Challenges and Solutions 2019

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1 Mondrian

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Projekt: 4TU.AMI

1.1 Challenge

Mondrian, the painter-elf, has designed a square-shaped Christmas card and has subdivided it into 100 square-shaped cells in a 10×10 pattern, as shown in Figure 1.

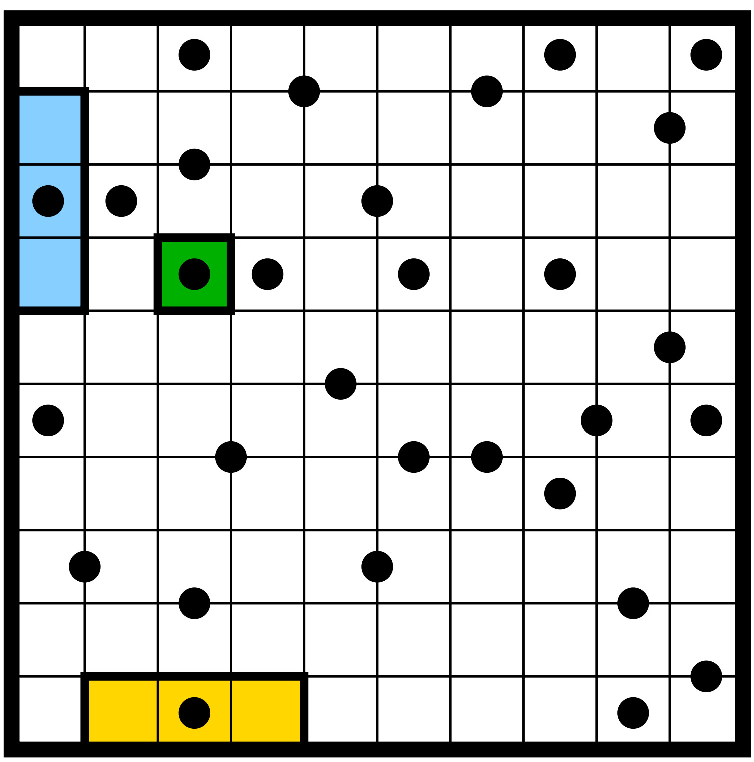
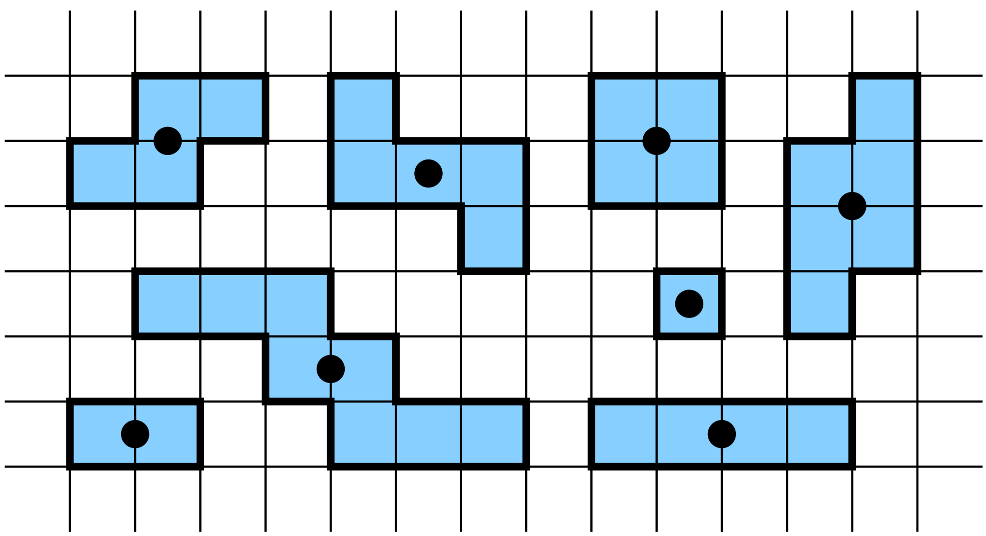


Figure 1: Mondrian's *unfinished* Christmas card

Furthermore, Mondrian has drawn 30 black circles on his card. Around each of these 30 black circles, Mondrian has constructed a painted region; the blue region, the green region, and the yellow region are shown in picture 1.

- Each painted region consists of one or more cells of the 10×10 pattern.
- Each cell belongs to exactly one region.
- The cells of each region are connected: You can reach every cell of the region from every other cell of the region, by making a sequence of horizontal and vertical steps within the region.
- The regions do not contain holes.
- Every region contains exactly one connected black circle in its interior. *No* additional circular segments are allowed in a region.
- Every region is rotationally symmetric: If you rotate the region by 180 degrees around its black circle, then the rotated region coincides with the original region.

Here are some examples of such rotationally symmetric regions with a black circle as its center:



Question: What is the area of the largest region on Mondrians *finished* Christmas card (cf. Fig. 1)?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. The largest region consists of 9 cells.
2. The largest region consists of 10 cells.
3. The largest region consists of 11 cells.
4. The largest region consists of 12 cells.
5. The largest region consists of 13 cells.
6. The largest region consists of 14 cells.
7. The largest region consists of 15 cells.
8. The largest region consists of 16 cells.
9. The largest region consists of 17 cells.
10. The largest region consists of 18 cells.

1.2 Solution

The correct answer is: 10.

Some cells contain a black circle, a clack semicircle or a black quadrant. These cells can be associated to the corresponding circle k—there are located in the region containing the circle k. For example, the cell in the upper right corner is associated to the circle in its inside, and the cell in the lower right corner is associated to the circle at it upper boundary.

If two cells are (horizontally or vertically) adjacent and if these cells are associated to different circles, then their mutual (horizontal or vertical) edge is part of the boundary of their associated regions. In phase one, we draw all such borderlines in Mondrian's Christmas card (see figure 2).

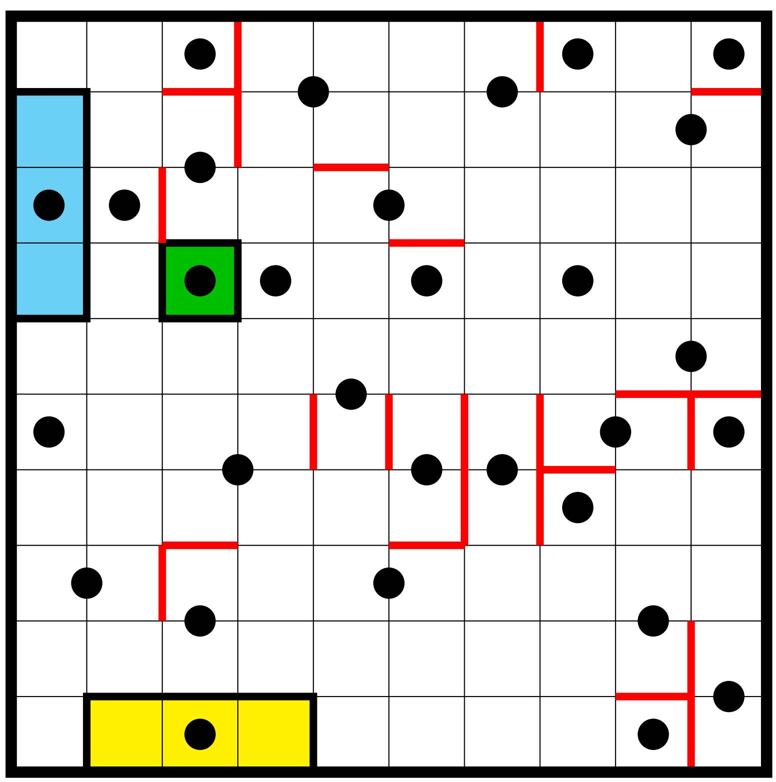


Figure 2: Borderlines of regions with a mutual edge

Now, we know some borderlines of the desired regions. Of course, all 40 edges forming the card's boundary belong to these known borderlines. Each borderline ℓ is associated to a cell c, and this cell c is associated to a black circle k. If one rotates l around k by 180 degrees, then one obtains a borderline ℓ'. Since every region is rotationally symmetric to itself, the borderline ℓ' is associated to the circle k as well.
In phase two, we draw in all these borderlines ℓ' that are rotationally symmetric to a already existing borderline ℓ'. In order to distinguish these new lines from the old borderlines, we display them in red (see figure 3).

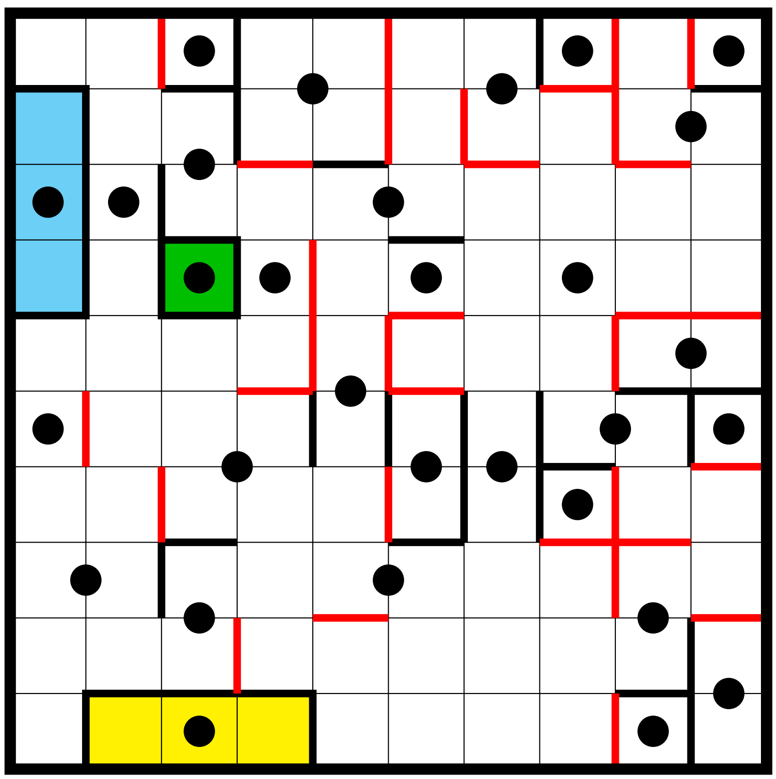


Figure 3: Rotationally symmetric borderlines

During the next phase, we first draw in all borderlines that are rotationally symmetric to the lines found in the previous phase. Afterwards, we are able to associated further cells uniquely to their corresponding regions. After several iterations, one obtains the following picture (see figure 4).

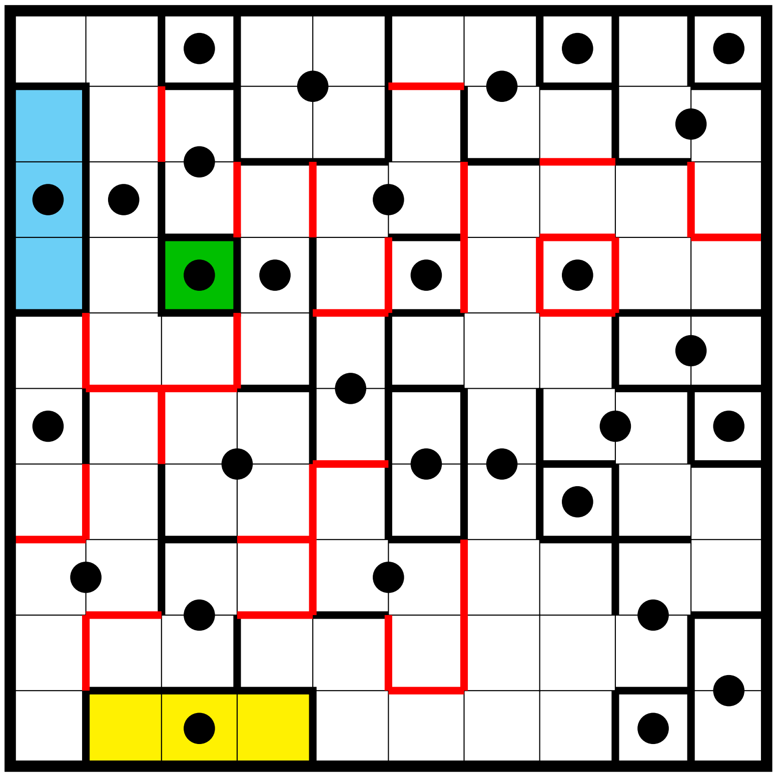


Figure 4: Borderlines gained by several iterations

The remaining three circles (blue) are associated to their corresponding regions via case analysis. The region with the largest area is composed of 18 cells; it is depicted in gray in the following picture (see figure 5).

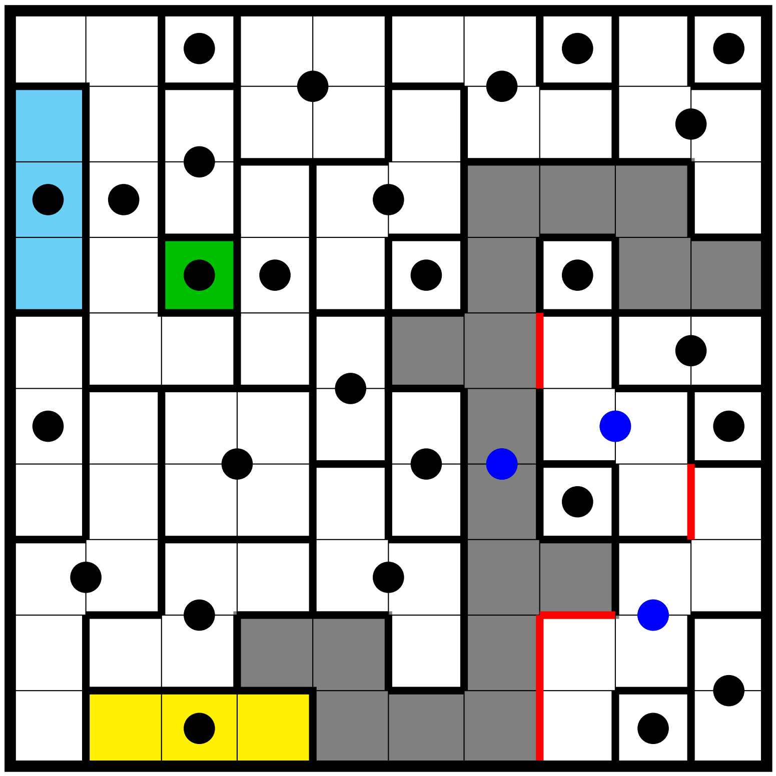


Figure 5: The finished Christmas card

2 Xmasium

Authors: Cor Hurkens (TU Eindhoven),
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Project: 4TU.AMI

2.1 Challenge

In one of Santa's research laboratories, the scientists have discovered a new chemical element and have named it *Xmasium* (by analogy with the famous elements Rubidium, Caesium, and Francium). Xmasium exists in three different types: There is α-Xmasium, β-Xmasium, and γ-Xmasium. If two Xmasium-atoms of different type collide, they sometimes merge into a single atom of the third type. If two Xmasium-atoms of the same type collide, they repel each other and nothing else happens.

Ruprecht puts 91 α-Xmasium-atoms, 25 β-Xmasium-atoms, and 4 γ-Xmasium-atoms into a cooking pot, covers it, and then leaves for lunch. When he returns, he notices that all Xmasium-atoms in the pot now are of the same type.

What is the largest possible number z of Xmasium-atoms in the pot at that moment?

Artwork: Friederike Hofmann

Possible answers:

1. This largest possible number satisfies $20 \leq z \leq 23$.
2. This largest possible number satisfies $24 \leq z \leq 27$.
3. This largest possible number satisfies $28 \leq z \leq 31$.
4. This largest possible number satisfies $32 \leq z \leq 35$.
5. This largest possible number satisfies $36 \leq z \leq 39$.
6. This largest possible number satisfies $40 \leq z \leq 43$.
7. This largest possible number satisfies $44 \leq z \leq 47$.
8. This largest possible number satisfies $48 \leq z \leq 51$.
9. This largest possible number satisfies $52 \leq z \leq 55$.
10. This largest possible number satisfies $56 \leq z \leq 59$.

