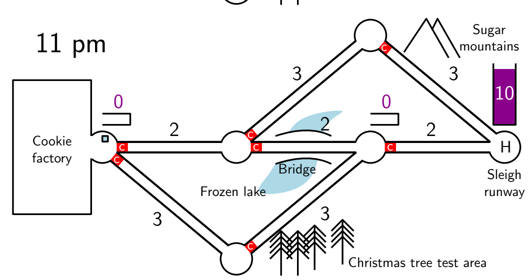
Bridge
Frozen lake
Christmas tree test area


11 pm
Cookie factory
0
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10
Sugar mountains
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Sleigh runway
Bridge
Frozen lake
Christmas tree test area

Summarising, we obtain the following values for all four strategies:

Strategy	5 pm	11 pm
Santa	3	9
Ruprecht	2	14
Optimal (Rudolph)	3	14
Elves	3	10

Finally, we can evaluate the given statements:

1. *Knecht Ruprecht's strategy is optimal, if the reindeers will pick up the cookies until 11 pm.*

   This statement is **true**, because according to Ruprecht's strategy there will be 14 elf-units arriving at the sleigh runway until 11 pm and this is the optimal value.

2. *According to the elves' workplan, the waiting time at the checkpoints amounts to at most two hours.*

   This statement is **true**, as one can learn from the description of the elves' workplan and the according figure.

3. *Subject to Rudolph's optimal strategy, there are elves arriving later at the sleigh runway than others, although they had to start their walk earlier.*

   This statement is **true**: elves leaving the cookie factory at 8 am along P_l will reach the sleigh runway not until 5 pm, whereas elves leaving the factory at 9 am along P_m will already arrive at 4 pm.

4. *If the reindeers decide to pick up the last cookie supply at 5 pm, then the elves' workplan is as good as Santa's strategy—more precisely, there will be the same amount of cookies arriving at the sleigh runway.*

   This statement is also **true** (see the table above).

5. *According to Santa's strategy, until 11 pm, 5 elf-units less will arrive at the sleigh runway than under Rudolph's optimal strategy.*

   This statement is also **true** (see the table above).

6. *Subject to the elves' workplan, some elves will use the upper path.*

   This statement is **true**, as one can learn from the description of the elves' workplan above.

7. *Santa's strategy is optimal, if the reindeers will start their last delivery at 5 pm.*

   This statement is **true**, because according to Santa's strategy there will be 3 elf-units arriving at the sleigh runway until 5 pm.

8. *According to the workplan of the elves, the sooner the elves leave the factory the sooner they will be at the sleigh runway.*

   This statement is **true**, as one can learn from the description of the elves' workplan above. In the elves' workplan, all elves decide to take the fastest crossing. If there were elves leaving the factory earlier and reaching the sleigh runway later, they could have taken a faster path, namely the path the other (faster) elves took.

9. *Subject to Rudolph's optimal strategy, until 11 pm, at least 1.5-times as many elves will arrive at the sleigh runway as under the elves' workplan.*

   This statement is **false**. According to the elves' workplan, 10 elf-units will reach the sleigh runway until 11 pm; subject to Rudolph's optimal strategy, 14 elf-units will arrive until 11 pm, that are only 1.4-times as many elves.

10. *According to Rudolph's optimal strategy and the elves' workplan, in a particular time interval, the elves move as under Knecht Ruprecht's strategy.*
    from 10 am to 3 pm move as under Ruprecht's strategy. Subject to the elves' workplan, elves leaving the cookie factory from 12 pm to 1 pm move as under Ruprecht's strategy.

11 Santa's Metro

Author: Niels Lindner
Project: MI7 – *Routing Structures and Periodic Timetabling*
Translation: Ariane Beier (MATHEON)

11.1 Challenge

Since its construction in 1899, the North Pole's metro network (NoPe) is Santa's pride and joy. That explains his extreme fondness of the huge network plan attached to his office walls, (see Fig. 4).

Whenever Santa's lad Nick comes to see him at his office, the two discuss vividly the beauty of network plans. To surprise Nick at his next visit, Santa plans to design a particular beautiful plan of the NoPe network. Both friends agree on the fact that a network plan needs to be *octilinear*, that is, the network plan needs to satisfy both of the following conditions:

1. All stations have to be positioned such that the slope of the edges (i.e. the straight lines between two stations) is an integer multiple of 45°—seen from the horizontal.

2. There exist no crossing edges.

Furthermore, there exist the following features that describe a network plan:

- *Skew edges*: An edge is called skew, if its angle of slope is 45°, 135°, 225° or 315°.

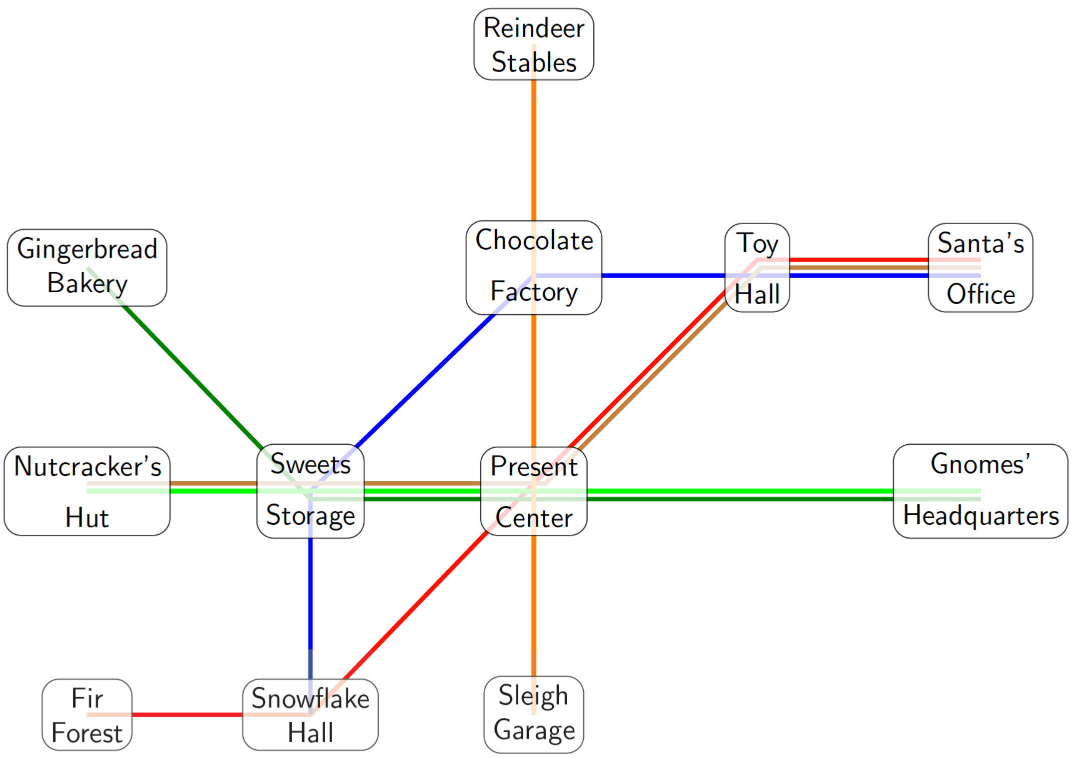


Figure 4: The NoPe network plan in Santa's office. The North Pole's metro network consists of 12 stations and 6 metro lines, which are composed of individual edges. There are possibly several parallel lines on one edge: for example, there are 3 lines on the edge between the stations Sweets Storage and Present Center.

- *Curvature number of a line*: The curvature number of a line is the number of stations that are not passed through at an angle of 180°—excluding the start and end point of the line.
- *Crossing of two lines*: Two lines l and m cross at a station S if the paths of l and m cross at S and l and m run parallel on at least one edge adjacent to S.

Note that the NoPe network plan in Santa's office is octilinear. Furthermore, there are 4 skew edges. The curvature number of the red line is 2; the *total curvature number* (i.e. the sum of the curvature numbers of all lines) is 7. For example, the green lines cross at station Sweets Storage, and the blue and red line cross at station Toy Hall. (In the figures, the crossing points are hidden beneath the stations.) In contrast, according to the above definition, the blue and orange line do not cross at the station Chocolate factory, because these lines neither run parallel before nor after the station.

Santa has asked all gnomes to contemplate about possible layouts of the NoPe network. While going through their statements and thinking about the layout of the most beautiful network plan, Santa has the suspicion that there might be a mistake.

Which of the following statements is false?

Possible answers:

1. If all stations remain exactly at the positions given in the NoPe network plan in Santa's office, then it is not possible to draw a layout of the NoPe network without crossing lines.
2. There exists an octilinear layout of the NoPe network, in which no two lines cross.

3. In every octilinear layout of the NoPe network, the sum of the curvature numbers of the two green lines is greater than 0.
4. There is no octilinear layout of the NoPe network, in which the blue line's curvature number is 0.
5. There exists an octilinear layout of the NoPe network, such that the blue line's curvature number is less than the brown line's curvature number.
6. In every possible octilinear layout of the NoPe network, the total curvature number (i.e. the sum of the curvature numbers of all lines) is at least 4.
7. There is an octilinear layout of the NoPe network, in which the 12 stations are positioned in a grid of size 4×3.
8. There exists an octilinear layout of the NoPe network with only two skew edges.
9. In every possible octilinear layout of the NoPe network, there are at least 3 lines running on skew edges.
10. Suppose the highly demanded metro line connecting the stations Sleigh Garage and Gingerbread Bakery via Sweets Storage was installed. Then, it would not be possible to draw an octilinear layout of the enhanced network.

Project relevance:

The project MI7 *Routing Structures and Periodic Timetabling* deals with the optimization of timetables, aiming to reduce the waiting time for passengers while also considering the choice of travel route in the process of building timetables. The visualization of transport networks gives rise to a number of hard problems, such as finding octilinear layouts and minimizing curvature and crossings.

11.2 Solution

The correct answer is: 9.

1. The dark green and the brown line will always cross, because the dark green line runs below the brown line on the edge Present Center – Sweets Storage and above the brown line to the left of the station Sweets Storage.

2. An octilinear layout without crossings is designed, for example, by

   (a) interchanging the brown and red line in-between the stations Present Center and Santa's Office,

   (b) placing the blue line above all other lines in-between Toy Hall and Santa's Office,

   (c) interchanging the stations Gingerbread Bakery and Nutcracker's Hut.

3. A positive curvature number always occurs, since both green lines run on the same edge to the right of the station Sweets Storage, whereas they run on two different edges on the left-hand side of this station.

4. The station Present Center is directly connected to the stations Snowflake Hall, Sweets Storage, Chocolate Factory and Toy Hall. Assuming the blue line was straight, then these four stations would lie on this straight line. In an octilinear layout, there would be only three edges emerging from the blue line connecting the four stations with the station Present Center. But this is impossible. Therefore, the blue line cannot run on one straight line and it must have a positive curvature number.

5. An octilinear layout, in which the blue line's curvature number is 1 and the brown one's is 2, can be obtained by translating the stations Chocolate Factory, Toy Hall, Santa's Office and Reindeer Stables to the left by the same offset, such that the blue line turns right at the Chocolate Factory at an angle of 90°.

6. The following turnouts each generate a curvature number of 1:

   - The turnout of the dark and light green line to the left of the station Sweet Storage.
   - The turnout of the brown and light green line to the right of the station Present Center.

- The turnout of the red and blue line to the left of the station Toy Hall.

In addition, the blue line cannot be straight in-between Toy Hall and Sweets Storage (see the proof of the 4th statement). Accordingly, the curvature numbers of all lines add up to at least 4. See Fig. 5 for an octilinear plan of the NoPe network with total curvature number 4.

7. To obtain an octilinear layout of the NoPe network in a grid of size 4×3, it suffices to make the following adjustments to the original plan in Santa's office (Fig. 4):

   - Place the station Reindeer Stables between Gingerbread Bakery and Chocolate Factory.
   - Place Santa's Office beneath Toy Hall, to the right of Present Center.
   - Place Gnomes' Headquarters beneath Santa's Office, to the right of Sleigh Garage.

8. The following triangles of stations in the NoPe network plan each generate a skew edge:

   - Snowflake Hall – Present Center – Sweets Storage,
   - Sweets Storage – Present Center – Chocolate Factory,
   - Present Center – Toy Hall – Chocolate Factory.

   Note that the first two triangles share a common non-skew edge. The total number of skew edges in the plan may be reduced by "straightening" the edges Snowflake Hall – Present Center and Chocolate Factory – Sweets Storage while "skewing" the edge Sweets Storage – Present Center. See Fig. 6 for an octilinear layout of the NoPe network with only 2 skew edges.

9. The statement is false:

   It is in fact possible to construct an octilinear layout, where only two lines run on skew edges. In Fig. 7 these are the blue and the orange line.

10. If the requested line was installed, the edge Sweets Storage – Present Center would be the edge of three triangles with third points given by the stations Chocolate Factory, Snowflake Hall and Sleigh Garage, respectively. Because of the octilinearity, these triangles would be isosceles and rectangular, the

edge Sweets Storage – Present Center being either a base or a leg. Therefore, there are only 6 triangles originating from this edge (see Fig. 8). Using any three of these triangles, there are two edges of different triangles that intersect, making it impossible to construct an octilinear layout (without crossing edges).