2.2 Solution

The correct answer is: 3.

We denote the number of α-atoms, β-atoms, and γ-atoms with a, b, and c, respectively. Whenever two Xmasium-atoms of different types merge into one Xmasium-atom of the third type, the *parity* (the property of an integer being divisible by 2) of all three numbers a, b, c changes at the same time: Thereby, two numbers are decreased by 1 and the third numbers is increased by 1.

Since one has $(a, b, c) = (91, 25, 4)$ at the beginning, a and b will always have the same parity, whereas c has the a different parity as b and c. Since at the end of the lunch break two of the three numbers a, b, c are equal to 0, we deduce that $a = b = 0$ and that c must be odd. In particular, all 91 α-atoms must have disappeared, and *at least* 91 fusions have taken place. Hence, of the $91 + 25 + 4 = 120$ atoms, there remain *at most* $120 - 21 = 29$.

Now, we describe a process with exactly 91 fusions:

- First, 25 α- and β-atoms collide:
$$(91, 25, 4) \to (66, 0, 29)$$
- Then, 29 α- and γ-atoms collide:
$$(66, 0, 29) \to (37, 29, 0)$$
- Then, 29 α- and β-atoms collide:
$$(37, 29, 0) \to (8, 0, 29)$$
- Then, 4 α- and γ-atoms collide:
$$(8, 0, 29) \to (4, 4, 25)$$
- Then, 4 α- and β-atoms collide:
$$(4, 4, 25) \to (0, 0, 29)$$

After these $25 + 29 + 29 + 4 + 4 = 91$ fusions, there are no α-atoms, no β-atoms, and exactly 29 γ atoms in the cooking pot. Hence, $z = 29$.

3 Sneaky Elves

Author: Ariane Beier (TU Berlin)
Project: MATH+ School Activities

3.1 Challenge

In reward for their good work today, the five sneaky elves Charlie, Kim, Luca, Mika, and Ulli received big box with finitely many, whole, delicious coconut macaroons from Santa himself. Worn out from the exhausting day, they carry the box home and go to bed immediately while yawning loudly. Now, the box sits on the elves' kitchen table, where it is guarded by the elves' dog Anouk.

At 10:34 p. m., Charlie sneaks into the kitchen, gives a macaroon from the box to Anouk, divides the remaining macaroons into five equal parts (while *not* dividing a single macaroon), pouches one part, rearranges the residual four parts into a big pile, and returns to bed with a cunning grin on the face. At 11:56 p. m., Kim tiptoes into the kitchen, gives a macaroon from the box to Anouk, divides the remaining macaroons into five equal parts (*without* splitting single macaroons), puts one of these parts into the pyjama pockets, shoves the other four parts into a pile, and slips off to bed looking very pleased. At 1:45 a. m., 3:17 a. m., and 4:23 a. m., Luca, Mika, and Ulli, respectively, repeat this procedure.

Despite the wakeful night, the elves are in a very good mood at 9:00 a. m. and meet for breakfast. They offer a breakfast macaroon to Anouk and split the remains of the coconut macaroons equally among themselves (again, dividing *none* of the single macaroons).

How many coconut macaroons does **Kim** *at least* get altogether?

Artwork: Frauke Jansen

Possible answers:

1. 35
2. 352
3. 3,522
4. 35,223
5. 352,230
6. 3,522,300
7. 35,223,004
8. 352.230.042
9. 3,522,300,421
10. 35,223,004,219

3.2 Solution

The correct answer is: 3.

Let n be the number of coconut macaroons at the beginning of the night. Let c, k, l, m, and u be the number of macaroons that are stolen during the night by Charlie, Kim, Luca, Mika, and Ulli, respectively. Futhermore, let j be the number of macaroons that are given to each of the five elves at breakfast. Then,

$$\begin{aligned} n &= 5c + 1, \\ 4c &= 5k + 1, \\ 4k &= 5l + 1, \\ 4l &= 5m + 1, \\ 4m &= 5u + 1, \\ 4u &= 5j + 1. \end{aligned}$$

By adding 4 to both sides of all six equations, one obtains:

$$\begin{aligned} n + 4 &= 5(c + 1), \\ 4(c + 1) &= 5(k + 1) \\ 4(k + 1) &= 5(l + 1), \\ 4(l + 1) &= 5(m + 1), \\ 4(m + 1) &= 5(u + 1), \\ 4(u + 1) &= 5(j + 1). \end{aligned}$$

Multiplying the left- and right-hand sides of the equations, respectively, yields:

$$4^5(n+4)(c+1)(k+1)(l+1)(m+1)(u+1) = 5^6(c+1)(k+1)(l+1)(m+1)(u+1)(j+1).$$

Since $c, k, l, m, u \geq 0$, both sides of the above equation can be divided by $(c + 1)(k + 1)(l + 1)(m + 1)(u + 1)$, and it remains:

$$4^5(n + 4) = 5^6(j + 1).$$

In order for n and j being integers, $n+4$ needs to be divisible by $5^6 = 15.625$ and $j + 1$ by $4^5 = 1.024$. Hence, the smallest possible positive integer that satisfies the equations is $n = 5^6 - 4 = 15.621$. Additionally, for $n = 15.621$, the other unknowns

$$c = 3.124,\ k = 2.499,\ l = 1.999,\ m = 1.599,\ u = 1.279,\ j = 1.023,$$

yield positive integers.

Therefore, the desired number of coconut macaroons obtained by Kim is

$$k + j = 2.499 + 1.023 = \mathbf{3.522}.$$

We note that the coconut macaroons are either very, very small or the elves have huge hiding places for their theft. Futhermore, we hope that the elves are rationing their macaroons and do not get stomach sick...

4 Soccer

Author: Hennie ter Morsche (TU Eindhoven)

Project: 4TU.AMI

4.1 Challenge

Four soccer teams from the cities Icetown, Frostville, Glacierhampton, and Coldbury have participated in the Christmas soccer tournament. During this tournament, each team played exactly one match against each of the other three teams. A win yields three points, a draw one point, and a loss zero points. No two of these six matches ended with the same result. (For example: If some match ended in 3:2, then none of the other five matches ended with 2 goals for one team and 3 goals for the other team.) Here is the final table of the tournament:

Team	Wins	Draws	Losses	Goals : Goals against	Points
Icetown	2	0	1	5 : 1	6
Frostville	2	0	1	3 : 5	6
Glacierhampton	1	0	2	5 : 6	3
Coldbury	1	0	2	4 : 5	3

Which of the following statements is true?

Artwork: Friederike Hofmann

Possible answers:

1. Coldbury has won its match against Glacierhampton 1:0.
2. Coldbury has lost its match against Glacierhampton 0:1.
3. Coldbury has won its match against Glacierhampton 2:0.
4. Coldbury has lost its match against Glacierhampton 0:2.
5. Coldbury has won its match against Glacierhampton 3:0.
6. Coldbury has lost its match against Glacierhampton 0:3.
7. Coldbury has won its match against Glacierhampton 2:1.
8. Coldbury has lost its match against Glacierhampton 1:2.
9. Coldbury has won its match against Glacierhampton 3:1.
10. Coldbury has lost its match against Glacierhampton 1:3.

4.2 Solution

The correct answer is: 9.

During the tournament, each of the four teams played exactly once against the teams from the other three cities. Hence, there were $3+2+1=6$ games. The tournament's table suggests that no game terminated with a draw and $5+3+5+3+4 = 1+5+6+5 = 17$ goals were scored.

Now, we show that in each of the six games at most four goals were scored. To this end, we list all possible results that do not yield a draw and in that at most four goals are scored:

$$1{:}0,\ 2{:}0,\ 3{:}0,\ 2{:}1,\ 4{:}0, \text{ and } 3{:}1. \tag{1}$$

In other words, there are exactly six such results. Moreover, these results add up to exactly 17 goals. Hence, in none of the tournament's matches were more than four goals scored, and the six matches result in 1:0, 2:0, 3:0, 2:1, 4:0, and 3:1.

Below, we list some observations:

- *Frostville* has won two matches, lost one match, and scored a goal difference of -2.

  Suppose the two matches won yield a goal difference of $+x$ and $+y$, and the third match was lost with a goal difference of $-z$. Then, $x, y \geq 1$ and $z \leq 4$, and the goal difference is

  $$x + y - z \geq 1 + 1 - 4 = -2.$$

  This implies that $x = y = 1$ and $z = 4$. Hence, Frostville won two of its matches with a goal diffenrence of $+1$ and lost one of its matches with a goal difference of -4. From the list of the tournament's results (1), there are only the following ones possible:

  $$\text{2 wins: 1:0, 2:1} \quad \text{and} \quad \text{1 loss: 0:4.}$$

- *Icetown* lost only one match and scored only one goal. Therefore, Icetown lost one match 0:1, and we already know that the winner of this match was Frostville.

During the other two matches, Icetown scored five goals in total and received not a single goal against. Since the (4:0)-match was lost by Frostville (and Icetown played only one match against Fostberg), Icetown has to win its matches with 2:0 and 3:0.

- Consequently, the **match between Glacierhampton and Coldbury yields the result 3:1 (or 1:3)**.
- *Coldbury* scored four goals in total. Since Coldbury scored at least one goal during the match against Glacierhampton, it scored at most three goals against Frostberg. Hence, Coldbury did not win 4:0 against Frostville.
- Thus, Frostville lost 0:4 against the last remaining opponent Glacierhampton. For the game against Icetown then remains only a 0:2 defeat.

We summarize the results for *Frostville*:

Frostville lost 0:4 against Glacierhampton, won 1:0 against Icetown, and won its third match 2:1 against Coldbury.

We summarize the situation of *Coldbury*:

Goldbury won 1:2 against Frostville and lost either 0:2 or 0:3 against Icetown. Therefore, **Coldbury won its remaining match against Glacierhampton with 3:1**. The remaining match against Icetown was lost 0:2.

The above discussion can be summarized with the following table:

	Icetown	Frostville	Glacierhampton	Coldbury
Icetown	—	0 : 1	3 : 0	2 : 0
Frostville	1 : 0	—	0 : 4	2 : 1
Glacierhampton	0 : 3	4 : 0	—	1 : 3
Coldbury	0 : 2	1 : 2	**3 : 1**	—

5 Number Game

Authors: Aart Blokhuis (TU Eindhoven)
Gerhard Woeginger (TU Eindhoven)

Project: 4TU.AMI

5.1 Challenge

Santa Clause tells the three super-smart elves Alpha, Beta, and Gamma, "Each of you has received a card that carries a number from $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Your three numbers are pairwise distinct, and the largest one equals the sum of the two smaller ones. Now, please have a look at your cards with your number, but don't show it to the two others!"

Alpha, Beta and Gamma stare at their cards for some time.

1) After some time **Alpha** says, "According to my knowledge, there are at most eight possible candidates for Beta's number."

2) Then, **Beta** says, "According to my current knowledge, there are exactly three possible candidates for Gamma's number."

3) **Gamma** shouts, "I see! Now, I know Alpha's number."

4) **Alpha** thinks about it and says, "I still don't know Beta's number."

5) **Beta** exclaims, "I see! Now, I also do know Alpha's number."

Of course, we would like to know: What is Alpha's number?

Artwork: Frauke Jansen

Possible answers:

1. Alpha's number is 1.
2. Alpha's number is 2.
3. Alpha's number is 3.
4. Alpha's number is 4.
5. Alpha's number is 5.
6. Alpha's number is 6.
7. Alpha's number is 7.
8. Alpha's number is 8.
9. Alpha's number is 9.
10. Alpha's number is 10.

5.2 Solution

The correct answer is: 5.

We denote Alpha's number by A, Beta's number by B, and Gamma's number by C. First, we note: If one knows two of the three numbers A, B, C, then the third number is either the sum or difference of these two numbers. Consequently:

- The two numbers 4 and 8 cannot be contained in $\{A, B, C\}$ at the same time.
- The two numbers 5 and 10 cannot be contained in $\{A, B, C\}$ at the same time.

Now, we analyse the statements of the three elves successively.

What follows from the first statement:

If $A = 4$, then Alpha can immediately deduce that $B \notin \{4, 8\}$. And analogously, Alpha can deduce from $A = 5$, $A = 8$, $A = 10$, that $B \notin \{5, 10\}$, $B \notin \{4, 8\}$, $B \notin \{5, 10\}$, respectively.

If $A \in \{4, 5, 8, 10\}$, then according to Beta there are at most eight candidates for B. Table 1 shows the examples for triples (A, B, C) with $A \in \{1, 2, 3, 6, 7, 9\}$, from which we deduce that there are nine candidates for B for each $A \in \{1, 2, 3, 6, 7, 9\}$.

What follows from the second statement:

Like we did, the elf Beta deduced that $A \in \{4, 5, 8, 10\}$, and to his knowledge, there are now exactly three candidates for C. Table 2 lists all possible situations.

We see that there are six candidates $3, 4, 5, 6, 7, 9$ for C if $B = 1$. In the cases $B = 2$, $B = 3$, $B = 6$, $B = 8$, $B = 10$, there are five, five, four, two, two candidates for C, respectively. Only for $B \in \{4, 5, 7, 9\}$, there are exactly three candidates for C.

What follows from the third statement:

	B=1	B=2	B=3	B=4	B=5	B=6	B=7	B=8	B=9	B=10
A=1	—	(1,2,3)	(1,3,2)	(1,4,5)	(1,5,4)	(1,6,7)	(1,7,6)	(1,8,9)	(1,9,8)	(1,10,9)
A=2	(2,1,3)	—	(2,3,1)	(2,4,6)	(2,5,7)	(2,6,4)	(2,7,9)	(2,8,6)	(2,9,7)	(2,10,8)
A=3	(3,1,4)	(3,2,1)	—	(3,4,1)	(3,5,2)	(3,6,9)	(3,7,4)	(3,8,5)	(3,9,6)	(3,10,7)
A=6	(6,1,5)	(6,2,4)	(6,3,9)	(6,4,2)	(6,5,1)	—	(6,7,1)	(6,8,2)	(6,9,3)	(6,10,4)
A=7	(7,1,6)	(7,2,5)	(7,3,4)	(7,4,3)	(7,5,2)	(7,6,1)	—	(7,8,1)	(7,9,2)	(7,10,3)
A=9	(9,1,8)	(9,2,7)	(9,3,6)	(9,4,5)	(9,5,4)	(9,6,3)	(9,7,2)	(9,8,1)	—	(9,10,1)

Table 1: Candidates B for $A \in \{1, 2, 3, 6, 7, 9\}$.

	B=1	B=2	B=3	**B=4**	**B=5**	B=6	**B=7**	B=8	**B=9**	B=10
A=4	(4,1,**3**) (4,1,**5**)	(4,2,**6**)	(4,3,**1**) (4,3,**7**)	—	(4,5,**1**) (4,5,**9**)	(4,6,**2**) (4,6,**10**)	(4,7,**3**)	—	(4,9,**5**)	(4,10,**6**)
A=5	(5,1,**4**) (5,1,**6**)	(5,2,**3**) (5,2,**7**)	(5,3,**2**) (5,3,**8**)	(5,4,**1**) (5,4,**9**)	—	(5,6,**1**)	(5,7,**2**)	(5,8,**3**)	(5,9,**4**)	—
A=8	(8,1,**7**) (8,1,**9**)	(8,2,**6**) (8,2,**10**)	(8,3,**5**)	—	(8,5,**3**)	(8,6,**2**)	(8,7,**1**)	—	(8,9,**1**)	(8,10,**2**)
A=10	(10,1,**9**)	(10,2,**8**)	(10,3,**7**)	(10,4,**6**)	—	(10,6,**4**)	(10,7,**3**)	(10,8,**2**)	(10,9,**1**)	—

Table 2: Candidates C.

Gamma knows that $A \in \{4, 5, 8, 10\}$ and that $B \in \{4, 5, 7, 9\}$. We quickly discuss the all cases for C:

- For $C \in \{7, 8, 10\}$, there is *no combination* (A, B, C) with $A \in \{4, 5, 8, 10\}$ and $B \in \{4, 5, 7, 9\}$.
- For $C = 1$, there are *four possibilities* for A:

$$A = 4 \text{ (with } B = 5),$$
$$A = 5 \text{ (with } B = 4),$$
$$A = 8 \text{ (with } B = 7 \text{ or } B = 9),$$
$$A = 10 \text{ (with} B = 9).$$

  For $C = 3$, there are *three possibilities* for A:

$$A = 4 \text{ (with } B = 7),$$
$$A = 8 \text{ (with } B = 5),$$
$$A = 10 \text{ (with } B = 7).$$

  For $C = 9$, there are *two possibilities* for A:

$$A = 4 \text{ (with } B = 5),$$
$$A = 5 \text{ (with } B = 4).$$

  Hence, in theses cases Gamma is not able to determine the number A.

- For $C = 2$, only $A = 5$ and $B = 7$ are possible.

  For $C = 4$, only $A = 5$ and $B = 9$ are possible.

  For $C = 5$, only $A = 4$ and $B = 9$ are possible.

  For $C = 6$, only $A = 10$ and $B = 4$ are possible.

Since Gamma knows Alpha's number, we conclude that $C \in \{2, 4, 5, 6\}$. Thus, the possible triples (A, B, C) are

$$(5, 7, 2), \quad (5, 9, 4), \quad (4, 9, 5) \quad \text{und} \quad (10, 4, 6).$$

What follows from the fourth statement:

Alpha knows that the only remaining candidates for (A, B, C) are

$$(5, 7, 2), \quad (5, 9, 4), \quad (4, 9, 5) \quad \text{und} \quad (10, 4, 6).$$

- If $A = 4$, Alpha could deduce $B = 9$.
- If $A = 10$, Alpha could deduce $B = 4$.

Hence, $A = 5$ holds.

What follows from the fourth statement:

As we did, Beta deduced $A = 5$.

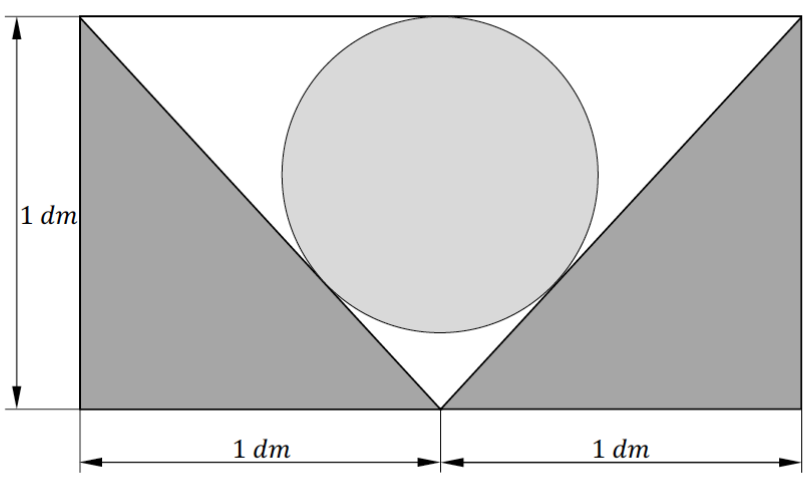
6 Snowball Fight

Author: Oliver Kaufmann (Immanuel-Kant-Gymnasium, HU Berlin)

6.1 Challenge

The ten Christmas elves Elisa, Frida, Gustav, Heinrich, Ida, Johann, Karla, Ludwig, Marta, and Norwin had a lot of work to do during the past weeks: they tinkered, painted, and sawed. But now, all presents are wrapped. Proud of their work, they high-fived each other—when Elisa looks out of the window and notices, “It is snowing a lot!”
“Let’s go outside and have a snowball fight!” Gustav says eagerly. Our ten Christmas elves put on warm clothes, run outside, and have a long and merry snowball fight.

At some point, Ida notices, “Folks! This spherical snowball fits perfectly into the snow-free rain gutter. The snowball touches each of the rain gutter’s edges at exactly one point and lines up precisely with its top.”
This amazing discovery causes the Christmas elves to abruptly end their snowball fight. Since they do not only have a distinct sense of delightful Christmas presents, but also a deep love for Mathematics, they immediately want to calculate the snowball’s volume. To this end, they use the following figure showing the cross section of the rain gutter and snowball (*not to scale*).



All elves begin do think and calculate eagerly. However, they all end up with different results. Who is right?

Illustration: Julia Nurit schönnagel

Possible answers:

1. After a little while, Elisa calculates:
$$V = \frac{\pi}{6}\, dm^3.$$
2. Frida needs a bit more time, but finally she states the following result:
$$V = \sqrt{3}\,\pi\, dm^3.$$
3. Gustav puzzles over the problem patiently and is confident about his result:
$$V = \frac{\sqrt{2}}{6}\,\pi\, dm^3.$$
4. Heinrich is very excited about his result, since he solved such a problem not long ago. He quickly obtains:
$$V = \frac{\pi}{3}\left(\sqrt{2}-1\right) dm^3.$$

5. Ida is very concentrated while calculating. In the end, she gets:
$$V = \frac{4}{3}\pi\left(\sqrt{2}-1\right)^2 dm^3.$$
6. Johann is very absorbed, smiles, and states his result:
$$V = \frac{4}{3}\pi\left(5\sqrt{2}-7\right) dm^3.$$
7. Karla follows a different approach and obtains the following volume formula:
$$V = \left(\frac{20\sqrt{3}-28}{3}\right)\pi\, dm^3.$$
8. Ludwig is very passionate about this task; so he makes his own optimized sketch and is able to present the following result using geometrical and analytical approaches:
$$V = \frac{4}{3}\pi\left(3\sqrt{2}-4\right) dm^3.$$
9. Marta trusts in her mathematical skills and is surprised about the volume she calculated:
$$V = 1\, dm^3.$$
10. Norwin leans back, looks at the sketch for quite a long time, and says, "Without further information, one cannot solve this problem!"

6.2 Solution

The correct answer is: 6.

A **possible solution path** is following an analytical-geometrical approach. To this end, we define a coordinate system ($1\,LE = 1\,dm$) and label certain points (see Fig. 6).

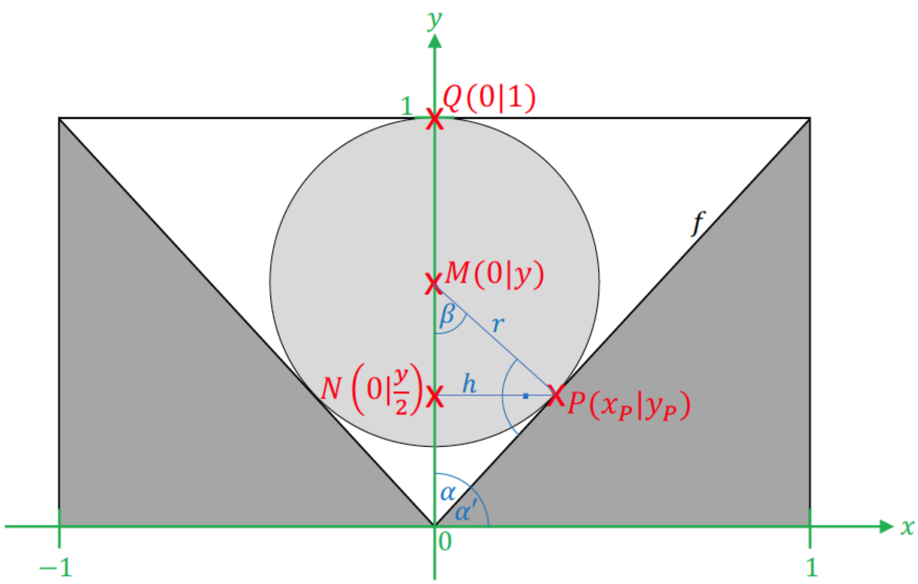


Figure 6: Cross section of the rain gutter and the snowball

First, we want to calculate the radius r of the snowball. For this purpose, we define the following points:

- $\mathbf{M(0 \mid y)}$: the center of the snowball, located on the y-axis,
- $\mathbf{Q(0 \mid 1)}$: point of contact of the snowball and the (imaginary) top edge of the rain gutter, located on the y-axis,
- $\mathbf{P(x_P \mid y_P)}$: point of contact of the snowball and the edge of the rain gutter.

The line segment f can be viewed as part of the graph of a linear function with slope 1. Therefore, the angles α and α' satisfy:

$$\alpha = \alpha' = 45^\circ.$$

Since f is tangential in P at the circle with radius r, r and f are orthogonal, i. e. $\angle MPO = 90°$. It follows that

$$\beta = 180° - \angle MPO - \alpha' = 180° - 90° - 45° = 45°.$$

Hence, the triangle $\triangle MOP$ is isosceles with a right angle at P. Therefore, it holds

$$r = d(M, P) = d(0, P).$$

Furthermore, we conclude that the altitude h on the edge $\overline{MO}$ (with base point N) divides the triangle $\triangle MOP$ into two smaller congruent triangles, where the side $\overline{MO}$ gets bisected, i. e.

$$N = \left(0 \mid \frac{y}{2}\right) \qquad \text{and} \qquad y_P = \frac{y}{2}.$$

Since f is part of the linear function given by $f(x) = x$, it follows:

$$x_P = y_P = \frac{y}{2},$$

Using the Pythagorean theorem, we get for the radius r:

$$\begin{aligned} r &= d(0, P) \\ &= \sqrt{x_p^2 + y_p^2} \\ &= \sqrt{\left(\frac{y}{2}\right)^2 + \left(\frac{y}{2}\right)^2} \\ &= \sqrt{2\,\frac{y^2}{4}} \\ &= \frac{y}{\sqrt{2}}. \end{aligned}$$

On the other hand,

$$r = d(M, Q) = 1\,dm - y,$$

and hence

$$1\,dm - y = \frac{y}{\sqrt{2}}.$$

If we solve the above equation for y, we get

$$y = \frac{\sqrt{2}}{\sqrt{2}+1}\,dm = \frac{\sqrt{2}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\,dm = (2-\sqrt{2})\,dm,$$

and therefore

$$r = 1\,dm - y = \Big(1-(2-\sqrt{2})\Big)\,dm = \Big(\sqrt{2}-1\Big)\,dm.$$

Finally, we determine the snowball's volume using the formula for the volume of a ball with radius r:

$$\begin{aligned}
V &= \frac{4}{3}\,\pi\,r^3 \\
&= \frac{4}{3}\pi\left(\sqrt{2}-1\right)^3 dm^3 \\
&= \frac{4}{3}\pi\left(\sqrt{2}^3 - 3\sqrt{2}^2 + 3\sqrt{2} - 1\right) dm^3 \\
&= \frac{4}{3}\pi\left(2\sqrt{2} - 6 + 3\sqrt{2} - 1\right) dm^3 \\
&= \frac{4}{3}\pi\left(5\sqrt{2} - 7\right) dm^3
\end{aligned}$$

Thus, only Johann calculated the snowball's volume correctly. All other given answers are not correct, which is shown by simple calculations.

Another **way to solve this task** uses the fact that the cross section of the snowball is the inscribed circle of the triangle given by the rain gutter's cross section (see Fig. 7).

For the radius r of the incircle of a right-angled triangle $\triangle ABC$ with hypotenuse c, we are able to prove that

$$r = \frac{a \cdot b}{a+b+c}$$

For this purpose, we partition the triangle $\triangle ABC$ into the three smaller triangles $\triangle ACM$, $\triangle CBM$, and $\triangle ABM$ (see Fig. 8).

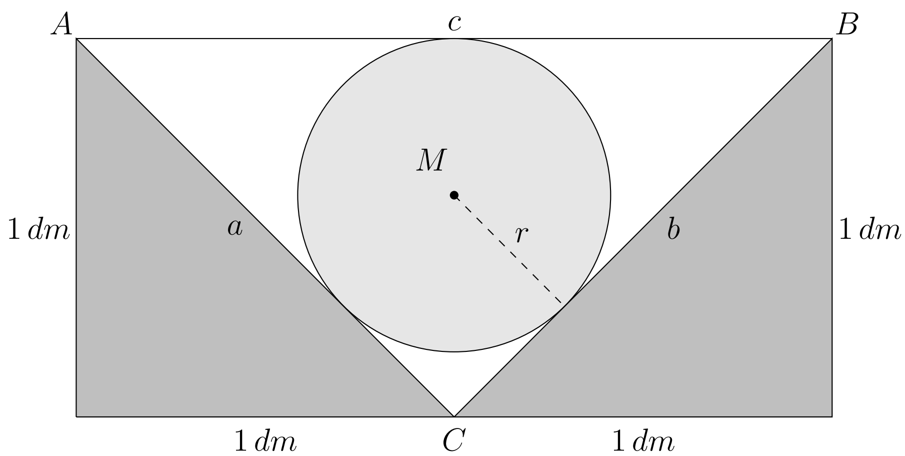


Figure 7: Cross section of the snowball in the rain gutter

For the area of the triangle $\triangle ABC$ and the smaller triangles $\triangle ACM$, $\triangle CBM$ and $\triangle ABM$, one obtains

$$area(\triangle ABC) = area(\triangle ACM) + area(\triangle CBM) + area(\triangle ABM) \qquad (2)$$

and

$$area(\triangle ABC) = \frac{1}{2}\, a \cdot b,$$

$$area(\triangle ACM) = \frac{1}{2}\, a \cdot h_a = \frac{1}{2}\, a \cdot r,$$

$$area(\triangle CBM) = \frac{1}{2}\, b \cdot h_b = \frac{1}{2}\, b \cdot r,$$

$$area(\triangle ABM) = \frac{1}{2}\, c \cdot h_c = \frac{1}{2}\, b \cdot r.$$

Plugging in these results into Equation (2), we get

$$\frac{1}{2}\, a \cdot b = \frac{1}{2}\, a \cdot r + \frac{1}{2}\, b \cdot r + \frac{1}{2}\, c \cdot r$$
$$= \frac{1}{2}\, r\,(a + b + c).$$

The above equation is equivalent to

$$r = \frac{a \cdot b}{a + b + c}.$$

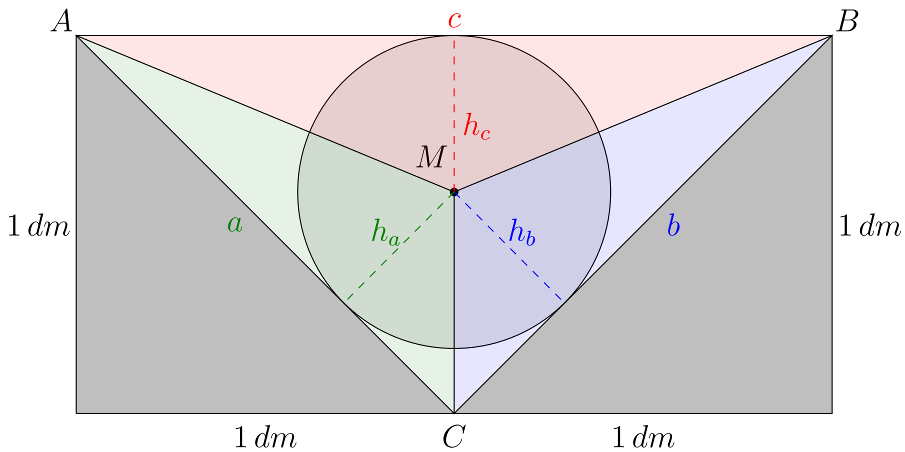


Figure 8: The three smaller triangles within $\triangle ABC$

Now, we are able to determine the radius r of our snowball using the following formula:

$$\begin{aligned} r &= \frac{a \cdot b}{a+b+c} \\ &= \frac{\sqrt{2}\,dm \cdot \sqrt{2}\,dm}{\sqrt{2}\,dm + \sqrt{2}\,dm + 2\,dm} \\ &= \frac{1}{\sqrt{2}+1}\,dm \\ &= \frac{\sqrt{2}-1}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}\,dm \\ &= \left(\sqrt{2}-1\right)dm. \end{aligned}$$