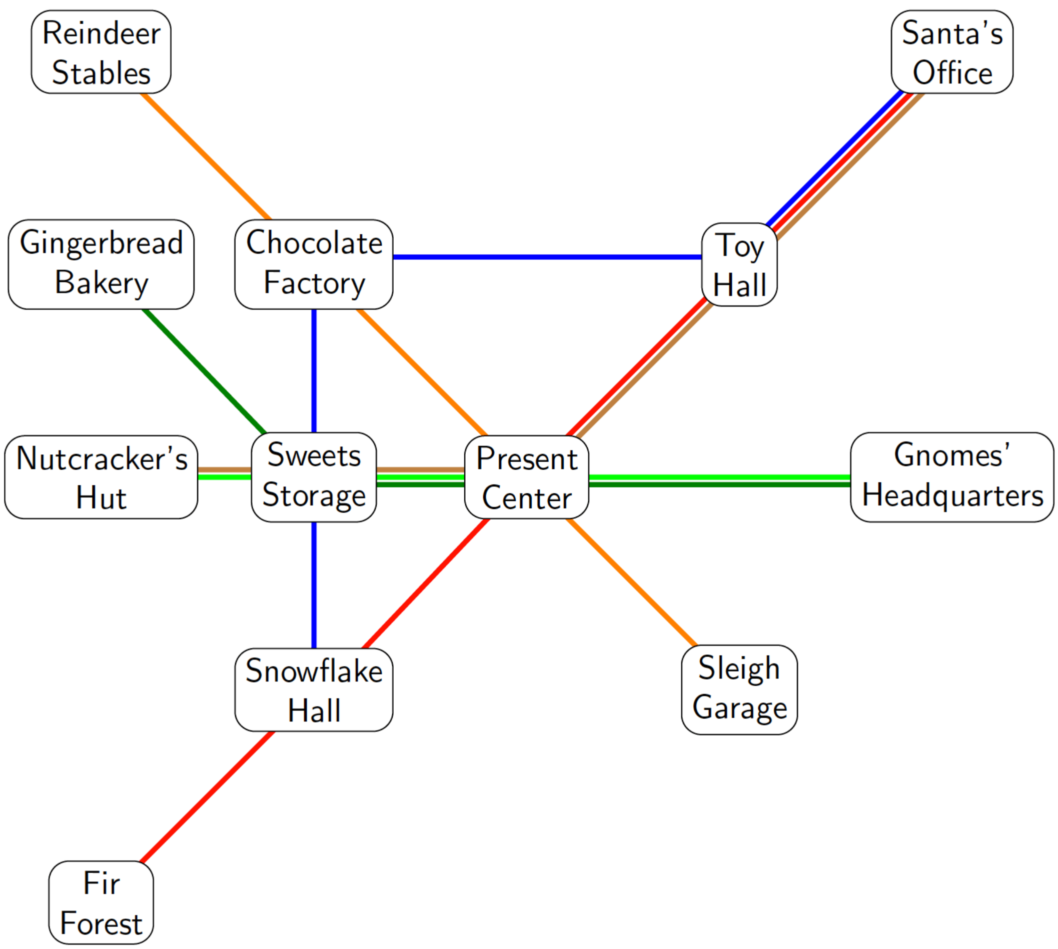


Figure 5: Ad statement 6: octilinear layout with minimum total curvature.

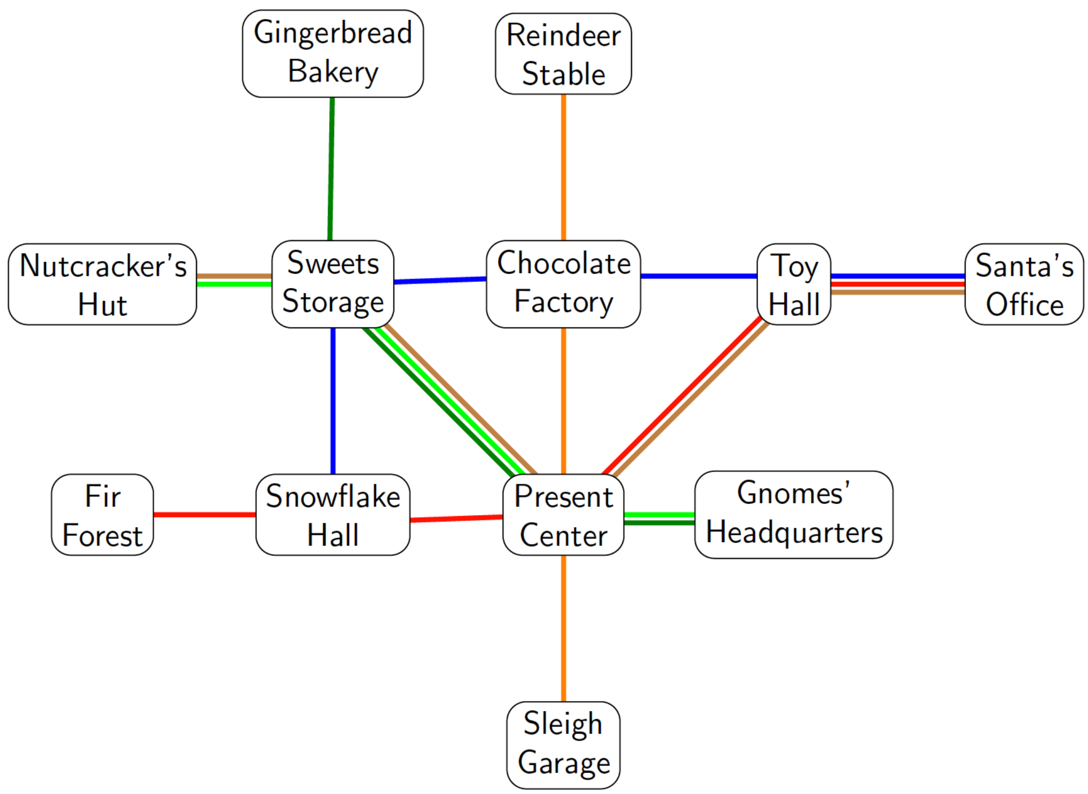


Figure 6: Ad statement 8: octilinear layout of the NoPe network with a minimum number of skew edges.

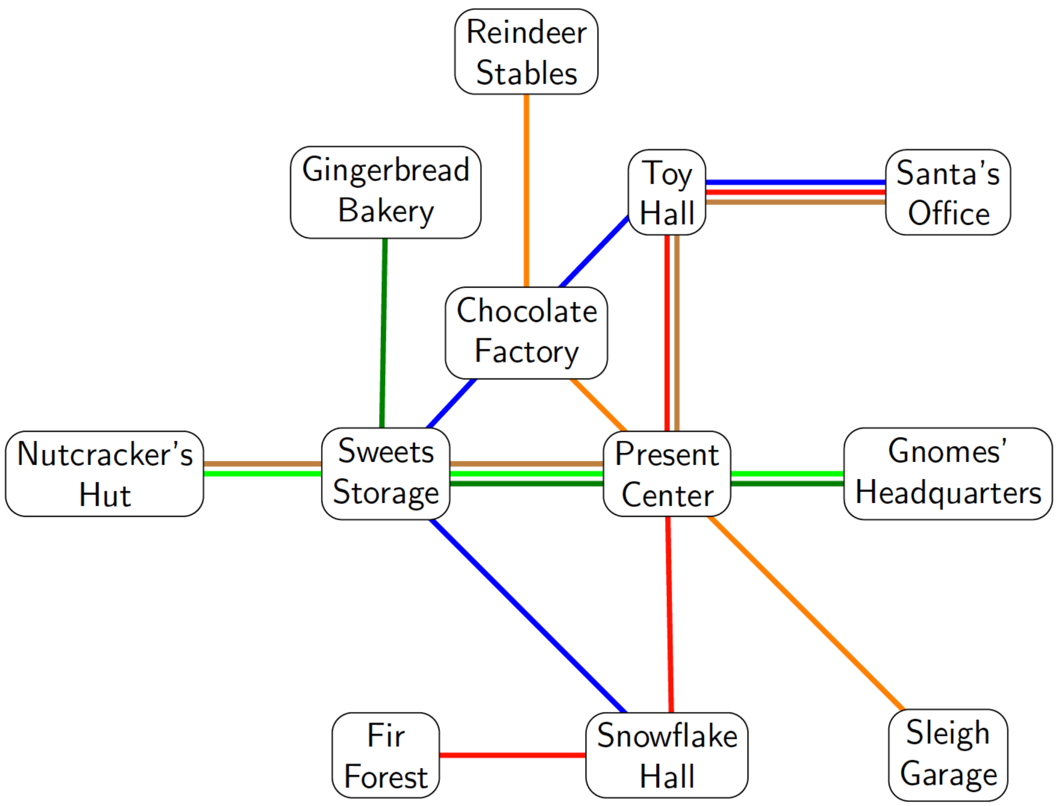


Figure 7: Ad statement 9: octilinear layout with only two lines running on skew edges.

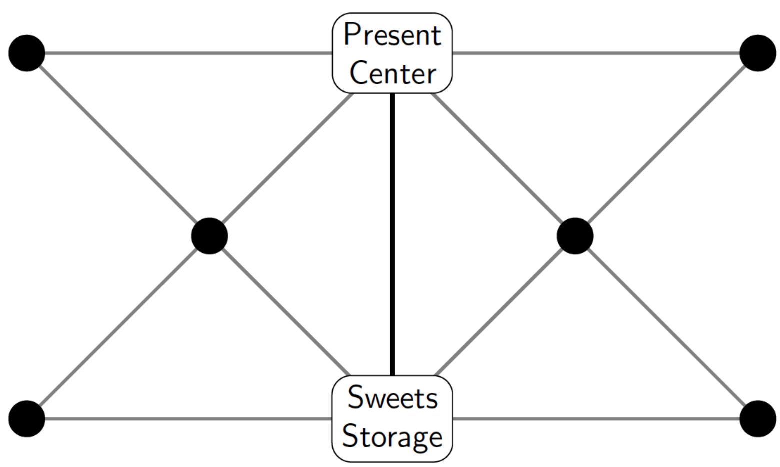


Figure 8: Ad statement 10: the 6 triangles originating from the edge Sweets Storage – Present Center.

12 Target

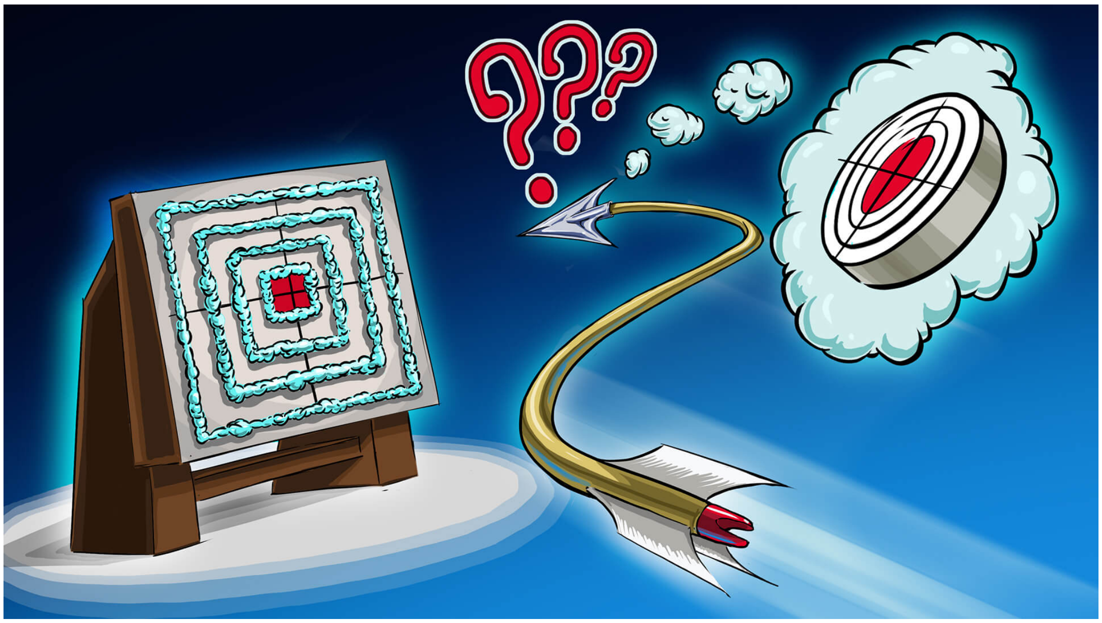
Authors: Hennie ter Morsche (TU Eindhoven), Gerhard Woeginger (TU Eindhoven), Falk Ebert (HU Berlin)

12.1 Challenge

Knecht Ruprecht took part in the IronGnome 2017—probably, the hardest competition in sports history. In the fourth part of the competition, bow and arrow had to be used to hit a far-away square target. Ruprecht remembers: *"I'm not a particularly good bowgnome. My arrows always hit the target, but my accuracy was so weak that all the points on the target were hit with equal probability."*

Hitting the target always yielded one point. Then, the distances of the point of impact to the four edges and to the center of the target were measured. If the point of impact was closer to the center of the target than to the nearest edge, an additional point was awarded.

Question: With what probability did Ruprecht score two points in his first shot (rounded to two decimals)?



Possible answers:

1. The probability is ≈ 0.16
2. The probability is ≈ 0.17
3. The probability is ≈ 0.18
4. The probability is ≈ 0.19
5. The probability is ≈ 0.20
6. The probability is ≈ 0.21
7. The probability is ≈ 0.22
8. The probability is ≈ 0.23
9. The probability is ≈ 0.24
10. The probability is ≈ 0.25

12.2 Solution

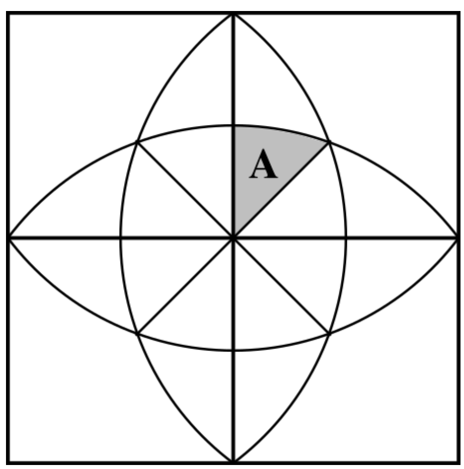
The correct answer is: 7.