As above, we obtain

$$V = \frac{4}{3}\pi\left(5\sqrt{2}-7\right)dm^3.$$

Remark: In general, one can show that the following holds for the radius r of the incircle of an arbitrary triangle $\triangle ABC$:

$$r = \frac{2 \cdot area(\triangle ABC)}{a+b+c}.$$

7 Dice

Authors: Judith Keijsper (TU Eindhoven)
Hennie ter Morsche (TU Eindhoven)

Project: 4TU.AMI

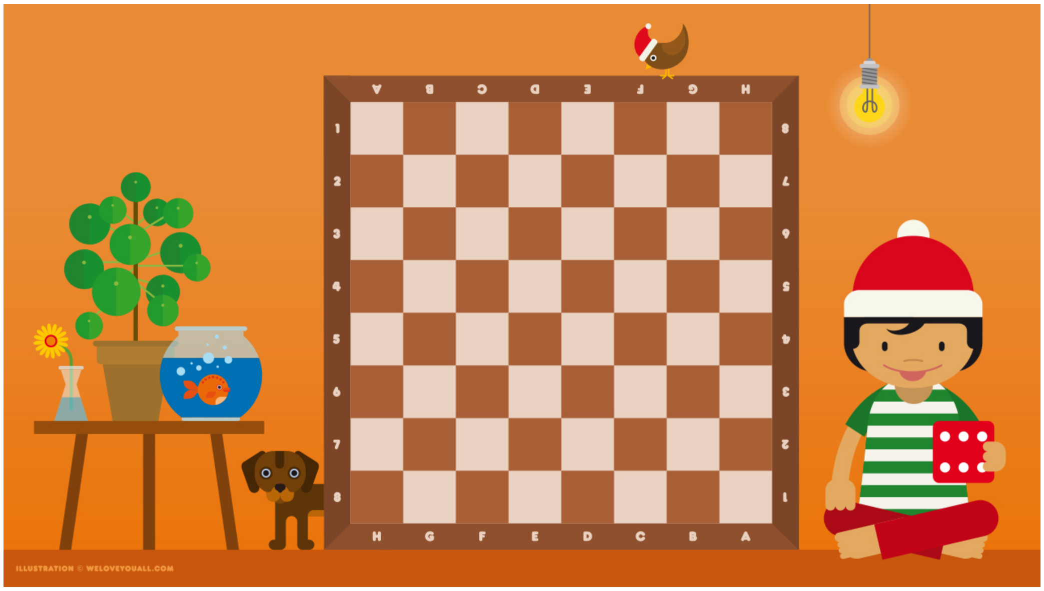
7.1 Challenge

The counting elf Zacharias owns a big red dice (the classical variant available all over the world). Its six faces show $1, 2, 3, 4, 5, 6$ pips, respectively, and the numbers of pips on two opposite faces add up to the sum 7. Almost dozing off, Zacharias is sitting in front of a huge 101×101 chessboard, when he suddenly notices that the squares on the chessboard are precisely the same size as the faces of his dice. All at once, he is wide awake.

Zacharias places his dice onto the southwesternmost square of the chessboard and memorizes the number of pips on the top face of the dice. Then, he tilts the dice to the adjacent square in the north or to the adjacent square in the east and again memorizes the number of pips on the top face. He keeps tilting and tilting and tilting and tilting the dice—again and again—always to the adjacent square in the north or in the east, until the dice finally reaches the northeasternmost square on the chessboard. Now, Zacharias has altogether memorized 201 numbers of pips, and he writes the *sum* of these 201 numbers into his notebook.

Then, he repeats the entire procedure, and again writes the sum of the resulting 201 numbers into his notebook. In this way, Zacharias proceeds for several days, filling his notebook with hundreds of sums.

Our question: What is the largest possible number of **different** sums that may show up in Zacharias' notebook?



Artwork: Friederike Hofmann

Possible answers:

1. The maximal number of different sums is 6.
2. The maximal number of different sums is 10.
3. The maximal number of different sums is 12.
4. The maximal number of different sums is 24.
5. The maximal number of different sums is 120.
6. The maximal number of different sums is 216.
7. The maximal number of different sums is 256.
8. The maximal number of different sums is 720.
9. The maximal number of different sums is 1006.
10. The maximal number of different sums is 1206.

7.2 Solution

The correct answer is: 2.

Let us first understand how the sums that Zacharias writes down in his notebook can be composed: Suppose the top face of the dice shows x pips at the moment. If Zacharias tilts the dice twice to the north or twice to the east, then $7-x$ pips show on the top face. If Zacharias first tilts the dice once to the north and then only to the east as often as he likes, the face with x pips always remains on the backside of the dice; the dice then shows neither x nor $7-x$ pips. If Zacharias first tilts the dice to the east and then only to the north as often as he likes, you can see with a symmetric argument that the dice shows neither x nor $7-x$ pips.

The following can be derived from the preceding paragraph: If, during a dice's journey over the entire chessboard, one only records the two faces with x pips and with $7-x$ pips, these two sides always alternate. Furthermore, between two occurrences of the number of pips "x", the number of pips "$7-x$" must always occur. Therefore, we know the approximate form of the dice's journey:

- The numbers 1 and 6 alternate (perhaps with other numbers in between). Thus, you can combine these two numbers in pairs with a sum of 7. At the very end, perhaps a single number 1 or a single number 6 remains unpaired.
- Similarly, you can summarize the numbers 2 and 5 in pairs with a sum of 7. At the end, a single number 2 or 5 may remain unpaired.
- Also, the numbers 3 and 4 can be combined in pairs with the sum 7. A single number 3 or 4 may remain unpaired.

Hence, the 201 numbers can be divided into 99 pairs with sum 7 and into three remaining numbers —two of these three numbers can form another pair with a sum of 7. (Note that it is not possible to have four or more remaining numbers, since out of any four of the numbers $1, 2, 3, 4, 5, 6$ there is at least one pair adding up to 7.)

The 99 pairs contribute exactly $7 \cdot 99 = 693$ to the total sum, and the three single numbers contribute at least $1+2+3=6$ and at most $4+5+6=15$.

Therefore, Zacharias may only have written down the ten values

$$693 + 6 = 699, \quad \dots \quad , 693 + 15 = 708$$

in his notebook.

Now, we give examples for the dice journeys, in which the sums are the 10 values between $693 + 6 = 699$ and $693 + 15 = 708$.

1. Let $x \in \{1, 2, 3, 4, 5, 6\}$:

   We start with the number x on the top face. Then, we roll the dice 100-times to the east, and afterwards 100-times to the north. The resulting sum is x plus $50 \cdot 7$ plus $50 \cdot 7$, and therefore amounts to $700 + x$.

   In doing so, we take care of the six values 701, 702, 703, 704, 705, 706.

2. Now, we consider a dice's vertex with the adjacent faces—in clockwise direction—with the numbers x, y, and z (where $x, y, z \in \{1, 2, 3, 4, 5, 6\}$):

   We start with the number x on the top face. After tilting the dice once to the north, one sees y on the top face. Tilting once to the east, one gets z on the top face.
   Then, to the east (plus $7 - y$), to the east (plus $7 - z$), to the north (plus $7 - x$), to the north (plus z), to the east (plus y), and to the north (plus x). These first nine steps yield a sum of $x + y + z + 21$.
   Now, we roll the dice another 96-times to the east and then 96-times to the north. In total, one gets

   $$x + y + z + 21 + 96 \cdot 7 = x + y + z + 693.$$

   Since a typical dice includes the four vertices

   $$\begin{aligned} (x, y, z) &= (1, 2, 3) \quad \text{with} \quad x + y + z = 1 + 2 + 3 = 6, \\ (x, y, z) &= (1, 4, 2) \quad \text{with} \quad x + y + z = 1 + 4 + 2 = 7, \\ (x, y, z) &= (3, 6, 5) \quad \text{with} \quad x + y + z = 3 + 6 + 5 = 14, \\ (x, y, z) &= (4, 5, 6) \quad \text{with} \quad x + y + z = 4 + 5 + 6 = 15, \end{aligned}$$

   this type of journey takes care of the remaining four values 699, 700, 707 and 708.

Summarizing, we get the **10 different sums**

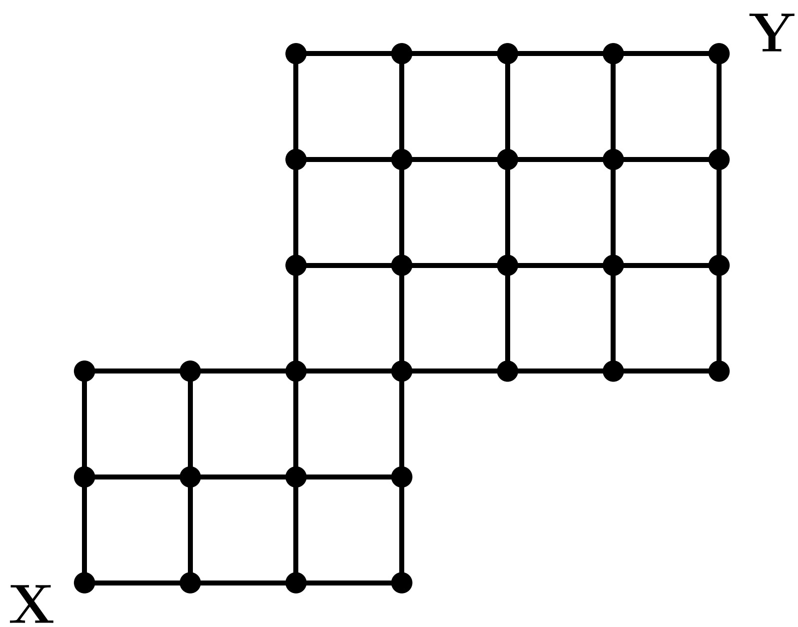
$$699, \dots, 708.$$

8 Meeting Point

Author: Cor Hurkens (TU Eindhoven)
Project: 4TU.AMI

8.1 Challenge

In the following figure, the distance between any two (horizontally or vertically) adjacent points is 1 km.



Ruprecht starts with his sledge in point X and drives to point Y; the Grinch starts at the same time with his sledge in Y and drives to X. Both sledges drive alongside the grid pictured above with the same constant velocity. Both choose—completely independently of each other—a route of length 11 km. These choices happen completely randomly, i. e. all possible ways of length 11 km from X to Y (and Y to X, resp.) are chosen with the same probability.

What is the probability p (rounded to the third decimal place) that Ruprecht and the Grinch meet each other on their routes?

Artwork: Frauke Jansen

Possible answers:

1. $p \approx 0.263$ holds.
2. $p \approx 0.268$ holds.
3. $p \approx 0.274$ holds.
4. $p \approx 0.279$ holds.
5. $p \approx 0.283$ holds.
6. $p \approx 0.288$ holds.
7. $p \approx 0.291$ holds.
8. $p \approx 0.296$ holds.
9. $p \approx 0.302$ holds.
10. $p \approx 0.307$ holds.

8.2 Solution

The correct answer is: 7.

If Ruprecht and the Grinch meet, then the meeting point is located exactly 5.5 km apart from both X and Y. The only possible meeting points are displayed in figure 9.

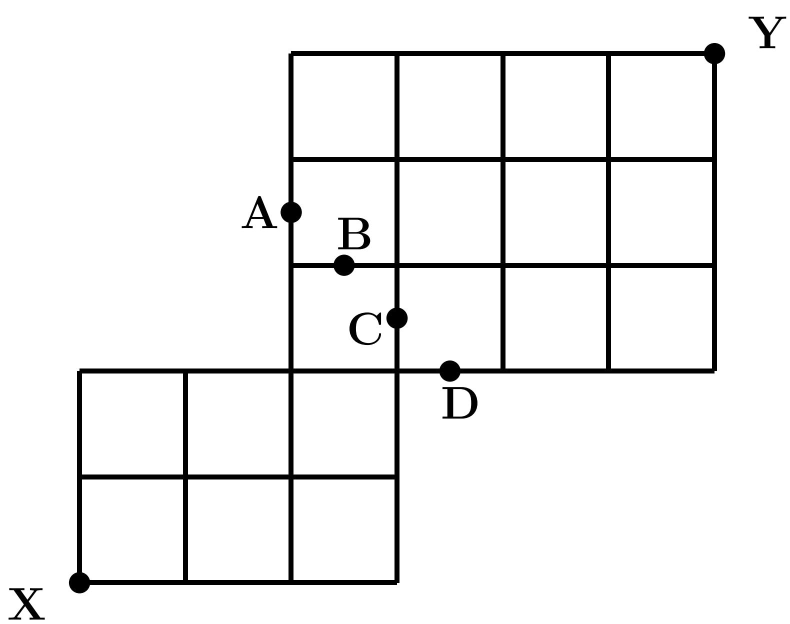


Figure 9: Mögliche Treffpunkte A, B, C, D.

To simplify the subsequent argument, we endow the grid with a coordinate system originating at X. Note that the number of shortest ways from the origin $X = (0, 0)$ to a grid point (n, m) is simply the sum of the number of shortest ways from X to $(n - 1, n)$ and the number of shortest ways from X to $(n, m - 1)$. For example, the number of shortest ways from X to the grid point $(1, 1)$ (see figure 10) is the sum of the number of shortest ways from X to $(0, 1)$ and the number of shortest ways from X to $(1, 0)$:

$$1 + 1 = 2.$$

Analogously, the number of shortest ways from X to the grid point $(3, 2)$ (see figure 10) is

$$6 + 4 = 10.$$

Figure 10 displays the number of shortest ways from X to other points in lower half of the grid:

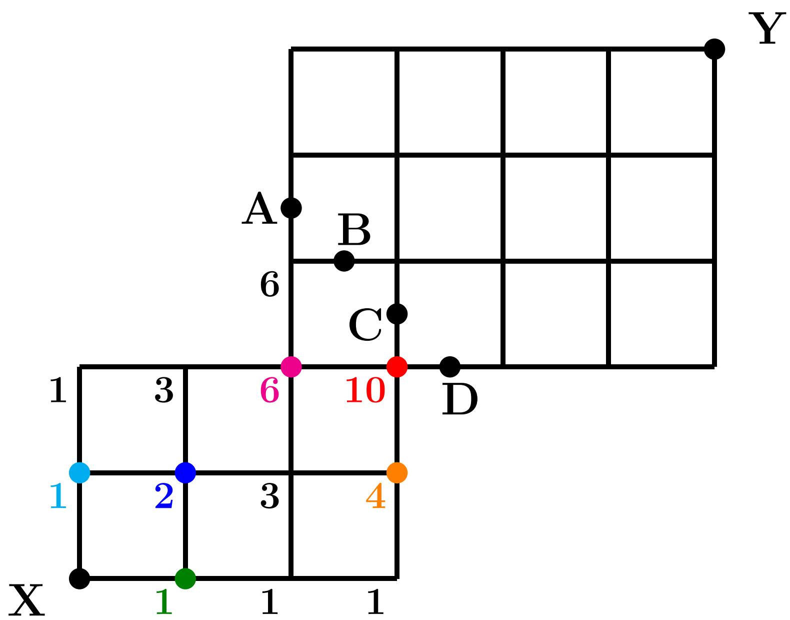


Figure 10: Anzahl der kürzesten Wege von X zu A, B, C, D.

Figure 11 shows the number of shortest ways from X to several points in the lower half of the grid, as well as the number of shortest ways from Y to several points in the upper half of the grid.

Hence, we deduce from figure 11 that there are exactly

- $6 \cdot 5 = 30$ ways of length 11 from X to Y through A;
- $6 \cdot 10 = 60$ ways of length 11 from X to Y through B;
- $10 \cdot 10 = 100$ ways of length 11 from X to Y through C;
- $10 \cdot 10 = 100$ ways of length 11 from X to Y through D.

Accordingly, the total number of ways of length 11 km from X to Y is

$$30 + 60 + 100 + 100 = 290.$$

Thus, the probability for both Ruprecht and the Grinch

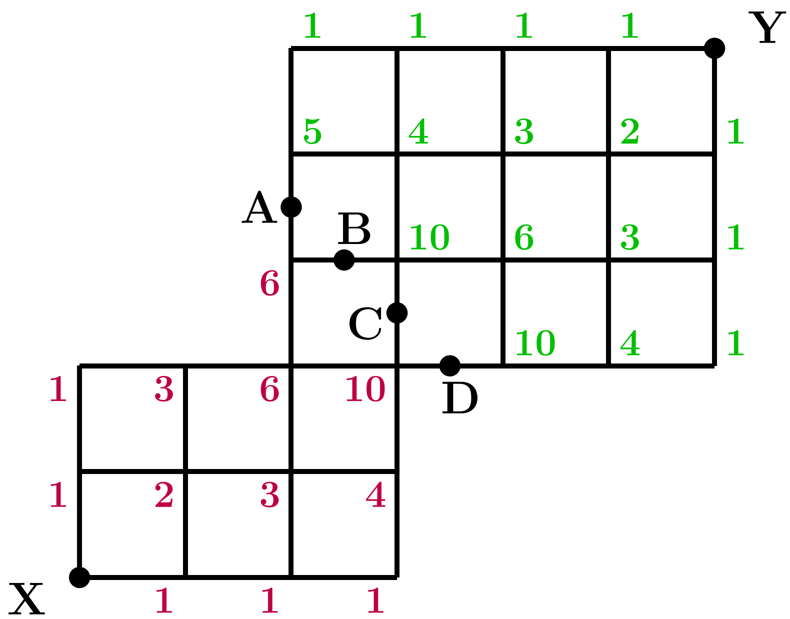


Figure 11: Anzahl der kürzesten Wege von X und Y zu A, B, C, D.

- driving through A is $\frac{30}{290} \cdot \frac{30}{290} = \left(\frac{3}{29}\right)^2$;
- driving through B is $\frac{60}{290} \cdot \frac{60}{290} = \left(\frac{6}{29}\right)^2$;
- driving through C is $\frac{100}{290} \cdot \frac{100}{290} = \left(\frac{10}{29}\right)^2$;
- driving through D is also $\frac{100}{290} \cdot \frac{100}{290} = \left(\frac{10}{29}\right)^2$.

In conclusion, the probability p is given by

$$\begin{aligned} p &= \left(\frac{3}{29}\right)^2 + \left(\frac{6}{29}\right)^2 + \left(\frac{10}{29}\right)^2 + \left(\frac{10}{29}\right)^2 \\ &= \frac{9 + 36 + 100 + 100}{841} \\ &= \frac{245}{841} \approx \mathbf{0{,}291}3198573. \end{aligned}$$

9 Palm Wine

Authors: Aart Blokhuis (TU Eindhoven)
Cor Hurkens (TU Eindhoven)

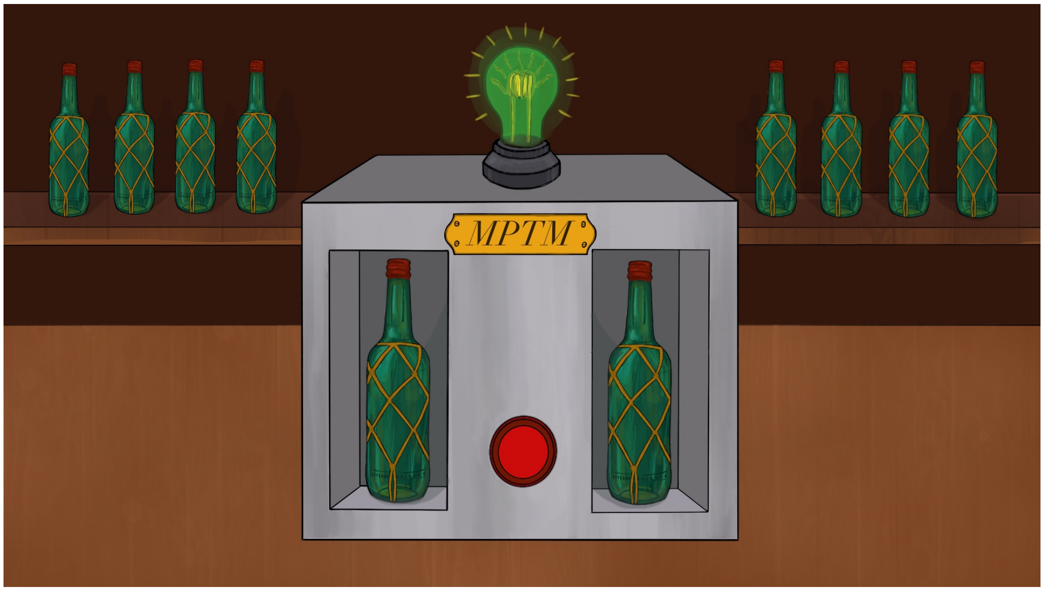
Project 4TU.AMI

9.1 Challenge

Ruprecht has ten indistinguishable sealed bottles in his wine cellar. Ruprecht knows that some even number $n \geq 2$ of these bottles contain delicious palm wine, whereas the other $10 - n$ bottles contain highly toxic acid. Unfortunately, Ruprecht has forgotten the exact value of n.

Ruprecht's Magical Palm Wine Testing Machine (MPWTM) has two compartments, a huge red button, and a light bulb. If he puts one bottle into each of the compartments and then presses the red button, the MPWTM wakes up and starts to work. After one hour, the bulb lights up in green or red: If the bulb lights up in green, then at least one of the two bottles contains delicious palm wine. If the bulb lights up in red, then neither of the two bottles does contain palm wine.

Ruprecht wants to give two bottles of palm wine to Santa Claus. How often does he have to use the MPWTM (in the worst case) in order to identify two bottles containing palm wine?



Artwork: Frauke Jansen

Possible answers:

1. 11 times.
2. 13 times.
3. 15 times.
4. 17 times.
5. 19 times.
6. 21 times.
7. 25 times.
8. 28 times.
9. 36 times.
10. 44 times.

9.2 Solution

The correct answer is: 4.

First, we show that 17 tests are **sufficient**:

Ruprecht denotes the ten bottles by $F_1, \ldots, F_{10}$ and puts the nine pairs

$$(F_1, F_2), (F_1, F_3), (F_1, F_4), (F_1, F_5), (F_1, F_6), (F_1, F_7), (F_1, F_8), (F_1, F_9)(F_1, F_{10}).$$

Case 1: One of the tests provokes the red light.

Ruprecht concludes that bottle F_1 contains acid. Since at least two bottles contain palm wine, at least two pairs $(F_1, F_j), (F_1, F_k)$, with $2 \leq j, k \leq 10,\ j \neq k$, provoke the green light. These bottles F_j and F_k contain palm wine, and Ruprecht can terminate the search after nine tests (in this fortunate case).

Case 2: All of the nine tests provoke the green light.

Ruprecht concludes that bottle F_1 contains the delicious palm wine. (Otherwise, the nine bottles $F_2, \ldots, F_{10}$ would contain palm wine, which would be a contradiction to the number of bottles containing palm wine being even.) Hence, Ruprecht puts F_1 aside and tests the eight pairs

$$(F_2, F_3), (F_2, F_4), (F_2, F_5), (F_2, F_6), (F_2, F_7), (F_2, F_8), (F_2, F_9), (F_2, F_{10}).$$

Case 2a: One of the eight tests provokes the red light.

As in case 1, Ruprecht concludes that F_2 contains acid and that at least one of the pairs (F_2, F_l), $2 \leq l \leq 10$, provokes the green light. This bottle F_l is a second bottle containing palm wine, and the search can be terminated after $9 + 8 = 17$ tests.

Case 2b: All of the eight tests provoke the green light.

As above, Ruprecht concludes that F_2 must contain palm wine. (Otherwise, the nine bottles F_1, and $F_3, \ldots, F_{10}$ would contain palm wine, which would be a contradiction to the number of bottles containing palm wine being even.) Knecht Ruprecht can terminate the search after $9 + 8 = 17$ tests and present Santa with the bottles F_1 and F_2.

Now, we will show that 17 tests are indeed—in a worst case—**necessary**:

To this end, let us assume that 16 tests have provoked the green light. In this case, Ruprecht cannot be sure that the two (fixed) bottles F_1 and F_2 contain palm wine.

Case 1: The pair (F_1, F_2) has been tested.

Since Ruprecht performed only 16 tests, one of the 16 tests

$$(F_1, F_3), (F_1, F_4), (F_1, F_5), (F_1, F_6), (F_1, F_7), (F_1, F_8), (F_1, F_9), (F_1, F_{10}),$$
$$(F_2, F_3), (F_2, F_4), (F_2, F_5), (F_2, F_6), (F_2, F_7), (F_2, F_8), (F_2, F_9), (F_2, F_{10})$$

was omitted. Denote this pair by (F_j, F_k), where $1 \leq j \leq 2$ and $3 \leq k \leq 10$. Then, the two bottles F_j (which is either F_1 or F_2) and F_k may contain acid, whereas the other eight bottles may contain palm wine.

Case 2: The pair (F_1, F_2) has *not* been tested.

In this case, the two bottles F_1 and F_2 may contain acid and the other eight bottles may contain palm wine.

In both cases, at least one of the bottles F_1, F_2 may contain acid. Hence, Ruprecht is not able to solve the problem with only 16 tests.

10 Rundeer

Author: Christian Hercher (Europa-Universität Flensburg)

10.1 Challenge

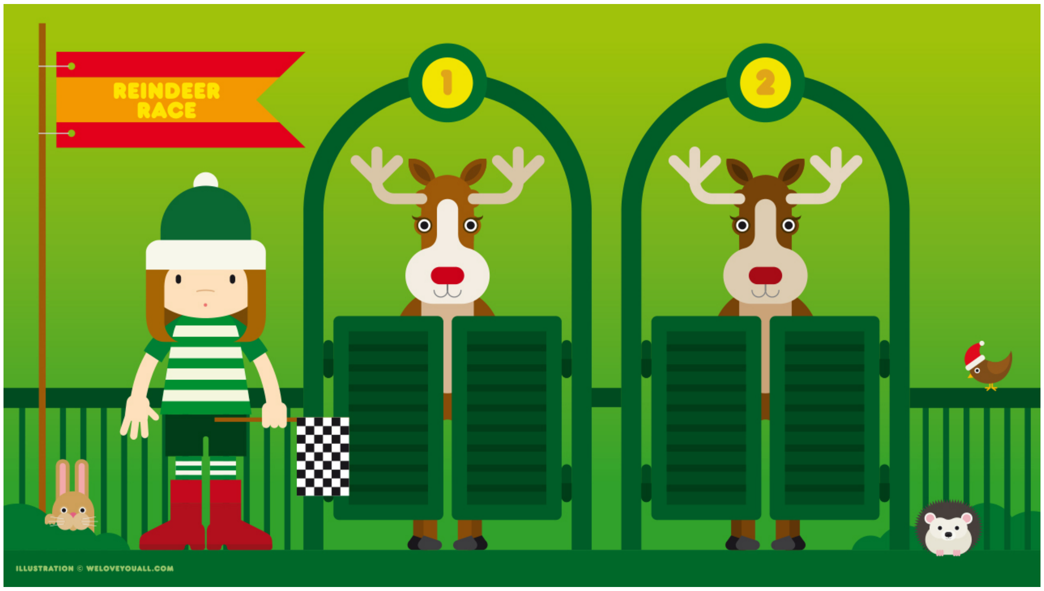
Christmas is approaching fast and with it the most important day of the year for the reindeer pulling Santa's sleigh. As every year, the reindeer quarrel about who should be at the front of the team. In order to make an objective choice, an individual time trial is scheduled, where the reindeer are supposed to individually run on a track one after the other. The fastest one wins the desired spot at the front of the team.

Of course, each reindeer knows the running times of the ones starting beforehand. All reindeer are equally strong; that is, if the reindeer plan to achieve a certain time, then they all accomplish this goal with the *success-probability* p—which is the same for each reindeer—whereas they all run out off breath and accordingly have to abort their race with the same probability $1-p$. (Of course, p decreases with increasing speed.) In case of a tie, the reindeer that started later is declared to be the winner.

(a) Rudolph runs against one other reindeer, but has to go first. In order to maximize the probability of winning the race, how should he choose his speed? (We are asking for the success-probability p corresponding to this planned speed.)

(b) This time, Rudolph has to run against his whole herd, which consists of (including Rudolph) ten reindeer. Again, Rudolph has to run first. If all reindeer pursue an optimal strategy, with what probability W will Rudolph win this race?

Follow-up question: Again suppose that Rudolph competes in a field consisting of ten reindeer—but now, he starts *last*. With what probability will he win this race?



Artwork: Friederike Hofmann

Possible answers:

1. (a) $p = \frac{1}{3}$ and (b) $0\,\% < W \leq 2\,\%$.
2. (a) $p = \frac{1}{3}$ and (b) $2\,\% < W \leq 4\,\%$.
3. (a) $p = \frac{1}{3}$ and (b) $4\,\% < W \leq 5\,\%$.
4. (a) $p = \frac{1}{2}$ and (b) $0\,\% < W \leq 2\,\%$.
5. (a) $p = \frac{1}{2}$ and (b) $2\,\% < W \leq 3\,\%$.
6. (a) $p = \frac{1}{2}$ and (b) $3\,\% < W \leq 4\,\%$.
7. (a) $p = \frac{1}{2}$ and (b) $4\,\% < W \leq 5\,\%$.
8. (a) $p = \frac{2}{3}$ and (b) $0\,\% < W \leq 2\,\%$.
9. (a) $p = \frac{2}{3}$ and (b) $2\,\% < W \leq 4\,\%$.
10. (a) $p = \frac{2}{3}$ and (b) $4\,\% < W \leq 5\,\%$.

10.2 Solution

The correct answer is: 6.

(a) In order to win the race, Rudolph needs to cross the finish line. He achieves this goal with the (still to be determined) probability p. If this is the case, then the second reindeer would start its race with the same speed since it would win in case of a draw. So that this does not happen, the second reindeer has to not finish the race, which happens at a with probability $1 - p$. Therefore Rudolph wins the race with probability

$$W(p) = p(1 - p).$$

Now, we need to maximize this probability (in Rudolph's interest). The graph of the function $W(p)$ is a downward opened parabola with roots 0 and 1. The parabola's vertex is $p = \frac{1}{2}$, and it is a maximum of the function (see Fig 12). Thus, Rudolph should run at a speed, which allows him to finish the race with probability $p = \frac{1}{2}$ in order to maximize his chance to win.

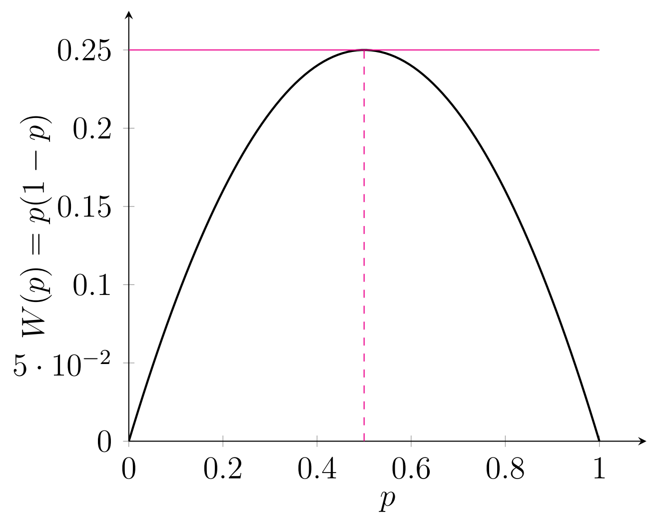


Figure 12: The probability function $W(p) = p(1 - p)$.