Solution (using integral calculus)

We embed the target in the square with $(\pm 1, \pm 1)$ corner points in the plane. A point (x, y) in the square has a distance of $\sqrt{x^2 + y^2}$ to the center of the square and the distances $1 - x$, $1 - y$, $1 + x$, $1 + y$ to the right, top, left, and bottom edges respectively. The point is therefore closer to the center than to the upper edge when $\sqrt{x^2 + y^2} < 1 - y$. This is equivalent to

$$y \;<\; \frac{1}{2}(1 - x^2).$$

This inequality describes the points below the parabola $y = \frac{1}{2}(1-x^2)$. The set M of points that are located closer to the center than to the nearest edge is therefore the intersection of four symmetrical parabolic areas. Or a little differently: The set M consists of eight symmetric copies of the area A in the following figure, where A is described by $y \leq \frac{1}{2}(1 - x^2)$, $x \geq 0$ and $x \leq y$.



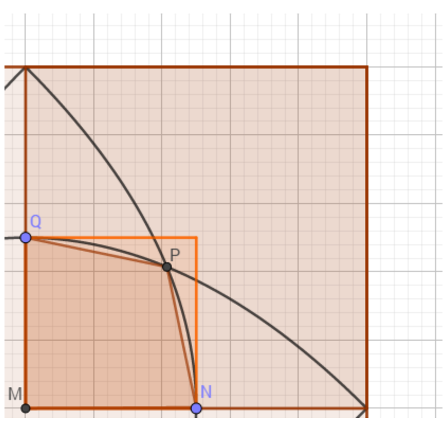
It is easy to calculate that the two limiting curves $y = \frac{1}{2}(1 - x^2)$ and $y = x$ intersect at the point $(\sqrt{2} - 1, \sqrt{2} - 1)$. Therefore, the area of A equals

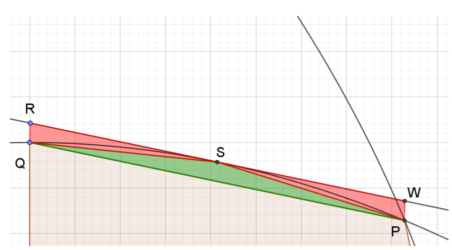
$$\int_0^{\sqrt{2}-1} \frac{1}{2}(1 - x^2) - x \,\mathrm{d}x \;=\; \frac{x}{2} - \frac{x^3}{6} - \frac{x^2}{2}\Big|_0^{\sqrt{2}-1} \;=\; \frac{2}{3}\sqrt{2} - \frac{5}{6}$$

The probability that Ruprecht scores two points for his shot is equal to the quotient of the area of M and the area of the square target. The area of M is eight times the area of A, and the area of the target is 4. The probability in question is therefore $8A/4 = (4\sqrt{2} - 5)/3 \approx 0.22$, and only **answer 7** is correct.

Solution (without integral calculus)

For those not familiar with integral calculus, the solution can also be determined by elimination. Consider the upper right quadrant of the target. The probability of hitting the part of M in this quadrant, let's call it $M_{\frac{1}{4}}$, is the same as the probability for hitting the inner part of the whole target. And as the area of the quadrant is 1, the probability we are looking for is the area of $M_{\frac{1}{4}}$.
$M_{\frac{1}{4}}$ is bounded by two squares, the smaller one with side length $\sqrt{2} - 1$ and the larger one with side length $\frac{1}{2}$. Hence, $(\sqrt{2} - 1)^2 < area(M_{\frac{1}{4}}) < \frac{1}{4}$. This already rules out **1, 2** and **10**. If we look at the quadrilateral $\square(MNPQ)$ for a better approximation, we obtain $area(\square(MNPQ)) = \frac{1}{2}(\sqrt{2} - 1)$ as a lower bound for $M_{\frac{1}{4}}$. This rules out **3, 4** and **5**.
We need to be even more precise. As a next step, we add triangles to the sides $\overline{NP}$ and $\overline{PQ}$ as in the following image.





The tip S will have its x-coordinate situated between P and Q at $x = \frac{1}{2}(\sqrt{2} - 1)$ and y-coordinate $\frac{1}{2}(1 - x^2) = \frac{1}{8} + \frac{\sqrt{2}}{4}$ and is located $\frac{3}{8} - \frac{\sqrt{2}}{4}$ above the midpoint of $\overline{PQ}$. The triangle ΔQPS has an area of $area(\Delta QPS) = \frac{1}{2}\left((\sqrt{2} - 1) \cdot (\frac{3}{8} - \frac{\sqrt{2}}{4})\right)$.
We need to add $area(\Delta QPS)$ twice to $area(\square(MNPQ))$ as there is another triangle attached to $\overline{NP}$. However, we also see, that adding the area of the

parallelogram $\square PWRQ$ which is twice $area(\Delta QPS)$ overestimates the area below the parabola. Hence, we obtain the following lower and upper bounds:

$$\begin{aligned} area(\square(MNPQ)) + 2area(\Delta QPS) &< area(M_{\frac{1}{4}}) \\ &< area(\square(MNPQ)) + 4area(\Delta QPS) \end{aligned}$$

which translates to

$$0.21599\ldots < area(M_{\frac{1}{4}}) < 0.22487...$$

and only leaves **7** as the correct answer.

13 The best card trick ever?

Author: Adrian Ruf (University of Oslo)
Translation: Clara Jansen (MATHEON), Ariane Beier (MATHEON)

13.1 Challenge

Gnome magician Michael is quite excited and claims: “Santa! I have developed an incredible card trick—the best card trick ever!”
“Keep calm, my dear friend!” Santa replies. “Let me see!”
Tensely, Michael begins to shuffle a standard 52-card deck (4 suits: $\diamondsuit$, $\heartsuit$, $\spadesuit$, $\clubsuit$; each suit includes 13 ranks: ace, king, queen, jack, and the numbers 2 to 10). Then, he asks Santa to pick 5 arbitrary cards.
“But do not show these cards to me, rather give them to my assistant gnome Kleber.”
After Santa did as told, Michael goes on: “Now, Kleber will give me—one after another—four of your five chosen cards, and he will return the fifth card to you.”—‘I see! There is the $\spadesuit$7, then the Queen of $\heartsuit$, then the $\diamondsuit$3, and finally the $\clubsuit$8. Now, there is only one card in your hand—a card only you and Kleber know. And this card, dear Santa, is the King of $\spadesuit$.”
Santa is stunned, because indeed he holds the King of $\spadesuit$ in his hand. He gives thanks to the gnome magician for this impressive presentation and congratulates him on the invention of such a terrific card trick.
Of course, Santa wants to know, how Michael managed to detect the card in his hand. For days, he agonizes over the method behind the card trick. Santa is pretty sure that the cards were not marked, that Michael did not put up cameras,

and that the whole trick was played absolutely fair.

Santa contemplates: Assistant gnome Kleber handed four cards to Michael. Thus, there were 48 cards left—only one of them being Santa's card. There was no secret information given to Michael, but the order in which the cards were handed to him by Kleber may have played a crucial role. Anyway, there are only $4! = 24$ possibilities to arrange 4 different cards—not enough to unambiguously encode one of the $52 - 4 = 48$ remaining cards of the deck.

Several weeks pass and the production of Christmas gifts suffer from Santa's distraction, when finally he has a dazzling idea... He calls the magician to his office and says: "Michael, finally I know how your card trick works!"

"Well done! But I expected no less of a man who has dealt with tricky Christmas problems for fourteen years now."

Santa continues: "Kleber has encoded the information on the remaining card in my hand only through the order of the four cards he gave to you, right? Other secret signs have not been exchanged between you and your assistant, have they? Otherwise, I would have noticed." Michael nods.

"The crucial point, however, is that the remaining fifth card carries some additional information. Since Kleber was allowed to choose this card, he has been able to simplify your quest. For instance, among each five cards of a standard 52-card deck, there are at least two having the same suit. And from these two cards he can choose one." Again, Michael nods approvingly.

"Well, my dear friend! Anyhow, this is not the best card trick ever. I claim that one can perform this card trick with even more than 52 mutually distinct cards from which the opponent has to pick five. Indeed, this trick would be better than yours, and I would not even need the four suits—as long as the cards are mutually distinguishable." The magician's jaw drops, and he does not know how to respond.

Are you able to help Michael and figure out which of the following statements concerning Santa's enhanced card trick is true?

Possible answers:

1. Santa is wrong, because this tricks works with at most $n = 52$ cards.
2. The trick can be managed with at most $n = 54$ mutually distinct cards.
3. The trick can be managed with at most $n = 60$ mutually distinct cards.
4. The trick can be managed with at most $n = 120$ mutually distinct cards.
5. The trick can be managed with at most $n = 124$ mutually distinct cards.
6. The trick can be managed with at most $n = 125$ mutually distinct cards.
7. The trick can be managed with at most $n = 255$ mutually distinct cards.
8. The trick can be managed with at most $n = 256$ mutually distinct cards.
9. The trick can be managed with at most $n = 257$ mutually distinct cards.
10. The trick can be managed with arbitrarily many mutually distinct cards.

13.2 Solution

The correct answer is: 5.

For the curious among you, here is how the trick of the gnome magician works. In fact, there are only $4! = 24$ possibilities to order four cards, but Santa missed the fact that the assistant gnome makes one more decision, namely the decision which of the five cards Santa keeps. And this is the trick. Since Santa picks 5 cards, but there only are four suits ($\diamondsuit$,$\heartsuit$,$\spadesuit$ and $\clubsuit$), he definitely picks two cards of the same suit. The assistant gnome, for example, can give one of these cards to the gnome magician and thereby inform him about the suit. (In the example given in the challenge, this pair was the $\spadesuit$ 7 and the King of $\spadesuit$. Accordingly, the assistant gnome gave the $\spadesuit$ 7 to magician Michael.)
So when Michael gets the first card, he already knows the suit and only has to decode the value of the card in the following. Overall there are 13 card-values (ace, 2, ..., 10, jack, queen and king). But since the magician already has seen one card of this suit, only twelve possible values are left.
Through manipulation of the order in which the assistant gives the cards to the magician, he can encode $3! = 6$ messages, not enough to collect one out of twelve cards. So there is more cleverness necessary.
The remaining choice the assistant makes is which of the cards of the same suit he gives to magician and which of them he gives Santa. If one interprets the card-values as numbers from 1 to 13, then it is always possible to add to one of the two card(-value)s a number between 1 and 6 such that the result modulo 13 is the card-value of the other card. For example, if one has the card-values ace (corresponding to 1) and king (corresponding to 13), then one can add 1 to the number 13 and gets $14 \mod 13 = 1$, the value of the other card. (Note that it will not work the other way around.)
Thus, the assistant first looks for a pair with the same suit among Santa's five cards. Then he chooses the card of the two to whose card-value one can add a number between 1 and 6 such that one gets (modulo 13) the card-value of the other card. He then gives this card to the magician. Afterwards, he only has to encode the corresponding number between 1 and 6 through the order of the other three cards to make the trick work.
For this last step, there are many possibilities. For example, the two gnomes can agree on a reference order of all 52 cards (for example, sorted by card-value ace, 2, 3, ..., 10, jack, queen, king, and these blocks of 13 cards each then sorted by

suit, e.g. ♣♢♡♠), such that there always is a smallest, a middle and a biggest card among each 3 cards. Then, for example, one can agree that if the smallest of the three cards is shown as the first/second/third of the three cards, this means that the wanted number is in one of the sets {1,2},{3,4},{5,6}. If the middle card is shown before the biggest card, this can encode the first number of the corresponding set. And if the middle card is shown after the biggest, this can mean the corresponding second number.
In the example from the challenge, this would look like the following: Santa picked the cards ♠ 7, Queen of ♡, ♣ 8, ♢ 3 and King of ♠. The assistant first looks for the pair of the same suit, here ♠ 7 and King of ♠. If one adds a 6 to the card-value of ♠ 7, one gets 13, the card-value of the king. Then he has to encode the number 6. Of the left 3 cards the ♣ 8 is the smallest, ♢ 3 the middle and Queen of ♡ the biggest card with respect to the reference order above. The assistant hands to the magician the smallest card at last and thereby shows that the wanted number is one of the set {5,6}. The middle card is handed to the magician after the biggest, so he can decode that the wanted number is the second of the set, namely the 6.

Now to the improved trick of Santa:

In the original trick, the assistant encodes a message for magician Michael through an ordered set of four cards. There are $\frac{52!}{48!}$ of these ordered sets. Since Michael sees four cards and identifies the fifth, the information which has to be encoded is an unordered set of five cards. There are $\binom{52}{5} = \frac{52!}{5!47!}$ of these unordered sets.
So it is very well possible to improve the trick because the quantity of messages is

$$\frac{\frac{52!}{48!}}{\frac{52!}{5!47!}} = \frac{120}{48} = 2{,}5$$

times bigger than the quantity of situations to be encoded.
This proportion cannot be better (smaller) than 1 if the trick shall work. So now, we are looking for n such that

$$\frac{\frac{n!}{(n-4)!}}{\frac{n!}{5!(n-5)!}} = \frac{5!(n-5)!}{(n-4)!} = \frac{120}{n-4} \overset{!}{=} 1.$$

Consequently, $n = 124$, which suggests answer possibility number 5. But is there an algorithm for 124 cards? Yes, there is. Here is one possibility:

First, we number the cards from 0 to 123 (for example, in the way described above). For simplicity, we even can suppose that the cards are numbered from 0 to 123. Santa then picks five cards $c_0 < c_1 < c_2 < c_3 < c_4$. The assistant then leaves the card c_i with Santa, where $i = c_0 + c_1 + c_2 + c_3 + c_4 \mod 5$.
Afterwards, the assistant renumbers the cards he is not going to show to Michael (that are the cards still being in the deck and the card staying with Santa) by crossing the four cards he is going to show to the gnome magician out of the original numbering. The new numbering then ranges from 0 to 119.
We denote the sum of the four cards being shown to the magician by s. Then the value of the wanted card is congruent to $-s \mod 5$. If one divides the remaining 120 cards into intervals of the length 5,

$$(0{,}1{,}2{,}3{,}4), (5{,}6{,}7{,}8{,}9), \ldots, (115{,}116{,}117{,}118{,}119),$$

one determines that in every interval there is only one card which could be the wanted card (see figure 9). So there are only 24 possibilities left.