(b) Again, Rudolph has to finish the race in order to have a chance to win. He achieves this goal with probability p. If any of the nine other reindeer finishes the race after Rudolph, Rudolph does *not* win the race. Thus, for every reindeer starting after Rudolph, only Rudolph has reached the finish line by now and it will therefore choose the same speed as Rudolph (since it would win in case of a tie). Each of the nine reindeer will fail to finish with a probability of $1-p$. All nine reindeer will therefore fail to reach the finishing line with a probability of $(1-p)^9$. This results in the following probability of Rudolph winning,

$$W(p) = p(1-p)^9,$$

which we now want to maximise.

Anyone knowing differential calculus is able to determine the derivative of W,

$$W'(p) = (1-p)^9 - 9p(1-p)^8 = (1-p)^8(1-p-9p) = (1-p)^8(1-10p),$$

and its zeros $p_{10} = \frac{1}{10}$ and $p_1 = 1$. Then, by inserting the zero p_0 of W' into the second derivative of W,

$$W''(p) = 18(5p-1)(1-p)^7,$$

one verifies that $p_1 = \frac{1}{10}$ is indeed a maximum of W:

$$W''(p_0) = 18\left(5\tfrac{1}{10} - 1\right)\left(1 - \tfrac{1}{10}\right)^7 = -\frac{43046721}{10000000} = -4{,}3046721 < 0.$$

Finally, the function value is calculated:

$$W = W(p_0) = \tfrac{1}{10}\left(1 - \tfrac{1}{10}\right)^9 = \frac{387420489}{10000000000} = 0{,}0387420489 = 3{,}87420489\,\%$$

Hence, Rudolph wins the race with a probability of about $W = 3{,}9\,\%$.

If you are not (yet) able to differentiate, you can also analyse the graph of the function W (see Figure 13) and narrow down the winning probability W: $3\,\% < W \leq 4\,\%$.

Follow-up question: Here, one has to consider the strategy of all reindeer, which changes depending on the results of the reindeer running in front of it:

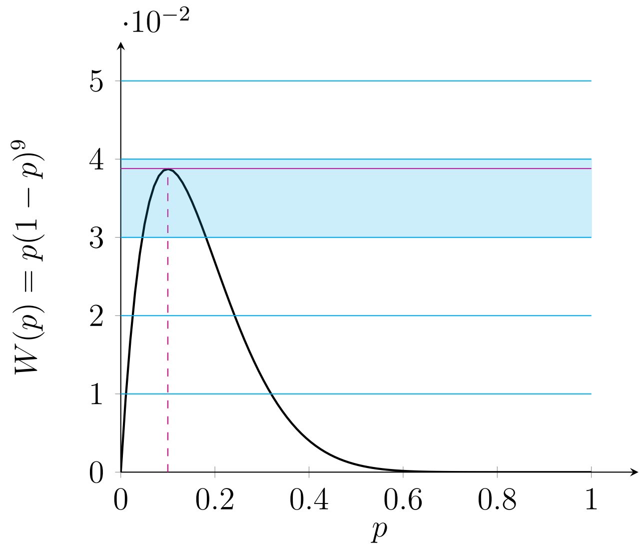


Figure 13: The probabilty function $W(p) = p(1-p)^9$.

To this end, consider the reindeer with the starting grid k, $1 \leq k \leq 10$. If none of the first $k-1$ reindeer has reached the finish line, then the k-th reindeer is in the same situation as a reindeer with the first starting place in a race with a total of $10-(k-1) = 11-k$ participating reindeer. It should therefore—analogous to (a) and (b)—run at a speed such that the probability

$$W(p) = p(1-p)^{11-k-1} = p(1-p)^{10-k}$$

is maximal. In order to maximise W, one calculates the derivative W (as done in (b)),

$$W'(p) = ((k-11)p+1)(1-p)^{9-k},$$

and its zeros $p_k = \frac{1}{11-k}$ and $p_* = 0$. Since

$$W''(p_k) > 0,$$

p_k is indeed a maximum of the probability function W with

$$W(p_k) = \tfrac{1}{11-k}\left(1 - \tfrac{1}{11-k}\right)^{10-k}.$$

The k-th reindeer should run with the speed that corresponds to the success-probability $p_k = \frac{1}{11-k}$. If this k-th reindeer reaches the finish line, then the succeeding reindeer will also choose this speed (corresponding to the success-probability p_k), since they would win in case of a tie. If the k-th reindeer does not finish its race, then reindeer # $(k+1)$ will choose its speed corresponding to the success-probability $p_{k+1} = \frac{1}{10-k}$. And so on and so forth.

Accordingly, Rudolph will win the race if he runs as fast as the first reindeer that came in (before him). The probability that the first reindeer *and* Rudolph will finish is

$$\tfrac{1}{10} \cdot \tfrac{1}{10}.$$

The probability that the first reindeer will not finish, but the second reindeer *and* Rudolph will finish is

$$\tfrac{9}{10} \cdot \tfrac{1}{9} \cdot \tfrac{1}{9} = \tfrac{1}{10} \cdot \tfrac{1}{9}.$$

The probability that the first and second reindeer will not finish, but the third *and* Rudolph will, is

$$\tfrac{9}{10} \cdot \tfrac{8}{9} \cdot \tfrac{1}{8} \cdot \tfrac{1}{8} = \tfrac{1}{10} \cdot \tfrac{1}{8}.$$

And so on and so forth. The probability that the first $k-1$ reindeer will not make it to the finish line, but the k-th *and* Rudolph will

$$\tfrac{1}{10} \cdot \tfrac{1}{11-k},$$

where $1 \leq k \leq 9$. The probability that none of the reindeer will finish before Rudolph, but Rudolph will is

$$\tfrac{9}{10} \cdot \tfrac{8}{9} \cdot \dots \cdot \tfrac{1}{2} \cdot 1 = \tfrac{1}{20}.$$

In total, the probability that Rudolph will win the race of the ten reindeer is

$$\tfrac{1}{10} \cdot \left(\tfrac{1}{10} + \tfrac{1}{9} + \cdots + \tfrac{1}{2} + 1\right) = \tfrac{7381}{25200} = 0{,}2928\overline{968253} \approx 29{,}29\,\%,$$

and thus is a good 7.5 times as high as the probability that he will win if he starts first.

11 Piece of Cake

Author: Frits Spieksma (TU Eindhoven)
Project: 4TU.AMI

11.1 Challenge

Ruprecht has prepared a huge cake and invited 100 elves to a cake eating party. The elves are numbered from 1 to 100 and visit Ruprecht one by one in the afternoon.

- The elf with number 1 receives 1 % of the cake.
- The elf with number 2 receives 2 % of the rest of the cake.
- The elf with number 3 receives 3 % of the rest of the cake.
- The elf with number 4 receives 4 % of the rest of the cake.
- And so on.
- The elf with number k receives k % of the rest of the cake.
- And so on.
- The elf with number 99 receives 99 % of the rest of the cake.
- The elf with number 100 finally receives 100 % and hence all the remaining cake.

The elf with number N has received the largest piece of cake. Which of the following statements about this number N is correct?

Artwork: Frauke Jansen

Possible answers:

1. The unit digit of N is 1.
2. The unit digit of N is 2.
3. The unit digit of N is 3.
4. The unit digit of N is 4.
5. The unit digit of N is 5.
6. The unit digit of N is 6.
7. The unit digit of N is 7.
8. The unit digit of N is 8.
9. The unit digit of N is 9.
10. The unit digit of N is 0.

11.2 Solution

The correct answer is: 10.

By S_k we denote the size of the cake received by the k-th elf, and we call R_k the remaining part of the cake after the elf with the number k has received his part.

One can easily see that $S_1 = 1/100$ and $R_1 = 99/100$. Since the k-th elf receives exactly $k\%$ of R_{k-1}, one has

$$R_k = \left(1 - \frac{k}{100}\right) R_{k-1} = \frac{100-k}{100} R_{k-1}$$

for $k \geq 2$. Together with $R_1 = 99/100$, one has

$$R_k = \frac{99}{100} \cdot \frac{98}{100} \cdot \frac{97}{100} \cdots \frac{100-k}{100} = \frac{99!}{(99-k)!\ 100^k}$$

for $1 \leq k \leq 99$. Since the k-th elf receives exactly $k\%$ of R_{k-1} the following also applies for $1 \leq k \leq 100$:

$$S_k = \frac{k}{100} R_{k-1} = \frac{k}{100} \cdot \frac{99!}{(100-k)!\ 100^{k-1}} = \frac{99! \cdot k}{(100-k)!\ 100^k}. \tag{3}$$

Now, we want to determine for which k the inequality

$$S_k < S_{k+1} \tag{4}$$

holds. Because of (3), one has

$$
\begin{aligned}
& & S_k &< S_{k+1} \\
&\iff & \frac{99!\cdot k}{(100-k)!\,100^k} &< \frac{99!\cdot(k+1)}{(99-k)!\,100^{k+1}}, \\
&\iff & \frac{k}{(100-k)!} &< \frac{k+1}{(99-k)!\cdot 100}, \\
&\iff & \frac{100k}{(100-k)(99-k)!} &< \frac{k+1}{(99-k)!}, \\
&\iff & 100k &< (k+1)(100-k) \\
&\iff & 100k &< 100k-k^2+100-k \\
&\iff & k(k+1) &< 100
\end{aligned}
$$

Hence, (4) holds for $1 \le k \le 10$. Analogously, the inequality

$$S_k > S_{k+1}$$

holds for $k \geq 10$.

This means that the seize of the pieces strictly increases from the first to the tenth piece of cake and strictly decreases afterwards (that is, from the tenth to the hundredth piece of cake). In conclusion, the tenth elf receives the largest piece of cake.

12 Hat Challenge 2019

Authors: Aart Blokhuis (TU Eindhoven)
Gerhard Woeginger (TU Eindhoven)

Project: 4TU.AMI

12.1 Challenge

Santa Claus says to Atto, Bilbo, and Chico, "My dear super-smart elves! As every year, the Mathematical Advent Calendar is going to pose a tricky puzzle with colored hats on the heads of elves. For this reason, I will invite you for coffee and cake tomorrow afternoon."
"Great, we love to come!", answer Atto, Bilbo and Chico.

Santa Claus is pleased, "Well, then tonight I am going to prepare three blue hats, three yellow hats, and three red hats. Tomorrow, I will put one hat on each of your heads such that none of you can see his own hat color. However, you will see the hat colors of the other two elves, but you are not allowed to exchange any information with each other."

Santa continues: "Then, simultaneously, each of you must show a certain number of fingers.

- A **single finger** means that you guess that your own hat is **blue**,
- **two fingers** mean that you guess **yellow**,
- and **three fingers** mean that you guess **red**.
- **Zero or four or more fingers** mean that you prefer not to guess.

If at least one of you guesses wrong, the game is over and you have to leave. If none of you guesses wrong and at least one of you guesses the right color, then you get a tasty piece of *Sacher torte* and a nice cup of coffee."

Atto wants to know, "How are you going to choose our hat colors?"
"They are chosen randomly such that each of the 27 color combinations arises with the same probability", answers Santa Claus.

Bilbo asks, “Will we be able to see the six hats that you do not use?”
“No!”, answers Santa Claus, “The unused hats will be hidden in the closet!”

And Chico asks, “What is going to happen, if none of us guesses?”
“Then, the game is over and you have to leave”, says Santa Claus, “Coffee and cake will be served only if at least one of you guesses right and none of you guesses wrong. Do you understand?”

The elves start to ponder. They discuss and they think. They think and they discuss. Then, they discuss some more, and then they think some more. Eventually they manage to develop an amazing strategy that maximizes the number M of color combinations for which coffee and cake will be served.

The question is of course: How big is this number M?

Artwork: Friederike Hofmann

Possible answers:

1. $M = 9$.
2. $M = 10$.

3. $M = 11$.
4. $M = 12$.
5. $M = 13$.
6. $M = 14$.
7. $M = 15$.
8. $M = 16$.
9. $M = 17$.
10. $M = 18$.

12.2 Solution

The correct answer is: 7.

Why M=15 is possible:

The following strategy allows the elves to have success in at least 15 out of 27 possible cases: Each elf looks at the hats of the two other elves.

- If none of the hats is blue, he guesses BLUE.
- If both hats are blue, he guesses YELLOW.
- If exactly one of the hats is blue, he does NOT guess at all.

Now we want to examine the 27 possible color combinations within this strategy:

- There are exactly 8 color combinations without any blue hat:

  YYY, YYR, YRY, RYY, RRY, RYR, YRR, RRR.

  In these 8 cases, all three elves guess BLUE as the color of their hats, and all of them guess wrong.

- There are exactly 12 color combinations where exactly one elf has a blue hat:

  BYY, BYR, BRY, BRR, YBY, YBR,
  RBY, RBR, YYB, YRB, RYB, RRB.

  In these 12 cases, only the elf with the blue hat guesses a color, and he guesses the correct color BLUE.

- There are exactly 3 color combinations with two blue hats and one yellow hat:

  BBY, BYB, YBB.

  In these 3 cases, the elf with the yellow hat guesses his hat's color, and he guesses the correct color YELLOW.

- There are exactly 3 color combinations with two blue hats and one red hat:

  BBR, BRB, RBB.

  In these 3 cases the elf with the red hat guesses his hat's color. He guesses YELLOW and thus is wrong.

- In the only remaining case BBB, each elf has a blue hat. All elves guess YELLOW and all are wrong.

This strategy guarantees, indeed, coffee and cake in $M = 12 + 3 = 15$ color combinations.

Why M=16 is *not* possible:

By contradiction, we assume, there exists a strategy S, which guarantees the elves to be successful in a minimum of 16 cases. For Atto, there are exactly nine different color combinations he could possibly see on Bilbo's and Chico's heads:

xBB, xYY, xRR, xBY, xYB, xBR, xRB, xYR, xRY,

where x is a placeholder for the color of Atto's hat. The strategy S determines for each of the nine color combinations, whether Atto should guess BLUE, YELLOW, RED or not guess at all. If Atto decides to place a tip, he is correct in one case and wrong in two other cases (since his tip meets his hat's color only once). If Atto places a tip in $a \geq 6$ of the nine color combinations, he guesses wrong in $2a$ cases, which gives $M \leq 27 - 2a \leq 15$. Thus, Atto guesses for $a \leq 5$ of the possible nine color combinations and, in particular, he guesses correctly in $a \leq 5$ of, in total, 27 possible situations.

Symmetrical arguments show that Bilbo guesses correctly in $b \leq 5$ out of 27 possible situations, and that Chico places a correct tip in $c \leq 5$ out of 27 possible cases. Since each elf can place a maximum of five tips, the inequality $M \leq 5 + 5 + 5 = 15$ holds.

13 Organizing Gingerbread

Author: Falk Ebert (Herder-Gymnasium Berlin)

13.1 Challenge

It smells delicious in the Christmas bakery "Tasty Pastry" at the North Pole, where the world's best gingerbread, cookies, and Christmas stollen are baked.

Packing elf Paul is assigned to pack the most tasteful gingerbread (which are almost cuboidal, 5 cm wide and 15 cm long[1]) into big boxes. Each of the normed boxes is 40 cm wide and 40 cm long. Paul always packs layers of several gingerbread with parchment paper between the layers. It is up to him to decide how to arrange the biscuits in each layer. However, it is a matter of honour for Paul to arrange the gingerbread as efficient as possible, i. e. such that there are no other arrangements that fit more gingerbread.

He notices one fact right away: in every cake layer there is always a small empty space left which he can not fill, regardless of how he arranges the gingerbread. After packing several boxes, he observes that when packing most efficiently, there occurs the same phenomenon in each layer.

Which phenomenon does Paul notice?

[1] Since the hight of gingerbreads is not important for the solution, it is not stated here.

Artwork: Frauke Jansen

Possible answers:

1. There is exactly one possibility for the position of the empty region.
2. There are exactly four possible positions for the empty region.
3. There are exactly eight possible positions for the empty region.
4. There are exactly 16 possible positions for the empty region.
5. There are extactly 64 possible positions for the empty region.
6. The empty space always touches the boundary of the box.
7. There is always a gingerbread-long distance (15 cm) between the empty region and a side of the box.
8. The empty region is never connected.
9. The empty region always has a quadratic shape with side length 10 cm.
10. The empty region is always **L**-shaped.