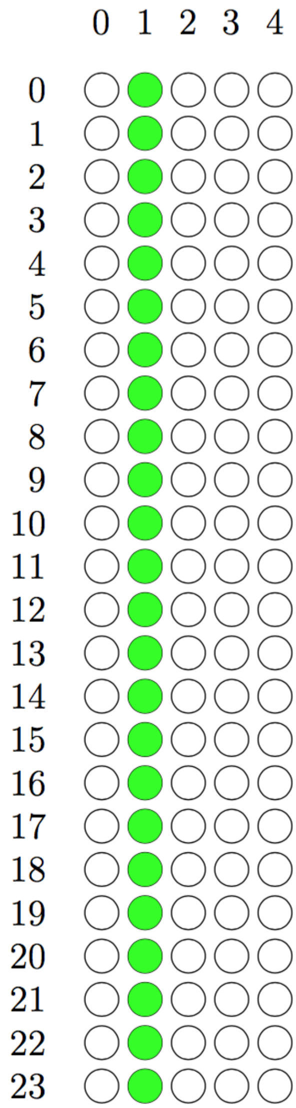


Figure 9: Here $s \mod 5 = 4$ (or $-s \mod 5 = 1$, respectively) and the 24 possibilities are marked green

For the order in which the four cards are shown, there also are exactly $4! = 24$ possibilities. So in preparation of the trick magician Michael and his assistant only have to agree on a bijection between the permutations of the four cards and the intervals. For this purpose, every table assigning the permutations to the intervals suffices. But also the following formula is sufficient:
If the p-th interval (p between 0 and 23) is to be encoded, then the assistant gnome deconstructs p as $p = d_1 1! + d_2 2! + d_3 3!$.
The number d_i, $i = 0, 1, 2, 3$ stands for the number of cards smaller than the $4-i$-biggest standing on the right side of the $(4-i)$th-biggest number.
For example, if the magician gnome gets the four cards

$$37, 7, 94 \text{ and } 61$$

in this order, he can compute d_0 by counting how many cards are smaller than the 4th-biggest (i.e., the smallest, in this case the 7) standing right of it. This is, of course, zero because it is already the smallest card. To compute d_1, he counts the number of cards that are smaller than the third-biggest, the 37, and lying to the right to it. This is only the 7, so $d_1 = 1$. On the right side of the second biggest card, the 61, there is no smaller one, so $d_2 = 0$. To finally compute d_3, the magician counts how many numbers smaller than the biggest number, the 94, lying right to it. This is only the 61, hence $d_3 = 1$. With this the magician computes $p = 1 \cdot 1! + 0 \cdot 2! + 1 \cdot 3! = 7$. If he sums up the four numbers, he gets $s = 199$ and $-s \mod 5 = 1$. Consequently, the wanted card is the first card (not the zeroth, but the first) in the seventh interval ((35,36,37,38,39)), hence 36. Here it is to keep in mind that this is the 36-th card according to the new numbering. In the numbering the four shown cards, especially the 7 and the 37, were crossed out. Therefore, the wanted card has the number 38 according to the original numbering.

References

[1] Michael Kleber. The best card trick.
http://www.apprendre-en-ligne.net/crypto/magie/card.pdf, 2002.

14 Concert

Author: Onno Boxma (TU Eindhoven)

14.1 Challenge

This evening 26 gnomes are visiting the Christmas concert. They have tickets for the 26 seats in row 1, which are numbered 1,2,...,26. Every gnome has the number of his seat printed on his entrance ticket. The gnomes enter the concert hall in alphabetical order: first Atto, then Bilbo, then Chico, then Dondo and so on, and at the very end Ziggo.

Atto has lost his ticket and completely randomly (more precisely, uniformly distributed) takes a seat on the first row. Bilbo, Chico and Dondo also have lost their ticket and also take a seat completely randomly among the empty seats in the first row. The next 22 gnomes still have their tickets and all behave as follows: they first go to the seat for which they have a ticket. If that seat is still free, then they take it; otherwise they randomly take one of the empty seats in the first row.

Let p denote the probability that Ziggo gets the seat for which he has a ticket. Which of the following statements about p is correct?

Possible answers:

1. $p \leq 0.002$
2. $0.002 < p \leq 0.004$
3. $0.004 < p \leq 0.008$
4. $0.008 < p \leq 0.016$
5. $0.016 < p \leq 0.032$
6. $0.032 < p \leq 0.064$
7. $0.064 < p \leq 0.128$
8. $0.128 < p \leq 0.256$
9. $0.256 < p \leq 0.512$
10. $0.512 < p$

14.2 Solution

The correct answer is: 8.

We show two possible solution methods.

First approach: Denote the seat numbers of Atto, Bilbo, Chico, Dondo and Ziggo respectively with A, B, C, D and Z. We make two observations.

- Ziggo can only be sitting in one of the five seats A,B,C,D,Z: Each other seat will at the latest be occupied by the person who had the corresponding seat number.
- When one of the first 25 gnomes randomly chooses a seat, his decision makes no distinction between the five seats A, B, C, D and Z.

This implies that Ziggo will completely randomly get one of the five seats A, B, C, D and Z. Hence the probability that Ziggo gets his own seat equals $p = 1/5$.

Second approach: We generalize the problem from $n = 26$ gnomes to an arbitrary number $n \geq 5$ gnomes. The first four gnomes randomly take a seat, as before, and the remaining $n-4$ gnomes behave as indicated above. Let $P(n)$ denote the probability that the last gnome gets his own seat. It is easily seen that $P(5) = \frac{1}{5}$: The first four gnomes randomly occupy four seats, and the fifth gnome hence has equal probabilities $\frac{1}{5}$ to be seated in any of the five seats.
Now let $n \geq 6$. Consider the epoch at which Espo, the fifth gnome, enters the concert hall. We distinguish two cases.

- In the first case, Espo's seat is free. He takes it, and plays no role in the remainder of the reasoning. We might as well remove Espo and his seat from the whole story, reducing it to a story with $n-1$ gnomes. Hence the probability that the last gnome gets his own seat then equals $P(n-1)$.
- In the second case, Espo's seat is taken by one of the first four gnomes; let's say, by Atto. Now 5 seats are taken: Espo's plus four random seats. We can now let Atto and Espo change seats, and again remove Espo and his seat from the whole story. We are now back in the situation in which the first four gnomes randomly took a seat from a set of $n-1$ seats, and again $P(n-1)$ equals the probability that the last (of the $n-1$) gnome gets his own seat.

The first case occurs with some probability q, and the second with probability $1 - q$. Hence

$$P(n) \;=\; qP(n-1) + (1-q)P(n-1) \;=\; P(n-1).$$

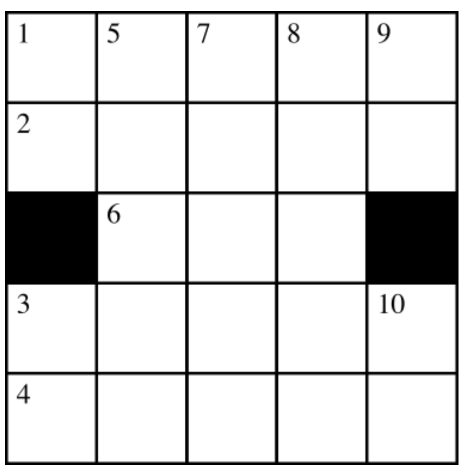
It follows by induction that $P(n) = P(5)$ for all $n \geq 5$.

15 Cross number puzzle

Authors: Hennie ter Morsche (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

15.1 Challenge

This puzzle asks you to fill each of the empty little squares in the following figure with a decimal digit. These digits then yield five horizontal and seven vertical numbers, none of which must begin with the digit 0.



For finding the five horizontal and twelve vertical numbers, you have to solve clues of the form *"xyz: abcd"*, where *"xyz"* states the numeral of the clue number (for instance for the clue 8-horizontal, the corresponding English numeral would be *xyz=eight*), whereas *"abcd"* states the numeral of the number to be entered (for instance if the number 2017 is to be entered, then the corresponding English numeral would be *abcd=two thousand and seventeen*).

Please note, however, that the decimal numbers in the following twelve clues are not presented in English, but in the Old High Elvish language. We stress that Old High Elvish uses a highly structured and absolutely logical system of numeral words. And a final (but crucial!) remark: One of our bureaucracy-elves has brought the horizontal and the vertical clue list into alphabetical order.

Horizontal clues:
Afri: farapuk-gemubol-luxadim-gemu
Fara: luxadim-faratek-kolm
Gemu: osmopuk-zabodim
Kolm: zabopuk-gemubol-zabodim-afritek-rucu
Zabo: afripuk-devebol-rucudim-gemutek-zabo

Vertical clues:
Deve: zabotek
Kolm: zabotek-fara
Luxa: devepuk-luxadim-gemutek-gemu
Osmo: gemupuk-kolmdim-afritek
Rucu: rucupuk-zabobol-faradim-zabotek-luxa
Zabo: afritek-osmo
Zabotek: rucutek-gemu

We denote by H the sum of the five horizontal numbers and by V the sum of the seven vertical numbers. Our question is: What is the value of the difference $H - V$?

Possible answers:

1. Farapuk-kolmbol-afridim-luxatek-osmo
2. Rucubol-luxadim-rucutek-kolm
3. Kolmpuk-devebol-gemudim-afritek-afri
4. Luxapuk-zabobol-kolmdim-luxatek-fara
5. Gemupuk-gemubol-afridim-zabotek
6. Kolmpuk-afridim-osmotek
7. Luxapuk-osmobol-gemudim-zabotek-deve
8. Zabopuk-devebol-devedim-faratek
9. Osmopuk-osmobol-faradim-luxatek-rucu
10. Rucupuk-farabol-osmotek-fara

15.2 Solution

The correct answer is: 5.

Since the horizontal hints in the figure are numbered with 1, 2, 3, 4, 6, the five wordnumbers afri, fara, gemu, kolm, zabo must represent a permutation of these five numbers. Since the down hints are numbered with 1, 3, 5, 7, 8, 9, 10, the seven wordnumbers deve, kolm, luxa, osmo, rucu, zabo, zabotek must belong to these numbers.
In the horizontal and the vertical hints (also in the possible answers) the subwords afri, deve, fara, gemu, kolm, luxa, osmo, rucu, zabo occur repeatedly .
Often these subwords are provided with the suffixes -puk, -bol, -dim, -tek. The suffixes will therefore denote the corresponding places in the decimal system: Tens, hundreds, thousands and tens of thousands.

- The word zabotek occurs once as hint number (as zabotek-down) and once as the number to be entered (at deve down). Since every number to be entered has at least two digits, and because 10 is the only two-digit hint number, we conclude zabotek=10. Therefore, **zabo= 1** and the suffix -tek indicates the tens position.

- The numbers 1 and 3 are the only ones to appear both horizontally and vertically as a hint number. Since only the words zabo (=1) and kolm appear in both of the lists, we conclude that **kolm= 3**.

- The number zabotek= 10 (from the hint deve-down) is to be entered either at 1-down, at 3-down, at 9-down or at 10-down. But 1-down and 3-down are ruled out, otherwise 2-across or 4-across would start with the digit 0.

  10-down is also eliminated, because then rucutek - gemu (and not zabotek) can be entered. Therefore, 9-down contains the number zabotek= 10, and there holds **deve= 9**.

- 1-across has the hint number zabo= 1 and therefore contains the number afripuk - devebol - rucudim - gemutek - zabo. Since Old High Gnomian is a thoroughly logically structured system of numberwords and since the suffix -tek denotes the tens digit, the remaining three suffixes -puk, -bol, -dim denote (in this order) the tenthousands position,the thousands positione and the hundreds position. The five digits in 1-acrossare therefore: afri, 9, rucu, gemu, 1.

- We now know that 5-down starts with the digit deve= 9. The only such vertical hint is Luxa with devepuk - luxadim - gemutek - gemu. Therefore, **luxa= 5**. Since in the hint luxa-down, there is no number with suffix - bol the thousand digit of the number to be entered is equal to 0. The five digits in 5-down are accordingly: 9, 0, 5, gemu, gemu.
- 6-across starts with the number 5 = luxa. The only compatible horizontal hint is fara-across with luxadim -faratek-kolm

  Therefore, **fara= 6**, and the three digits in 6-across are 5, 6, 3.
- osmo-down with gemupuk - kolmdim - afritek contains as the only vertical hint the digit kolm= 3; we deduce **osmo= 8**. Now you can easily identify the remaining three digits: **gemu= 2** and **afri= 4** and **rucu= 7**.

The following table summarizes the most important Old Hign Gnomian number words in the range from 1 to 90000 together:

Digit	×1	×10	×100	×1000	×10000
1	zabo	zabotek	zabodim	zabobol	zabopuk
2	gemu	gemutek	gemudim	gemubol	gemupuk
3	kolm	kolmtek	kolmdim	kolmbol	kolmpuk
4	afri	afritek	afridim	afribol	afripuk
5	luxa	luxatek	luxadim	luxabol	luxapuk
6	fara	faratek	faradim	farabol	farapuk
7	rucu	rucutek	rucudim	rucubol	rucupuk
8	osmo	osmotek	osmodim	osmobol	osmopuk
9	deve	devetek	devedim	devebol	devepuk

The following figure shows the only possible solution of the cross number puzzle:

^{1}4	^{5}9	^{7}7	^{8}2	^{9}1
^{2}8	0	1	0	0
	^{6}5	6	3	
^{3}1	2	1	4	^{10}7
^{4}6	2	5	0	2

Therefore, W = 205,033 and S = 182,623. Use the table to translate the difference $W - S$ = 22,410 into the numberword gemupuk - gemubol - afridim - zabotek. **Answer 5** is correct.