13.2 Solution

The correct answer is: 2.

We can partition the base of the box into 64 small squares of size 5 cm x 5 cm. The gingerbreads can be partitioned into three small squares of size 5 cm x 5 cm as well (s. Abb. 14).

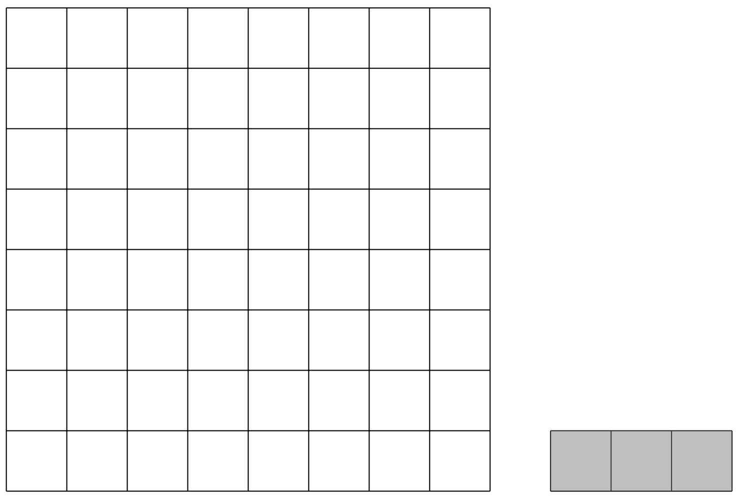


Figure 14: On the left: base of the box. On the right: A gingerbread.

We notice immediately (like Paul does) that 64 leaves the rest 1 when you divide it by 3, i. e. $64 = 21 \cdot 3 + 1$. Therefore, we can fit a maximum of 21 gingerbreads into one layer. An example of how this can be achieved is shown in Figure 15.

We show that there are exactly four possibilities to fit 21 gingerbreads into the box: To this end, we color the base of the box with three different colors as shown in Figure 16.

We observe that one gingerbread—regardless of how we put it in the box (vertical or horizontal)—always covers exactly one pink, one blue, and one yellow square. In total, we count 21 blue, 21 yellow, and 22 pink squares. Thus, fitting 21 gingerbreads into one layer inside the box leaves an empty pink square. Since the packing problem is both mirror symmetric and rotationally symmetric, the only candidates for the empty pink square are those,

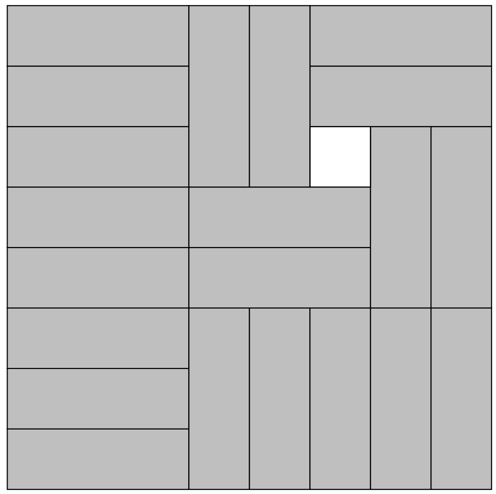


Figure 15: One possibility for fitting 21 gingerbreads into one layer.

who are transformed into a pink square again by reflection or rotation (by 90°, 180° resp. 270°). For example, assume that the bottommost left, pink square remains blank, then this is equivalent to the uppermost left, *yellow* square remain blank—but we already eliminated this possibility. Hence, there are only exactly **four squares**, which might remain empty at most efficient packing (see Figure 17). In Figure 15, we have seen, that it is actually possible to pack the 21 gingerbreads into the box in a way that leaves one of the four pink squares empty.

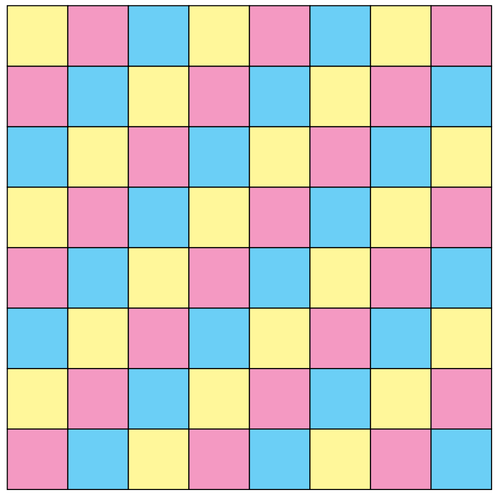


Figure 16: Coloring of the base of the box in three different colors.

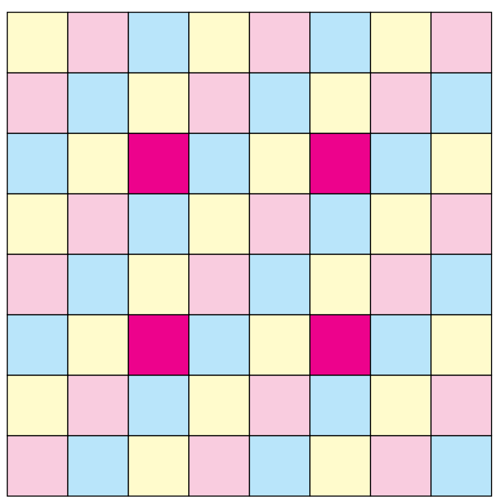


Figure 17: The dark pink squares are the only possible positions for the empty region.

14 Top Hat

Author: Jacques Resing (TU Eindhoven)
Project: 4TU.AMI

14.1 Challenge

The probability elves Waff, Weff, and Wiff have put 900 paper slips with non-negative integers into a top hat. Santa Clause draws a paper slip from the top hat, looks at the integer on the slip, and puts the slip back into the hat.

Waff says, "If you repeat this many times, then the expected value of the drawn numbers will be 1."
Weff says, "And the expected value of the squares of the drawn numbers will be 2."
Wiff says, "And the expected value of the cubes of the drawn numbers will be 5."

Santa Clause ponders about these statements, "Since all paper slips contain non-negative integers, a lot of them must contain the integer zero."
Waff, Weff and Wiff reply, "Yes, indeed! Zero is our favorite number."

And we would like to know: What is the smallest possible number N of paper slips in the top hat that do contain the integer zero?

Artwork: Friederike Hofmann

Possible answers:

1. This smallest possible number is $N = 280$.
2. This smallest possible number is $N = 289$.
3. This smallest possible number is $N = 295$.
4. This smallest possible number is $N = 300$.
5. This smallest possible number is $N = 312$.
6. This smallest possible number is $N = 314$.
7. This smallest possible number is $N = 324$.
8. This smallest possible number is $N = 333$.
9. This smallest possible number is $N = 345$.
10. This smallest possible number is $N = 352$.

14.2 Solution

The correct answer is: 4.

Let A_n be the number of slips labelled with the integer $n \geq 0$. According to the challenge, it holds

$$
\begin{aligned}
1 \cdot 900 &= 0 \cdot A_0 + 1 \cdot A_1 + 2 \cdot A_2 + 3 \cdot A_3 + 4 \cdot A_4 + \cdots + n \cdot A_n + \cdots\,, \\
2 \cdot 900 &= 0 \cdot A_0 + 1 \cdot A_1 + 4 \cdot A_2 + 9 \cdot A_3 + 16 \cdot A_4 + \cdots + n^2 \cdot A_n + \cdots\,, \\
5 \cdot 900 &= 0 \cdot A_0 + 1 \cdot A_1 + 8 \cdot A_2 + 27 \cdot A_3 + 64 \cdot A_4 + \cdots + n^3 \cdot A_n + \cdots\,.
\end{aligned}
$$

We multiply the first equation by 11 and the second by -6, and then add the two to the third equation:

$$4 \cdot 900 = \sum_{n \geq 1} (11n - 6n^2 + n^3)\, A_n.$$

Since

$$11n - 6n^2 + n^3 = (n-1)(n-2)(n-3) + 6 \geq 6$$

holds for all $n \geq 1$, we obtain

$$
\begin{aligned}
4 \cdot 900 = \sum_{n \geq 1} (11n - 6n^2 + n^3)\, A_n &\geq 6 \sum_{n \geq 1} A_n \\
\iff \qquad 600 &\geq \sum_{n \geq 1} A_n
\end{aligned}
$$

Hence, *at most* 600 slips are labelled with a number $n \geq 1$, and *at least* 300 slips labelled with the number 0.

If one sets $A_0 = 300$, $A_1 = 450$, $A_3 = 150$, and all remaining $A_n = 0$, one obtains a situation in which exactly 300 slips are labelled with the number 0 and such that

$$
\begin{aligned}
\tfrac{1}{900}(0 \cdot A_0 + 1 \cdot A_1 + 0 + 3 \cdot A_3 + 0) &= \tfrac{1}{900}(450 + 3 \cdot 150) = 1, \\
\tfrac{1}{900}(0 \cdot A_0 + 1 \cdot A_1 + 0 + 9 \cdot A_3 + 0) &= \tfrac{1}{900}(450 + 9 \cdot 150) = 2, \\
\tfrac{1}{900}(0 \cdot A_0 + 1 \cdot A_1 + 0 + 27 \cdot A_3 + 0) &= \tfrac{1}{900}(450 + 27 \cdot 150) = 5.
\end{aligned}
$$

15 Rudolph's Timetable

Author: Niels Lindner (Zuse Institute Berlin)

Project: MATH+ Transition Project MI7:
Routing Structures & Periodic Timetabling

15.1 Challenge

A new railway network is being constructed at the north pole. Three trains are supposed to run: The **North Pole Express** (solid), the **Mail Train** (dashed), and the **Plum Pudding Train** (dotted), see Figure 18.

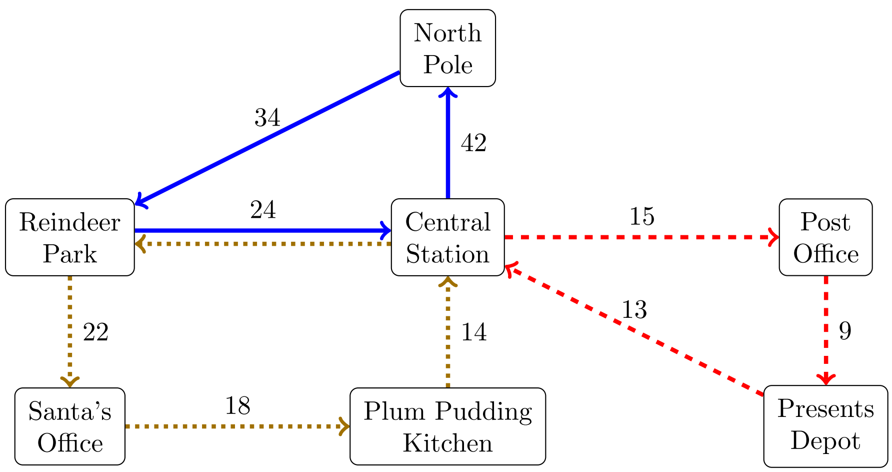


Figure 18: The North Pole Railway Network

The travel times between two neighboring stations have already been determined (see Figure 18). For example, the ride of the Mail Train from Central Station to Post Office takes 15 minutes. The trains run only in the depicted direction.

Santa Claus makes a daily trip from his home at the North Pole to Santa's Office. This is why, he does not want to wait too long when changing at Reindeer Park. Because Santa is not too deeply into timetabling, he asks Rudolph the Reindeer for help. Rudolph studied mathematics for some time and is able to create a timetable, i. e., arrival and departure times for every train at every station, satisfying the following properties:

(a) The timetable is repeated every T minutes.

(b) The arrival time at a station is computed as the sum of the departure time at the previous station and the travel time.

(c) Arrival and departure of a train a the same station can be scheduled at the same minute, but there might also be time in between.

(d) From the departure at Central Station until the next arrival at Central Station, every train needs at most T minutes.

(e) North Pole Express, Mail Train, and Plum Pudding Train always depart at the same time from Central Station.

(f) North Pole Express and Plum Pudding Train depart at the same time from Reindeer Park.

(g) T is the smallest natural number satisfying all these properties.

The reindeer Dasher, Comet, Cupid, Donner and Blitzen comment on Rudolph's timetable:

Dasher, "Awesome! The trains run every 100 minutes."

Comet, "Great! I do not need to wait for more than 30 minutes at Central Station when travelling from Reindeer Park to Presents Depot."

Cupid, "Excellent! All trains arrive at Central Station at the same time."

Donner, "Fantastic! The North Pole Express arrives at the same minute at Reindeer Park as the Plum Pudding Train departs."

Blitzen, "This timetable is stupid! I need longer than 90 minutes from Central Station to Plum Pudding Kitchen."

Which three reindeer are right?

Artwork: Friederike Hofmann

Possible answers:

1. Dasher, Comet, and Cupid.
2. Dasher, Comet, and Donner.
3. Dasher, Comet, and Blitzen.
4. Dasher, Cupid, and Donner.
5. Dasher, Cupid, and Blitzen.
6. Dasher, Donner, and Blitzen.
7. Comet, Cupid, and Donner.
8. Comet, Cupid, and Blitzen.
9. Comet, Donner, and Blitzen.
10. Cupid, Donner, and Blitzen.

Project reference:

The project *Routing Structures & Periodic Timetabling* deals with computing optimal timetables in public transit networks. This takes into account the impact a timetable has on the choice of passenger routes. Conversely, the different routes passengers can take are included into the timetable creation process. The main goal is to achieve shortest possible transfer times for all passengers.

15.2 Solution

The correct answer is: 9.

First, we calculate a lower bound for the T: Without loss of generality, one can assume that North Pole Express, Mail Train and Plum Pudding Train leave Central Station at minute 0. The common departure time Z of North Pole Express and Plum Pudding Train at Reindeer Park satisfies:

$$Z \geq 34 + 42 = 76 \qquad \text{and} \qquad Z \geq 24,$$

because these are the minimum travel times from Central Station to Reindeer Park. Moreover, North Pole Express and Plum Pudding Train have to reach Central Station again after at most T minutes. Hence,

$$Z + 24 \leq T \qquad \text{und} \qquad Z + 22 + 18 + 14 = Z + 54 \leq T$$

This implies

$$T \geq Z + 54 \geq 76 + 54 = 130.$$

In fact, one can construct a timetable with the given properties (a) to (g) repeating itself every $T = 130$ minutes:

North Pole Express	Arrival	Departure
Central Station		**0**
North Pole	42	42
Reindeer Park	**76**	**76**
Central Station	**100**	

Mail Train	Arrival	Departure
Central Station		**0**
Post Office	15	15
Presents Depot	24	24
Central Station	**37**	

Plum Pudding Train	Arrival	Departure
Central Station		**0**
Reindeer Park	**24**	**98**
Plum Pudding Kitchen	116	116
Central Station	**130**	

Let's turn to the reindeers:

- **Dasher is wrong:** As shown above, the smallest possible value for $T = 130 > 100$.

- **Comet is right:** From Reindeer Park to Presents Depot, one has to take either the North Pole Express and the Mail Train or the Plum Pudding Train and the Mail Train—both with a change at Central Station.

  In the above timetable, the waiting time at Central Station is 30 minutes when using the North Pole Express, and 0 minutes when taking the Plum Pudding Train.

  In general, the following holds: According to the above calculation, $T = 130$ implies $Z = 76$. Employing (b), the arrivals at Central Station are already determined to be $Z + 24 = 100$ for the North Pole Express and $Z + 54 = 130$ for the Plum Pudding Train. The Mail Train leaves Central Station at $0, 130, 260$, etc. Hence, the waiting time when changing from the North Pole Express to the Mail Train is 0 minutes, whereas the waiting time when changing from the Plum Pudding Train to the Mail Train is 30 minutes. Thus, indeed, when changing to the Mail Train at Central Station, one has to wait *at most* 30 minutes.