16 Workbench

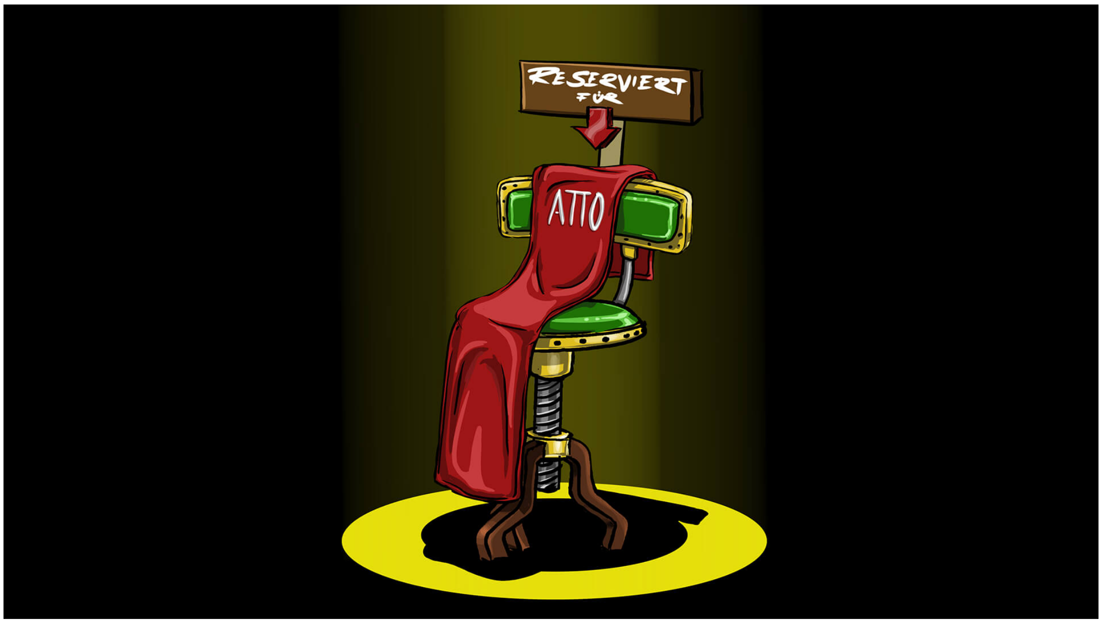
Authors: Cor Hurkens (TU Eindhoven), Jesper Nederlof (TU Eindhoven)

16.1 Challenge

The gnomes Atto, Bilbo and Chico work side-by-side at a small workbench with three working positions: One gnome sits on the left, one on the right, the third sits in the middle. None of them wants to sit at the exact same place on consecutive days. Furthermore, they do not want that a same seating arrangement (for instance: Atto left, Bilbo middle, Chico right) is ever used again. In this fashion they work for x days in a row.

Then Dondo shows up and joins the work force. The four gnomes move to a larger workbench with four positions: One gnome is seated in the North, one in the East, one in the South and the last one in the West of the bench. Like before, none of them wants to sit at the same position on consecutive days. And also they do not want that a seating arrangement (like Atto North, Bilbo East, Chico South, Dondo West) is ever repeated. In this fashion they work y days in a row.

Now we like to know: what is the largest possible sum $x + y$?



Possible answers:

1. The highest possible value for $x + y$ is 19.
2. The highest possible value for $x + y$ is 20.
3. The highest possible value for $x + y$ is 21.
4. The highest possible value for $x + y$ is 22.
5. The highest possible value for $x + y$ is 23.
6. The highest possible value for $x + y$ is 24.
7. The highest possible value for $x + y$ is 25.
8. The highest possible value for $x + y$ is 26.
9. The highest possible value for $x + y$ is 27.
10. The highest possible value for $x + y$ is 28.

16.2 Solution

The correct answer is: 9.

If one uses the permutation XYZ as a place assignment in the small workbench in one day, so only the two permutations YZX and ZXY are valid for the following day: If X works in the right place on the next day, Y must be left and Z must be in the middle. And if X works the next day in the middle, Z must be left and Y must be right.
In total, only the six permutations ABC, CAB, BCA and ACB, BAC, CBA come in question for a space assignment. If one of the three permutations ABC, CAB, BCA is used as the place allocation on the first day, on the later days only the other two permutations are allowed. Similarly, one of the three permutations ACB, BAC, CBA is obtained from only the other both permutations. It is easy to see that $x = 3$ is the largest possible value for x.
The following table shows a work plan for $y = 24$ days at the large workbench satisfying all the conditions desired by the wichtel (there are many others such workplaces for $y = 24$ days):

1:	ABCD	9:	CDAB	17:	DBCA
2:	DCAB	10:	DABC	18:	ADBC
3:	CBDA	11:	BCDA	19:	BACD
4:	BCAD	12:	CABD	20:	ACDB
5:	ADCB	13:	BDCA	21:	BDAC
6:	BADC	14:	DBAC	22:	DACB
7:	CDBA	15:	CADB	23:	CBAD
8:	ABDC	16:	ACBD	24:	DCBA

Therefore $y = 24$ is the largest possible value for y. In summary, the $x + y = 27$ is the largest value of the sum. Hence **response # 9** is correct.

17 Hats

Authors: Aart Blokhuis (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

17.1 Challenge

Father Christmas addresses the twelve most clever elves, Atto, Bilbo, Chico, Dondo, Espo, Frodo, Gumbo, Harpo, Izzo, Jacco, Kuffo, and Loco: *"My dear elves! Challenging brain teasers with hats on elves' heads have a long and rich tradition in the mathematical advent calendar. For this reason, I invite you once again to a merry afternoon with coffee and cake tomorrow."—"Great, we love to come!"*, the twelve elves shout.

Father Christmas continues: *"This evening I will sew into the lining of twelve elf hats, twelve pieces of paper, with twelve different numbers,* 1,2,3,...,12*. Tomorrow I'll put a hat on the head of each one of you. The pieces of paper are completely hidden inside the hats, so none of you will know the number of any of the hats. Then two elves can raise their hands, and I will tell them the number that is hidden in one of their two hats. After this again two elves can raise their hands, and again I will give them one of the two numbers. Then another two elves, and again I give a number. This goes on until you decide to stop. Then each one of you should tell me your number, but it is not enough to just guess, you should have a convincing argument why your number is correct. As soon as I have heard all numbers, I send away the elves with an incorrect number and also those whose argument is not convincing. Only the elves that had the number correct and had a valid argument are then invited in the main room and get a cup* *of coffee and a large piece of cherry pie."*

Kuffo asks: *"Is it allowed for the other elves to listen, when one of us gives his number and his reasoning?"—"Yes, that is allowed"*, Father Christmas answers. *"You can also discuss things among each other."*

Frodo asks: *"Can the same pair of elves raise their hands a second time?"—"That is allowed too"*, Father Christmas replies.

Gumbo asks: *"And when the same pair of elves raises their hands for the second time, will you then give a different number or can it be the same?"—"Could be different, but could also be the same"*, says Father Christmas with a grin.

The super intelligent elves start to deliberate; they discuss matters and think; they think and discuss things, discuss a bit more and think even more. Finally, they work out a strategy that maximizes the number N of elves that is guaranteed to get their coffee and cake.

Now our question: How large is this number N?

Possible answers:

1. $N = 3$
2. $N = 4$
3. $N = 5$
4. $N = 6$
5. $N = 7$
6. $N = 8$
7. $N = 9$
8. $N = 10$
9. $N = 11$
10. $N = 12$.

17.2 Solution

The correct answer is: 7.

Why $N \geq 9$ holds. Consider an arbitrary group W,X,Y,Z of four elves. From these four we can build six pairs WX, WY, WZ, XY, XZ and YZ. Since there are four numbers and six pairs of elves, father Christmas has to give the same number k for two of the pairs. These pairs must have an elf in common, and this elf has then of course the number k sewn into his hat. So, as long as there are at least four elves with an unknown number, we can find out the number of one of them. So we can find out the number of at least nine elves.

Why $N \leq 9$ holds. We consider the following bad situation for the elves, where the nine elves D,E,F,G,H,I,J,K,L (in this order) get the numbers $4, 5, 6, 7, 8, 9, 10, 11, 12$. In case one of the elves in a pair belongs to this set then father Christmas gives its number (in case both of them belong to the set, he just gives one of them). There are only three pairs of elves that do not involve one of the nine elves $D, E, F, G, H, I, J, K, L$: the pairs AB, BC and CA. In case of the pair AB father Christmas gives the number 1, for the pair BC number 2, and for CA number 3.
The elves are now unable to determine the numbers for A, B and C. One possibility is that A, B, C have the number 1,2,3, another is that A, B, C have the numbers $3, 1, 2$. So only nine elves are guaranteed to get coffee and cake.

18 Lost presents

Author: Cor Hurkens (TU Eindhoven)

18.1 Challenge

Santa Claus always knew that his assistant Ruprecht was not the brightest candle. But today, he completely outdid himself. After sending him on a round-trip along the circular road that passes all present factories and also Santa's office to fetch the already finished presents, he came back with 6 presents missing. He claimed to have lost them somewhere on the road.
"Well, go and look for them!" Santa Claus said.
"But I don't know where they are. They can be anywhere. And it's a pretty long road," Ruprecht replied.
"Come on, it's only 660 meters long. You can do it! But you can not cut short, because there are meters of snow on the roadside."
Ruprecht actually did walk the road and came back.
"So, where are the lost presents?" Santa asked.
"You only told me to go look for them. I know exactly where they are."
"Well, then off you go again on your 660 meters. And this time, bring me my presents," Santa said exasperated.
And Ruprecht left again.

"Actually," Santa mused, "he does not even have to walk the 660 m. If he knows exactly where all the presents are and if he stays at the position of the last present, he can get by with less."

Question: What is the length M (in meters) of the shortest trip that Ruprecht has to make in order to collect (not return) all six presents—independent of his starting position and independent of the positions of the six presents?

Possible answers:

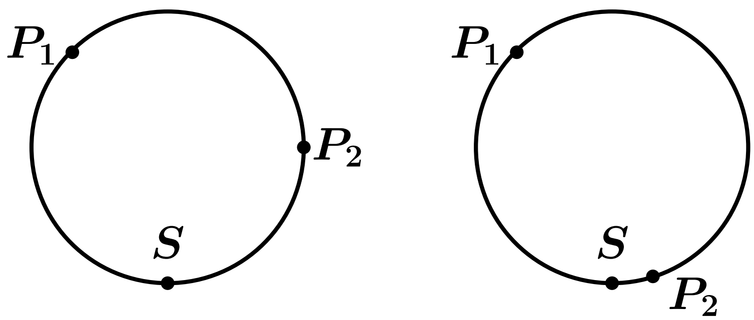
1. $555 < M \leq 565$.
2. $565 < M \leq 575$.
3. $575 < M \leq 585$.
4. $585 < M \leq 595$.
5. $595 < M \leq 605$.
6. $605 < M \leq 615$.
7. $615 < M \leq 625$.
8. $625 < M \leq 635$.

9. $635 < M \leq 645$.
10. $645 < M \leq 655$.

18.2 Solution

The correct answer is: 8.

Let's consider a smaller example with two presents first. We call Ruprecht's starting point S. Ruprecht could walk all the roundtrip from S and take all the presents with him. A slightly shorter way results if he stops at the last present. In general, one can describe Ruprecht's path through two points P_1 and P_2, where S, P_1, P_2 in this order are lying on the road:



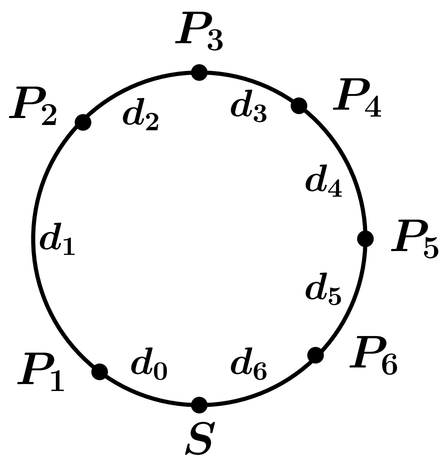
In the left setting, Ruprecht walks from S to P_2 to P_1 (counterclockwise) while the piece from P_1 to P_2 remains completely unvisited. In the example on the right, it is more sensible to first visit P_2, then pass S again and go to P_1. The shortest path will be one of

$$|SP_1| + |P_1P_2|,$$
$$|SP_2| + |P_1P_2|,$$
$$2|SP_1| + |SP_2|,$$
$$2|SP_2| + |SP_1|,$$

whichever is least.

Obviously, $|SP_1| + |P_1P_2|$ beats $|SP_2| + |P_1P_2|$ if $|SP_1| < |SP_2|$. However, even $|SP_1| + |P_1P_2|$ can be shortened to $2|SP_2| + |SP_1|$ if $|SP_2| < \frac{|P_1P_2|}{2}$. With $|SP_1| + |P_1P_2| + |SP_2| = L$, this means that $2|SP_2| + |SP_1|$ is shortest if $|SP_2| < \frac{L}{4}$ and $|P_1P_2| < \frac{L}{2}$. If $|SP_2| < \frac{L}{4}$ and $|P_1P_2| \geq \frac{L}{2}$, then $|SP_2| + |P_1P_2|$ is shorter. The same analysis can be performed if $|SP_1| < \frac{L}{4}$. In any case, it can be shown, that no shortest path is longer than $\frac{3}{4}L$. In the case where $|SP_1| = |SP_2| = \frac{L}{4}$ all cases will yield a shortest path of $\frac{3}{4}L$.

In the following figure, we scale this reasoning up to six presents. We name their positions from S clockwise in turn with $P_1, \ldots, P_6$ and we indicate their distances with $d_0, d_1, \ldots, d_6$, exactly as shown in the following figure:



According to the specifications, one has $d_0 + d_1 + d_2 + \cdots + d_6 = 660$. We now discuss eight cases and show that Ruprecht can always get by with 630 meters:

1. If $d_0 \geq 30$, Ruprecht will run counterclockwise $660 - d_0 \leq 630$ meters from S to P_1.

2. If $d_0 < 30$ and $d_1 \geq 60$, then first Ruprecht runs clockwise from S to P_1, and then counterclockwise (over S) to P_2. The distance travelled is $660 + d_0 - d_1 < 630$ meters.

3. If $d_0 < 30$ and $d_1 < 60$ and $d_2 \geq 120$, then first Ruprecht runs clockwise from S to P_2, and then counterclockwise until P_3. The distance travelled is $660 + d_0 + d_1 - d_2 < 630$ meters.

4. If $d_0 < 30$ and $d_1 < 60$ and $d_2 < 120$ and $d_3 \geq 240$, then first Ruprecht will run clockwise from S to P_3, and then counterclockwise from P_3 to P_4. The distance travelled is $660 + d_0 + d_1 + d_1 + d_2 - d_3 < 630$ meters.

5. If $d_6 \geq 30$, we argue symmetrically to the first case.

6. If $d_6 < 30$ and $d_5 \geq 60$, we argue symmetrically to the second case.

7. If $d_6 < 30$ and $d_5 < 60$ and $d_4 \geq 120$, we argue symmetrically to the third case.

8. In the remaining case,the following seven inequalities must be satisfied: $d_0 < 30$ and $d_1 < 60$ and $d_2 < 120$ and $d_3 < 240$ and $d_4 < 120$ and $d_5 < 60$ and $d_6 < 30$. The sum of these seven inequalities is $d_0 + d_1 + d_2 + \cdots + d_7 < 660$, that contradicts the statement about the length of the road. So this case cannot happen at all.

Our analysis is complete: There is always a solution with at most 630 meters. Finally, we consider the situation with $d_0 = d_6 = 30$, and $d_1 = d_5 = 60$, and $d_2 = d_4 = 120$, and $d_3 = 240$. In this situation, every possible solution is at least 630 meters long. Summarized: **answer 8** is correct.

19 Festive assembly line work

Author: Erhard Zorn (TU Berlin)
Translation: Ariane Beier (MATHEON)

19.1 Challenge

Turmoil in the chocolate heart factory: the gnomes want to come out on strike to fight for more breaks!

In 2017, a new production line was put into operation. The line has a length of 1 km. In the middle, there is a narrow transport corridor, on which the gnomes can walk either right- or leftwards. The bakers place chocolate-coated hearts on trays. The gnomes have to carry these trays to an arbitrary end of the production line to a drying plant.

On the narrow transport corridor, the gnomes are not able to pass each other. Therefore, the operation instruction says the following: The 24 gnomes stand on arbitrary positions in the corridor and choose an arbitrary walking direction—either left- or rightwards. Once the factory manager gives the starting signal, the gnomes start walking at a speed of 1 m/s in their chosen direction. If two gnomes meet, they instantly—without any delay—reverse their direction and continue walking at the speed of 1 m/s.

A five-minute break starts when all gnomes have carried their trays to one of the drying plants at the ends of the production line. After the break, the transport starts again, as described above.

In the gnomes' opinion, these breaks are too rare and they threaten to come out on strike. Thus, Santa Claus is sent for to smooth down the differences between the gnomes and the factory manager. Santa raises his eyebrows and contemplates.

Are you able to help Santa? How long do the gnomes have to walk until they can take a break?

Possible answers:

1. Each gnome walks at most 600 seconds on the transport corridor.
2. Each gnome walks at most $\ln(2) \cdot 1000$ seconds on the transport corridor.
3. Each gnome walks at most $1/\sqrt{2} \cdot 1000$ seconds on the transport corridor.
4. Each gnome walks at most 800 seconds on the transport corridor.
5. Each gnome walks at most 1000 seconds on the transport corridor.
6. Each gnome walks at most 1100 seconds on the transport corridor.

7. Each gnome walks at most 1200 seconds on the transport corridor.
8. Each gnome walks at most $\sqrt{2} \cdot 1000$ seconds on the transport corridor.
9. Each gnome walks at most 2000 seconds on the transport corridor.
10. There might be gnomes that never leave the transport way.

19.2 Solution

The correct answer is: 5.

Suppose that the gnomes were indistinguishable. Then, one could imagine the meeting of two identic gnomes who do not turn around anymore but run through each other, keeping their walking direction. An outside observer would not be able to distinguish these two situations.

Each gnome would then walk on the transport corridor as long as (s)he reaches the end of the production line that (s)he was looking at the start. If the distance to this end at the start was x meters, then (s)he will leave the corridor after x seconds. Since a gnome is at most 1 kilometer away from the end of the production line, (s)he will leave the corridor after at most 1000 seconds.

But what about the *distinguishable* trays, the gnomes are carrying? It turns out that they are not relevant to the problem: Imagine that two meeting gnomes indeed reverse their walking direction but exchange their trays. Then, each tray would be moved with a constant speed of 1 m/s. As worked out above, any tray leaves the transport corridor after at most 1000 seconds. Since every tray is carried by a gnome, this holds true for any gnome.

20 Soap sleigh derby

Author: Marika Karbstein (ZIB)
Project: Flight Trajectory Optimization on Airway Networks
Translation: Ariane Beier (MATHEON)

20.1 Challenge

All through the week the gnomes were excited. Finally, the day of the famous big soap sleigh derby has come. The winner will be allowed to choose the movie at the next gnomes' movie night.

The rules of the race are the following: All gnomes start at the same time, and everybody has to drive 6000 meters east and 4000 meters north. Unfortunately, there is a strong wind on the race track. Without any wind, a soap sleigh moves at a speed of 100 meters per minute. If the soap sleigh has headwind, it moves at half speed, i.e. at a speed of 50 meters per minute. If the soap sleigh has tailwind, it is twice as fast as without wind. It is interesting to remark that a potential crossing wind has no effect on the soap sleigh—it will not move sidewards and will be as fast as without wind. All the speeds described apply to the whole race track.

One never knows for sure the precise time neither the direction of the wind blowing. However, it is certain that the wind will blow from either south, north, east or west and that the wind direction will change every 10 minutes. The starting time of the soap sleigh derby coincides with a change of the wind direction.

The gnomes only drive north or east and are allowed to change their direction as often as desire. However, they have to schedule their route in terms of the chronology of their change of directions. Gnomes departing from their schedule will be disqualified. An example for such a schedule is the following: 20 minutes east, 30 minutes north, 10 minutes east, 20 minutes north etc.
The first gnome having travelled 4000 meters north and 6000 meters east wins. It is fine to drive too far in one direction. The race will be finished, once one has travelled the necessary distance in either direction.

Unfortunately, Santa was not able to attend the soap sleigh derby. Though, he manages to be at the wild après-soap-sleigh party. On his way there, he meets packaging master Edward. At the very moment, Santa asks who won the race, he regrets his question: Edward does not only wrap the children's gift fancifully, but also has a habit of framing his answers as mysteries.
Edward's statements are the following:

a) No gnome needed more than 100 minutes to travel the required distances.

b) All gnomes changed directions only at multiples of ten minutes.

c) During the 100 minutes after the starting time, the wind blew as often from the south as from the west. Moreover, it blew as often from the north as from the east. The wind blew at most 30 minutes from each direction.

d) Only Smettbo, Raichu and Glutexo went north at first.

e) Glutexo was the only gnome not having headwind on any part of the route. After 30 minutes, he had tailwind for 20 minutes—and that was the only phase he had tailwind.

f) Habitak and Raichu changed their direction only once. After 40 minutes, they drove in the same direction for 10 minutes, having tailwind in this phase. While one of them had had tailwind during the preceding 10 minutes, the other one had tailwind during the following ten minutes.

g) For the first 80 minutes, Chaneira and Habitak followed the same route. Afterwards, Chaneira went on a different route than Habitak.

h) Nobody started having headwind. At the 90th minute, the wind changed his direction—coming from south beforehand and west afterwards.

i) Using Morse code to encrypt the driving direction, one could describe the first 90 minutes of Blitza's route by the string V C T, Evoli's by F C T, Smettbo's by X U I, Glutexo's by Y S N, and Ursaring's by S O S. Afterwards, three of these gnomes had to go on to reach the finish.

(j) Except for the first 10 minutes, Natu's route was completely different from Blitza's route.

Assume that there are no time delays at the change of directions, i.e. the soap sleighs change their direction instantly if desired. Furthermore, there were no accidents, the gnomes did not impede each other, and nobody needed to be disqualified.

Can you tell Santa, which gnome won the soap sleigh derby?

Possible answers:

1. Blitza
2. Evoli
3. Smettbo

4. Chaneira
5. Habitak
6. There is no unique solution to the problem.
7. Raichu
8. Ursaring
9. Natu
10. Glutexo

Project relevance:

The puzzle is inspired by the project “Flight Trajectory Optimization on Airway Networks” (www.zib.de/projects/flight-trajectory-optimization-airway-networks). The aim of this project is to compute flight paths, which minimize the fuel consumption and overfly charges between given airports of departure and arrival, as fast as possible. The fuel consumption does not only depend on the properties of the aircraft, but also on weather conditions and wind currents.

20.2 Solution

The correct answer is: 9.

Table 10 depicts the wind directions, the gnomes' changes of course, and the total times. Note:

- From statement a), one deduces that it is sufficient to regard the first 100 minutes of the race. Because of b), these 100 minutes can be devided into phases of 10 minutes each.
- The time of the first change of direction of all participating gnomes is a consequence of statement d).
- Since the first direction of movement is known, one concludes that the Morse signal "dot" encodes east and "dash" encodes north. Hence, one obtains the first nine directions of Blitza, Evoli, Smettbo, Glutexo und Ursaring. Statement j) gives Nato's route.
- The wind directions between minute 30 to 40 and 40 to 50 follow from statement e).
- From f), one infers the directions of movement of Habitak and Raichu, as well as the wind direction between minute 50 and 60.
- Then, the wind direction for the remaining phases follow from statements h), e), and c).
- When calculating the distances, one observes that Blitza, Evoli and Smettbo have not finished their race after 90 minutes. The final direction results from the fact that all participating gnomes finished after at most 100 minutes (statement a)).

	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100	
	→	↓	←	↑	→	↑	↓	←	↑	→	
Smettbo	↑ 1	→ 1	→ 0.5	↑ 2	→ 2	→ 1	↑ 0.5	→ 0,5	→ 1	↑ 1 [0.5]	95
Raichu	↑ 1	↑ 0.5	↑ 1	↑ 2	→ 2	→ 1	→ 1	→ 0.5	→ 1	→ 2 [0.25]	92.5
Glutexo	↑ 1	→ 1	↑ 1	↑ 2	→ 2	→ 1	→ 1	↑ 1	→ 1		90
Blitza	→ 2	→ 1	→ 0.5	↑ 2	↑ 1	→ 1	↑ 0.5	→ 0.5	↑ 2	→ 1 [1]	100
Evoli	→ 2	→ 1	↑ 1	→ 1	↑ 1	→ 1	↑ 0.5	→ 0.5	↑ 2	→ 2 [0,25]	92.5
Natu	→ 2	↑ 0.5	↑ 1	→ 1	→ 2	↑ 2	→ 1	↑ 1 [0.5]			75
Ursaring	→ 2	→ 1	→ 0.5	↑ 2	↑ 1	↑ 2	→ 1	→ 0.5	→ 1		90
Chaneira	→ 2	→ 1	→ 0.5	→ 1	→ 2	↑ 2	↑ 0.5	↑ 1	→ 1	↑ 1 [0.5]	95
Habitak	→ 2	→ 1	→ 0.5	→ 1	→ 2	↑ 2	↑ 0.5	↑ 1	↑ 2 [0.25]		82.5

Figure 10: Headline: Division of the first 100 minutes of the race with information on the wind direction. For each gnome, the direction of movement and the travelled distance (under the occuring wind conditions) in km is given for every 10-minute time interval. Numbers given in [ ]-brackets state the time fraction of the according 10-minute time interval actually needed to finish the race. The last number represents the total time.

21 Sorting presents

Authors: Ulrich Reitebuch (FU Berlin), Martin Skrodzki (FU Berlin)
Project: GV-AP16 – *Computational and structural aspects of point set surfaces*
Translation: Ariane Beier (MATHEON)

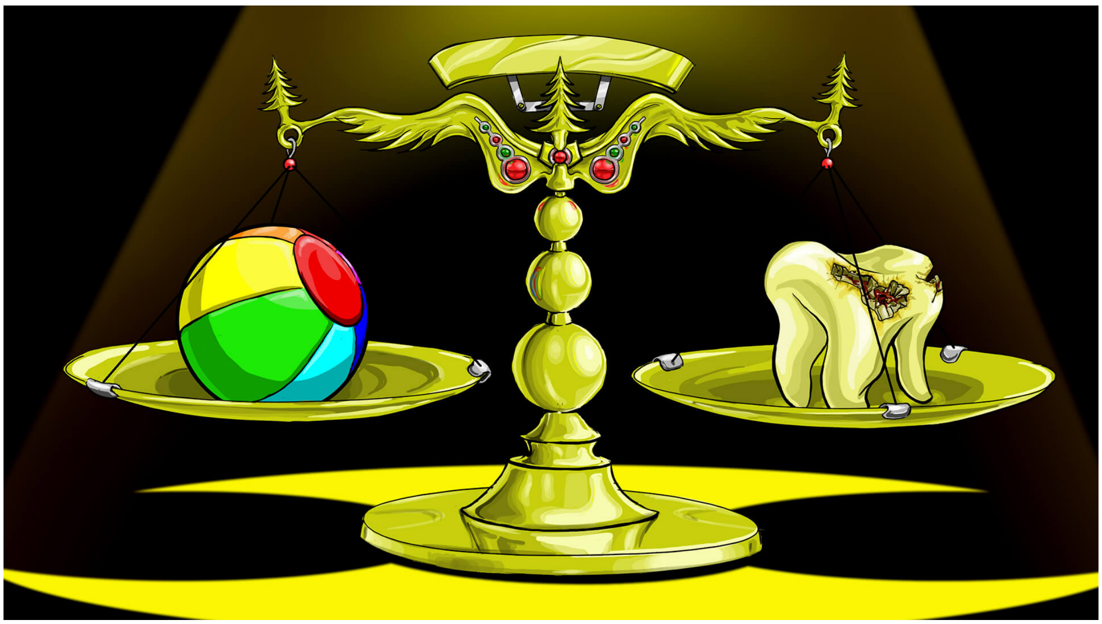
21.1 Challenge

Until now, the gnomes of the village used to sort the packages of Christmas presents by the number of toys inside. After several complaints filed by the World Dentist Association, the gnomes have to consider the amount of sweets in each package too. Thus, every package P gets a toy rating T_P and a sugar rating S_P. These ratings are distinct for any two packages. Now, the gnomes want to stack the packages according to the following rules:

- For two packages P and Q placed next to each other, one has: P lies to the left of Q if and only if $T_P < T_Q$.
- For two packages P and Q placed one top of each other, one has: P lies beneath Q if and only if $S_P < S_Q$.

A first delivery of nine cubic packages arrives; they are to be sorted in three rows, each consisting of three packages. The packages for the children are assembled such that less toys are compensated for with more sweets. The ratings (T_P,S_P) are the following: (1,9),(2,8),(3,7), . . . ,(9,1).

How many possibilities are there to stack these nine packages as a stack of size 3×3 according to the two stack rules above? (Note: It is out of question whether one can regard the stacks from both sides, because *left* is uniquely defined only from one side.)



Possible answers:

1. 1
2. 2
3. 3
4. 9
5. 11
6. 12
7. 13
8. 16

9. 21
10. 42

Project relevance:

This question arises when considering the combinatoric order of geometric point clouds. If one wants to order the n^2 points $(x_1,y_1),\ldots,(x_{n^2},x_{n^2})$, $n \in \mathbb{N}$, in a grid of size $n \times n$ such that the x-coordinate increases in the rows from left to right and the the y-coordinate increases in the columns from the bottom up, there exist only finitely many combinatoric possibilities to fill the grid—independent of the specific coordinates of the points. The grid is applied to determine neighbourhoods. This is of importance, for instance, for simulations of biological cells and physical particles or when processing large amounts of data provided by 3D scanners.

21.2 Solution

The correct answer is: 10.

To identify the correct answer, we rephrase the problem first: Assume we have found an order of the packages obeying both rules (see Figure 11, left). Now, we change the rating of each package from (T,S) to (T, $10 - S$) (see Figure 11, center). Then, each package is now rated (T,T). However, the order of the packages is not correct anymore and we have to interchange the row at the bottom with the row at the top (see Figure 11, right). With this transformation, any order of the given ratings (1,9),(2,8), . . . ,(9,1) may be converted into an order of of the ratings (1,1), (2,2), . . . , (9,9)—and vice versa. Thus, there is a bijection between these two sets of possible orders, and these sets need to be of the same size. If we know how many orders there exist for the ratings of the form (T,T), we also know this quantity for our original ratings (T,$10 - T$).

(1,9)	(2,8)	(7,3)
(3,7)	(5,5)	(8,2)
(4,6)	(6,4)	(9,1)

(1,1)	(2,2)	(7,7)
(3,3)	(5,5)	(8,8)
(4,4)	(6,6)	(9,9)

(4,4)	(6,6)	(9,9)
(3,3)	(5,5)	(8,8)
(1,1)	(2,2)	(7,7)

Figure 11: Left: order of the packages with original ratings. Center: change of rating. Right: correct order according to new rating.

Now, we know that $T_P = S_P$ for all packages P. Furthermore, $T_P \neq T_Q$ for any two packages P and Q. Since we are only concerned with the order, we can set $T_{P_1} = 1$, $T_{P_2} = 2$, ..., $T_{P_9} = 9$. The problem reduces to the question how the integers 1 to 9 can be written in a grid of size 3×3 such that the rows from left to right and the columns from bottom to top are ordered increasingly. Here, the total number of 9 packages is partitioned into (3,3,3).

More generally, one can consider, for a given positive integer N, a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 + \ldots + \lambda_n = N$ and $\lambda_i > \lambda_j$ whenever $i > j$. The grid with λ_i cells in the i-th row is called *Ferrers diagram* (see Figure 12, left). The number of possibilities to fill up this grid with the numbers from 1 to N, such that every row from left to right and every column from bottom to top is ordered increasingly, is provided by the *hook-length formula*:

$$\frac{N!}{\prod h_\lambda(i,j)},$$

where, for each cell (i,j) in the grid, the *hook* $H_\lambda(i,j)$ is the set of cells (a,b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The *hook-length* $h_\lambda(i,j)$ is the number of cells in the hook $H_\lambda(i,j)$ (see Figure 12, right).

In the special case $N = 9$ and $\lambda = (3,3,3)$, we have

$$h_\lambda(i,j) = (3 - i) + (3 - j) + 1,$$

and the number of possible orders is

$$\frac{9!}{\prod_{i=1}^{3} \prod_{j=1}^{3} ((3 - i) + (3 - j) + 1)} = 42.$$

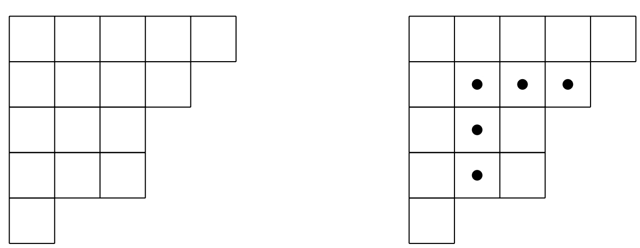


Figure 12: Left: Ferrers diagram for the partition $\lambda = (5,4,3,3,1)$ of $N = 16$. Right: according hook $H_\lambda(2,2)$ with hook-length $h_\lambda(2,2) = 5$.

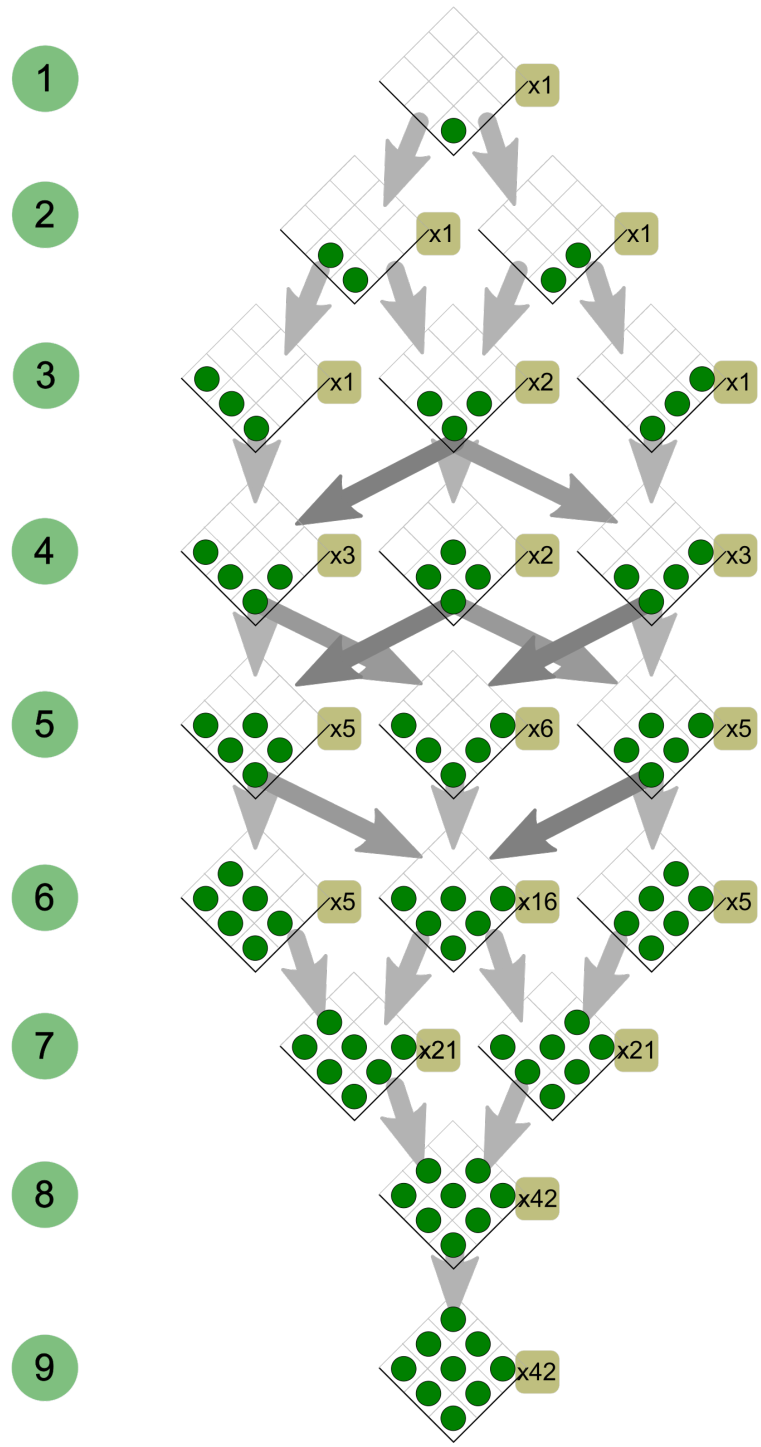
Further remarks: For a quadratic grid of side length 1, 2 or 3 it is known, for which specific ratings there are the least or the most possible orders, respectively. For $n \in \{1,2,3\}$, there are ratings that allow a unique order. This is not true for $n \geq 4$. It is conjectured that the rating $(1,1), \ldots (n^2,n^2)$ is the one which allows the most orders. This fact is known for $n \in \{1,2,3\}$, but remains unproven for $\neq 4$.

Alternative solution:

Those who are not acquainted with the hook-length formula are also able to compute the number of possiblities: We assume that the ratings of the packages have been already converted to (1,1), . . . , (9,9). Thus, we can represent each package by an integer from 1 to 9.

For a better understanding, we rotate the quadratic grid by 45° such that it stands on the bottom left vertex. We know let the packages fall—as under gravitational force—into the grid of size 3×3. The only possible position for package 1 to come to rest at is the bottom of the grid, because other packages are not allowed to be placed underneath the package with rating 1. Afterwards, we gradually drop the packages 2 to 9. These may settle at different positions in the grid. However, these positions are restricted by the edges of the grid and the packages that have already fallen into the grid.

In the following diagram, the possible valid configurations are displayed; the arrows indicate how one gets from one valid configuration to another. At times there are various ways to get to the same configuration. Therefore, the number of ways to get to a configuration is displayed besides the configuration itself. At the bottom, you can read off that there are 42 legit ways to get to a full configuration of our quadratic grid of size 3×3.



1
2
3
4
5
6
7
8
9
x1
x1
x1
x1
x2
x1
x3
x2
x3
x5
x6
x5
x5
x16
x5
x21
x21
x42
x42

22 Library

Authors: Aart Blokhuis (TU Eindhoven), Cor Hurkens (TU Eindhoven)

22.1 Challenge

Elf Bippo is the librarian of Father Christmas. When the elves have their supper, he tells them about his days work: *“Today I finally found some time to clean up the mess in the little corner room of the library. When I started* 70 *books where distributed unorderly on the floor, building* 70 *stacks of one book each. I started by putting one stack on top of another, reducing the number of stacks by one each time, until finally at the end of the day there was a single stack of* 70 *books in this corner room.”* Theso, the hypothesis elf, starts contemplating and meditating and then produces the following statements:

A. At some point in the day there were 3 stacks of books in the corner room containing together (exactly) 70 books.

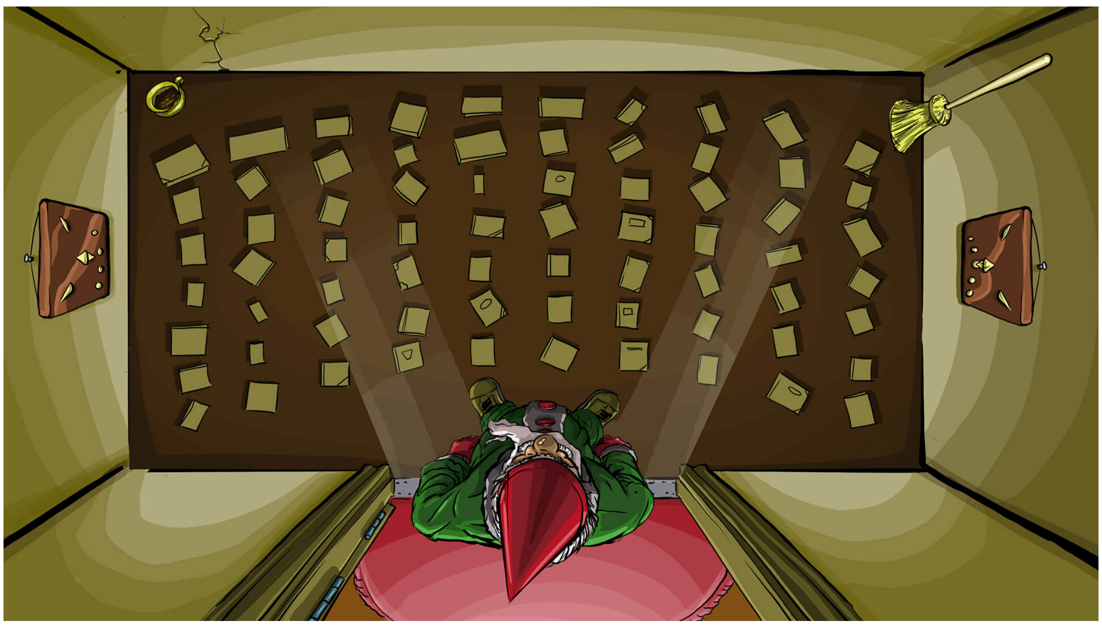
B. At some point in the day there were 3 stacks of books in the corner room containing together (exactly) 36 books.

C. At some point in the day there were 21 stacks of books in the corner room containing together (exactly) 42 books.

D. At some point in the day there were 13 stacks of books in the corner room containing together (exactly) 42 books.

Every stack of books in this problem contains at least one and at most seventy books.

Which of Theso's statements are true (no matter how Bippo produced his stacks), or false (for some way that Bippo stacks)?



Possible answers:

1. A is true, B, C, D are false.
2. A, B are true, C, D are false.
3. A, C are true, B, D are false.
4. A, D are true, B, C are false.
5. B, C are true, A, D are false.
6. B, D are true, A, C are false.
7. A, B, C are true, D is false.
8. A, B, D are true, C is false.

9. A, C, D are true, B is falsce.
10. A, B, C, D are all true.

22.2 Solution

The correct answer is: 3.

Statements A and C are true, statements B and D are false. We will now discuss and analyse the four statements in detail.

Statement A. The number of stacks reduces by one each time, so at some point there are exactly three stacks containing all 70 books, so statement A is **true**.

Statement B. This statement is **false**. We give a strategy for Bippo to avoid this constellation:
In phase 1 Bippo forms 7 stacks having 10 books each. In this phase every stack has at most 10 books, so there will be never three stacks with 36 books.
In phase 2 Bippo joins stacks in any way, in each stage the number of books in every stack will be a multiple of 10, so definitely not 36.

Statement C. After 21 steps there will be exactly 49 stacks of books in the corner room containing the 70 books. Out of these 49 stacks a certain number x still consist of a single book and the remaining $49 - x$ stacks have two or more books. This implies $1 \cdot x + 2(49 - x) \leq 70$, so $x \geq 28$. When we remove from the 49 stacks containing 70 books 28 stacks consisting of a single book, we are left with 21 stacks, containing a total of 42 books, so statement C is true.

Statement D. This statement is **false**.
We again give a strategy for Bippo to avoid this constellation:
In phase 1 Bippo forms 2 stacks with 2 books and 22 stacks with 3 books. In this first phase 13 stacks have at most $13 \cdot 3 = 39$ books, so never 42. In phase two Bippo starts by joining the two stacks with 2 books to a stack with 4 books, and calls this the special stack. After this Bippo simply puts a 3-book stack on the special stack until finally all books are on the special stack. In the second phase there are only 3-book stacks and the special stack with 1 mod 3 books, so not 36. So we don't have 13 stacks with 42 books altogether.

23 Mole at need

Authors: Falk Ebert (HU Berlin), Ariane Beier (MATHEON)
Translation: Clara Jansen (MATHEON)

23.1 Challenge

Last year, the little gnome Wilma got the best Christmas present ever: her own proper little mole. From the first second on, she was obsessed with the little rascal and has been calling him Marco since. (Disclaimer: Moles are completely unsuitable pets for human children!) However, Wilma underestimated Marco's need to roam freely. If he gets bored in Wilma's room, he slips away into the basement, looking for tasty insects. On his excursions, he already got lost on several occasions. It is generally known that moles are almost completely blind and find their way by touch and scent. But with all the fragrant delicious treats that are cooked and baked in Wilma's house during Christmas time, the poor mole's nose is completely overwhelmed.

Today, it happened again... Marco is sitting in the center of the completely dark basement (see Fig. 13) at point S. He starts walking northwards. As long as Marco does not encounter any obstacle, he walks on straight ahead. If he encounters a wall, he starts following that wall in a randomly (more precisely: equiprobably) chosen direction. If he subsequently loses the contact to the wall, he continues in a straight line. If he walks into a corner, he randomly chooses to follow that corner or to turn around and go back in the direction he came from, both with equal probability.

There are two exits from the basement, Z_1 and Z_2, which he reaches with probabilities P_{Z_1} and P_{Z_2}. What is the ratio $P_{Z_1} : P_{Z_2}$?

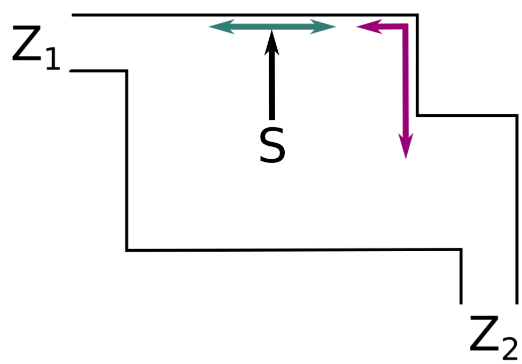


Figure 13: Basement of Wilma's house. S: Marco's starting place. Z_1 and Z_2: exits.

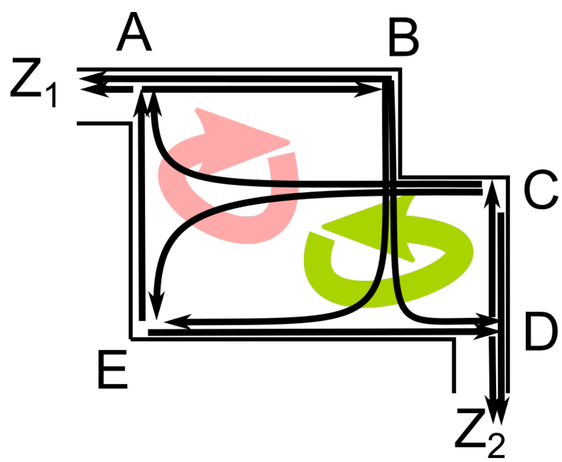
Possible answers:

1. $1:1$, i. e. both exits are equiprobable.
2. $2:1$, i. e. exit Z_1 is twice as probable as exit Z_2.
3. $3:1$.
4. $5:1$.
5. $6:1$.
6. $13:2$.
7. $16:3$.
8. $64:13$.
9. $128:25$.
10. The ratio cannot be determined with the given data.

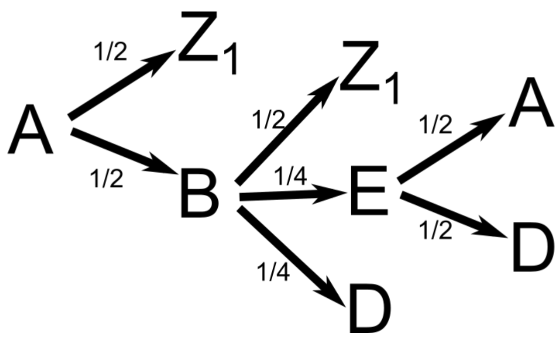
23.2 Solution

Correct answer: **4. 5:1**

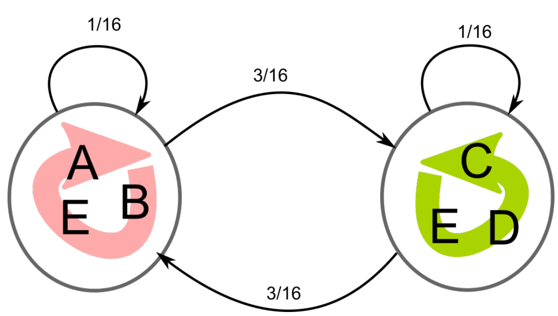
In the following figure, the possible ways of the mole are illustrated schematically.



From the points A and B there, is a chance of 50% to take the exit Z_1. Analogously, from the points C and D, there is a chance of 50% to take the exit Z_2. One has to observe that point A can only be reached from below, since coming from B, the mole does not reach any obstacle and therefore goes straight to exit Z_1. Analogously, B can only be reached from point A. The inverse situation appears at the points C and D. If the mole does not go from B to the exit Z_1, he hits the wall below and will go to point D or point E with the same probabilities. Overall, one can consider the the mole's path in the following way: If he is in the cycle A-B-E, he can escape through exit Z_1 from A or B, and from B or E he can reach point D. Besides, from E he can go back to A. This justifies the description as a cycle. Point D on the other hand is part of the cycle D-C-E. Only from this cycle one can reach the exit Z_2. It is apparent that both cycles, A-B-E and D-C-E, behave in a symmetric way. In this way, A corresponds to D, and B corresponds to C. The corner E belongs to both cycles.
So it is enough to make some observations on the cycle A-B-E. For D-C-E the results are analog. Without loss of generality, we may assume that the mole starts its path in E moving to A (because the possibilities of going left or right are the same as in S).



From the tree diagrams above, we conclude that, if Marco is coming from A in the A-B-E cycle, he has two possibilities to go to Z_1. He uses them with a probability of $\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}$. With probability $\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}=\frac{3}{16}$ he goes to D and therefore into the cycle D-C-E. And with probability $\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}$ he gets back to A and therefore stays in the cycle A-B-E. From now on, we denote by a_k the probability that in the k-th step Marco is in the cycle A-B-E, whereas by d_k we denote the probability that he is in the cycle D-C-E in the k-th step. One step is characterized by the fact that Marco reaches point A or point D. The starting situation is $a_0 = 1$ and $d_0 = 0$ since he starts (in principle) at A. With the probabilities from the tree diagram we can work out the following transition scheme:



Consequently, Marco stays in one cycle with probability $\frac{1}{16}$ and goes to the other cycle with probability $\frac{3}{16}$. It holds:

$$a_{n+1} = \frac{1}{16}a_n + \frac{3}{16}d_n, \tag{13}$$

$$d_{n+1} = \frac{1}{16}d_n + \frac{3}{16}a_n. \tag{14}$$

We now define the sums of all a_n and d_n as

$$T_A = a_0 + a_1 + a_2 + \ldots,$$
$$T_D = d_0 + d_1 + d_2 + \ldots.$$

These two series are finite in spite of the fact that there are infinitely many summands. These series give us the average number of steps that Marco stays in every cycle. To determine the ratio of the probabilities of the exits, that is the probabilities we are looking for, we only have to compute the ratio $T_A : T_D$. To compute the value of the series, we make use of a trick:

$$T_A = a_0 + a_1 + a_2 + \ldots$$
$$T_D = d_0 + d_1 + d_2 + \ldots$$

$$T_A - a_0 = T_A - 1 = a_1 + a_2 + \ldots$$
$$T_D - d_0 = T_D = d_1 + d_2 + \ldots$$

The terms a_n and d_n on the right hand side can be brought into an other form with (13) and (14).

$$T_A - 1 = \frac{1}{16}(a_0 + a_1 + a_2 + \ldots) + \frac{3}{16}(d_0 + d_1 + d_2 + \ldots) \quad = \frac{1}{16}T_A + \frac{3}{16}T_D$$
$$T_D = \frac{1}{16}(d_0 + d_1 + d_2 + \ldots) + \frac{3}{16}(a_0 + a_1 + a_2 + \ldots) \quad = \frac{1}{16}T_D + \frac{3}{16}T_A$$

Out of this, we can form the linear system of equations:

$$-1 = -\frac{15}{16}T_A + \frac{3}{16}T_D,$$
$$0 = \frac{3}{16}T_A - \frac{15}{16}T_D$$

with the solutions $T_A = \frac{10}{9}$ and $T_D = \frac{2}{9}$. Hence we get the ratio $T_A : T_D = 5 : 1$. As a nice confirmation of this result, we can look at the sum $T_A + T_D = \frac{12}{9} = \frac{4}{3}$. That is, it is likely that Marco stays in the basement for $\frac{4}{3}$ cycles. And since the probability that he is going from a cycle to an exit is $\frac{3}{4}$, the probability that he went to an exit after these $\frac{4}{3}$ cycles is $\frac{4}{3} \cdot \frac{3}{4} = 1$.

24 Change of hall

Author: Judith Keijsper (TU Eindhoven)

24.1 Challenge

At the annual meeting of the brain gnomes, each of the 101 participating gnomes has a different intelligence quotient of $100, 101, 102, \ldots, 199, 200$.
One group of gnomes is meeting in the blue hall, while the rest of the group is meeting in the red hall. Gnome Adalbert (with intelligence quotient 124) changes from the blue one into the red hall. As a result, the average intelligence ratio in both of these halls increases by $1/3$.

How many gnomes were in the blue room **before** Adalbert's change of hall?

Possible answers:

1. There were 17 gnomes in the blue room.
2. There were 24 gnomes in the blue room.
3. There were 38 gnomes in the blue room.
4. There were 49 gnomes in the blue room.
5. There were 51 gnomes in the blue room.
6. There were 56 gnomes in the blue room.
7. There were 65 gnomes in the blue room.
8. There were 72 gnomes in the blue room.
9. There were 83 gnomes in the blue room.
10. There were 90 gnomes in the blue room.

24.2 Solution

The correct answer is: 10.

We denote the number of gnomes in the blue and red hall before Adalbert's change of hall with b and r, and the sum of the intelligence quotients of all gnomes in the blue and red room respectively with B and R. Then b+r = 101 and B+R = $100 + 101 + \cdots + 200 = 15150$. The change in the average intelligence quotients in the halls leads to

$$\frac{B-124}{b-1} = \frac{B}{b} + \frac{1}{3} \qquad \text{und} \qquad \frac{R+124}{r+1} = \frac{R}{r} + \frac{1}{3}.$$

The two equations can easily be brought into the following equivalence form:

$$b^2 + 371b - 3B = 0 \qquad \text{und} \qquad r^2 - 371r + 3R = 0.$$

Due to $r = 101 - b$ and $R = 15150 - B$, the second equation now returns $b^2 + 169b - 3B + 18180 = 0$. If we subtract this new equation from the first equation $b^2 + 371b - 3B = 0$, we obtain $202b = 18180$ and thus b = 90. So **answer #10 is correct**. (You can also calculate that $r = 11$ and $R = 1320$ and $B = 13830$. Then, for example, $r = 11$ gnomes with intelligence quotients $100, 101, \ldots, 107$ and $163, 164, 165$ could be in the red room).