- **Cupid's statement is false:** Because the North Pole Express and the Plum Pudding Train depart from Reindeer park at the same time Z, the arrival of the North Pole Express at Central Station is $Z + 24$, whereas the Plum Pudding Train cannot arrive before $Z + 54$. Hence, the two trains will *not* arrive at the same time at Central Station.

- **Donner is right:** $Z = 76$ is the earliest possible arrival time of the North Pole Express and the latest possible departure time of the Plum Pudding Train. Since both trains have to depart at the same time $Z = 76$, the North Pole Express arrives at the same minute ($Z = 76$) at Reindeer Park as the Plum Pudding Train departs.

- **Blitzen is right as well:** From Central Station to Plum Pudding Kitchen, it is necessary to travel via Reindeer Park (regardless of whether taking the North Pole Express or the Plum Pudding Train). Departing from Central Station at minute 0, the next departure from Reindeer Park is not until $Z = 76$. Therefore, it takes at least

$$Z + 22 + 18 = 76 + 40 = 116 > 90$$

minutes from Central Station to Plum Pudding Kitchen.

16 Rendezvous at New Moon

Author: Khai Van Tran (TU Berlin)

16.1 Challenge

As every new moon, the elves Yann and Max are appointed for a night walk and a subsequent puzzling night. Both elves live on the (almost infinitely) long and totally straight Christmas Street. Since they live two miles apart, they usually leave an hour before their appointed meeting time and walk one mile towards each other. After one hour, they meet in the middle of their houses (you wouldn't be faster on these little elf legs).

Today, they also leave their houses at exactly the same time. But, unfortunately, there has been a tropical storm (the climate change does not spare anyone), that has taken all the road signs and destroyed the information and telecommunications system.
Yann and Max can distinguish the two directions in which the road is heading; however, they do not know in which direction the other elf lives—the Christmas stress must have made them oblivious. Since it is a pitch-black night, they cannot use the sun's position for orientation. Furthermore, they cannot count on the North Star for orientation, as they already are quite near to the North Pole. Light and smoke signals are not a possibility either, since Yann and Max are not to disturb they neighbours on Christmas Street—in short, they cannot communicate with each other...

Now, there are different strategies for the elves to consider, in order to meet nonetheless:

(A) Each elf tosses a fair coin to determine in which direction to go for one mile. This procedure is repeated until the elves meet.

(B) First, each elf tosses a fair coin to determine in which direction to go for one mile. If they do not meet, then each elf turns and goes on for *one* mile. This procedure is repeated until the elves meet.

(C) First, each elf tosses a fair coin to determine in which direction to go for one mile. If they do not meet, then each elf turns and goes on for *two* miles. This procedure is repeated until the elves meet.

Since Yann and Max are bosom buddies, they can be absolutely sure to pick the same strategy. The question remaining is, how good each strategy is. Let a, b, and c be the average time (measured in hours) until the two elves meet if they choose strategy A, B, and C, respectively.

Which of the following statements is true?

Illustration: Friederike Hofmann

Possible answers:

1. $a < b < c$
2. $a < c < b$
3. $a < b = c$
4. $b < a < c$
5. $b < c < a$
6. $b < a = c$
7. $c < a < b$

8. $c < b < a$
9. $c < a = b$
10. $a = b = c$

16.2 Solution

The correct answer is: 8.

In each strategy, a fair toss of a coin determines the direction in which the elves run. So, each elf (regardless of what the other one does) will first move towards the other one with probability $\frac{1}{2}$ and will move away from the other one with probability $\frac{1}{2}$. Each of the four cases shown in table 3 will therefore most occur with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. In particular, there is a probability of $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ that the gnomes will run parallel to each other, i. e. have the same direction of movement.

	Max moves towards: $p = 1/2$	Max moves away: $p = 1/2$
Yann moves towards: $p = 1/2$	they move towards each other: $p = 1/2 \cdot 1/2 = 1/4$	they move parallel: $p = 1/2 \cdot 1/2 = 1/4$
Yann moves away:: $p = 1/2$	they move parallel: $p = 1/2 \cdot 1/2 = 1/4$	they move away from each other: $p = 1/2 \cdot 1/2 = 1/4$

Table 3: The cases that can happen in the first hour.

Let us first consider **strategy B**: In this case, the elves can only meet after one hour, if they move towards each other. The probability for this case to occur is $\frac{1}{4}$.
Otherwise, they do not meet and return to their starting point within two hours. The time they need to meet is again b hours. Thus,

$$\begin{aligned} & b = \tfrac{1}{4} \cdot 1 + \tfrac{3}{4} \cdot (2 + b) \\ \Leftrightarrow \quad & b = \tfrac{1}{4} + \tfrac{6}{4} + \tfrac{3}{4} b \\ \Leftrightarrow \quad & \tfrac{1}{4} b = \tfrac{7}{4} \\ \Leftrightarrow \quad & b = 7 \end{aligned}$$

Let us now look at **strategy C**: Again, there is a $\frac{1}{4}$ chance that Yann and Max will meet after one hour.

If both elves first move away from each other, they will have increased their distance to four miles after one hour. But then they both turn around and walk two miles towards each other and meet after three hours. The probability for this case to occur is again $\frac{1}{4}$.
With probability $\frac{1}{2}$ the two elves move parallel to each other for three hours and will meet during that time. After that their distance has not changed and the elves need c hours to meet. Thus,

$$\begin{aligned} && c &= \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot 3 + \tfrac{1}{2}(3+c) \\ \Leftrightarrow && c &= \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{6}{4} + \tfrac{1}{2}c \\ \Leftrightarrow && \tfrac{1}{2}c &= \tfrac{5}{2} \\ \Leftrightarrow && c &= 5 \end{aligned}$$

Now, consider **strategy A**: Again, there is a $\frac{1}{4}$ chance that the elves will meet after one hour.
There is a $\frac{1}{2}$ chance that the two elves will run parallel to each other for an hour and will not meet during that time. After that their distance has not changed and the elves need a hours to meet.
If both elves move a mile away from each other, their distance increases to four miles. To meet, they must first reduce their distance to two miles. Then, they would be in the starting position and they need a hours to meet. To reduce the distance from four miles to two miles takes just as long as to reduce the distance from two miles to zero, so a hours. So in this case the gnomes need $2a$ hours to meet. Hence,

$$\begin{aligned} && a &= \tfrac{1}{4}\cdot 1 + \tfrac{1}{2}(1+a) + \tfrac{1}{4}(1+2a) \\ \Leftrightarrow && a &= \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{2}a + \tfrac{1}{4} + \tfrac{1}{2}a \\ \Leftrightarrow && a &= a+1 \end{aligned}$$

The last equation is not valid for any finite real number a. Therefore, a cannot be a finite real number. One could almost think that a is a 8 that stumbled and *um* stayed in a lying position...

Anyway, we have

$$c < b < a.$$

Remark: The problem family from which this task originates is known in the literature under the name “Rendezvous on the Line”. It is assumed that there is *no* strategy whose expected value is below 4.25 hours. In this task, we have shown that a strategy with an expected value of 5 hours exists. There is even a strategy known whose expected value is just over 4.25 hours. This strategy could only be found with the help of computers and in it the elves repeat a rather complicated fifteen hour search pattern, which they execute with different probabilities. See, for example,

`http://garden.irmacs.sfu.ca/op/rendezvous_on_a_line`.

17 Fairy Lights

Author: Luise Fehlinger (HU Berlin)

17.1 Challenge

The production of fairy lights at the North Pole goes on very smoothly. The elves Dincho, Antonia, and Bogdan are very happy with their newly invented machine, which simplifies their work tremendously. Therefore, they invite all the other elves to a board game party.

Unfortunately, they do not look after the production anymore and the Grinch is able to sneak into the plant. He sabotages the new machine and additionally puts broken light bulbs, that were sorted out already, back into the supply of functioning light bulbs. Elf Sinan notices this at first, and subsequently all elves are very upset: what will the elf-in-chief, Nicolas, say if the fairy lights cannot be shipped on time? Immediately, the elves Daria and Thore begin to fix the machine. Elves Bendix and Jan start to test the light bulbs again and sort them accordingly.

But time is running. Elf Joana says, “Testing every light bulbs individually costs too much time. We should let the machine put together the fairy lights and test the assembled fairy lights afterwards.”
Elf Lennart agrees, “The light bulbs are broken with the small probability of 0.1 %—accordingly, many of the fairy lights will work if we just assemble them.”
Elf Kai decides, “We put together the fairy lights and then test the assembled fairy lights. Afterwards, we individually test all light bulbs of all the fairy lights that do not work. Finally, we exchange the broken light bulbs by functioning ones, that have already been checked by Bendix and Jan.”
Elf Phillipp takes a look at the schedule, “For the next shipment, we need 100 fairy lights consisting of 100 light bulbs each.”
Testing expert Sara adds, “For each test—regardless of assembled fairy lights or individual light bulb—we need 3 seconds.”

Now, everyone is hard at work. Meanwhile, the mathematics-loving elves amuse themselves by predicting the test outcomes. However, one elf is mistaken. Which one?

Illustration: Frauke Jansen

Possible answers:

1. Elf Taha says, "If we are lucky, then we are finished with testing after exactly 5 minutes."
2. Elf Marwan replies, "If we have bad luck, then we need 8 hours and 25 minutes for testing."
3. Elf Lara points out, "In the worst case—that is, if all of the fairy lights are broken—we would need 5 minutes less, when testing all light bulbs individually."
4. Elf Andi replies, "But a single one of the fairy lights is fully functioning with a probability of over 90 %."
5. Elf Katharina adds, "However, the best case—that is, if all the fairy lights are working—occurs only with a probability of less then 0.0005 %."
6. Elf Jonathan indicates that, "The worst case—i. e., that all fairy lights are broken—occurs with a much smaller probability of less than 10^{-200} %."

7. Elf Yura says, “Anyway, the probability of exactly two of the fairy lights being broken is not equal to the probability of exactly three of the fairy lights being broken.”
8. Elf Isabella adds, “The probability of exactly five of the fairy lights being broken is less than 5 %.”
9. Elf Fabian points out, “We can expect that less than eleven of the hundred fairy lights are broken.”
10. Elf Clemens concludes, “Hence, we can expect that we need less than an hour for testing all hundred of the fairy lights.”

17.2 Solution:

The correct answer is: 6.

1. **Taha is correct:** If the elves are lucky, all 100 fairy lights are functioning. Thus, one only has test the 100 fairy lights one by one. It takes them 3 seconds to do each test. That amounts to:

$$100 \cdot 3\,s = 300\,s = 5\,\mathrm{min}\,.$$

2. **Marwan is correct:** If the elves are unlucky, all 100 fairy lights are broken. Hence, one has to test the 100 fairy lights one after the other and then each of the 100 light bulbs of all 100 fairy lights. For each of these tests one needs 3 seconds. In total:

$$\begin{aligned} 100 \cdot 3\,s + 100 \cdot 100 \cdot 3\,s &= (1 + 100) \cdot 100 \cdot 3\,s \\ &= 101 \cdot 5 \cdot 60\,s \\ &= 505\,min \\ &= 8\,h\ 25\,min. \end{aligned}$$

3. **Lara is correct:** In the worst case, i. e. if all the light bulbs are broken, the elves—as just calculated—need 505 minutes for the testing. If they tested all the light bulbs individually, they would need only

$$100 \cdot 100 \cdot 3\,s = 500 \cdot 20 \cdot 3\,s = 500\,min.$$

Hence, they would be five minutes faster.

4. **Andi is correct:** Since one of the fairy lights consists of 100 light bulbs and each of the light bulbs is functioning with a probability of 99.9 %, the each one of the fairy lights is functioning with a probability of
$$\left(\frac{999}{1000}\right)^{100} \approx 0{,}905 = 90{,}5\,\% > 90\,\%$$
ganz.

5. **Katharina is correct:** In the best case all 100 light bulbs of all 100 light chains are working. Since a single light bulb is working with a probability of 99.9 %, all of the 100 fairy lights are functioning with a probability of
$$\left(\frac{999}{1000}\right)^{100\cdot 100} = \left(\frac{999}{1000}\right)^{10.000} \approx 0{,}000045 = 0{,}0045\,\% < 0{,}005\,\%$$
ganz.

6. **Jonathan is wrong:** The probability that *one* of the fairy lights is broken is the complementary probability to the probability that it is working. We calculated this probability in (4). Accordingly,tThe probability that all 100 fairy lights are broken is
$$\left(1 - \left(\frac{999}{1000}\right)^{100}\right)^{100} \approx 7 \cdot 10^{-103} = 7 \cdot 10^{-101}\,\% > 10^{-200}\,\%.$$

   Many calculators cannot process these small numbers. But we can approximate the result:
$$\left(1 - \left(\frac{999}{1000}\right)^{100}\right)^{100} \approx \left(0,095\right)^{100}.$$
   is between
$$0,01^{100} = 10^{-200} = 10^{-198}\,\%$$
   and
$$0,1^{100} = 10^{-100} = 10^{-98}\,\%.$$
   Even this very rough lower estimate is still above the value of Jonathan.

7. **Yura is correct:** The probability that one of the fairy lights is working or broken was calculated in (4) and in (6), respectively. The number of possibilities how the two or three defective fairy lights can be distributed over the 100 ones is $\binom{100}{2}$ or $\binom{100}{3}$. In total, the probability that exactly two or three of the 100 fairy lights are broken is

$$P(\text{genau 2 defekt}) = \binom{100}{2} \cdot \left(1 - \left(\frac{999}{1000}\right)^{100}\right)^2 \cdot \left(\left(\frac{999}{1000}\right)^{100}\right)^{98}$$
$$\approx 0{,}0025 = 0{,}25\,\%$$

und

$$P(\text{genau 3 defekt}) = \binom{100}{3} \cdot \left(1 - \left(\frac{999}{1000}\right)^{100}\right)^3 \cdot \left(\left(\frac{999}{1000}\right)^{100}\right)^{97}$$
$$\approx 0{,}0085 = 0{,}85\,\%.$$

8. **Isabella is correct:** The probability that one of the fairy lights is working or broken was calculated in (4) and in (6), respectively. The number of possibilities how the five defective fairy lights can be distributed over the 100 ones is $\binom{100}{5}$. In total, the probability that exactly five of the 100 fairy lights are broken is

$$\binom{100}{5} \cdot \left(1 - \left(\frac{999}{1000}\right)^{100}\right)^5 \cdot \left(\left(\frac{999}{1000}\right)^{100}\right)^{95} \approx 0{,}044 = 4{,}4\,\% < 5\,\%.$$

9. **Fabian is correct:** Since one of the fairy lights is broken with a probability of

$$1 - \left(\frac{999}{1000}\right)^{100}$$

we can expect that out of 100 fairy lights

$$100 \cdot \big(1 - \left(\frac{999}{1000}\right)^{100}\big) \approx 9{,}52 < 10$$

are broken. Of course this value does not have to be assumed exactly and every time. But if you bet on the number of defective fairy lights beforehand, this is a good value.

10. **Clemens is correct:** With the considerations of (9), we expect less than 10 defective fairy lights. In this case, the 100 fairy lights have to be tested and afterwards all light bulbs of the 10 defective ones. For these tests, the elves will need

$$(10 \cdot 100 + 100) \cdot 3\, s = 3300\, s = 55\, min < 1\, h.$$

18 It's Bredele time!

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Project: MATH+ Emerging Field EF3-5:
Direct Reconstruction of Biophysical Parameters Using Dictionary Learning and Robust Regularization

18.1 Challenge

Every year, not very well in advance, Santa's little helpers in the Alsace rack their little brains to find the perfect combination of *Bredele*, the Alsatian version of Christmas cookies, to pack them into 20 equally assorted gift bags.

Fortunately, there is the internet platform `www.bredele.alsace` which contains all popular Bredele recipes with ratings by gazillions of test persons from all over the world (see Fig. 19). Each recipe lists the ingredients needed for one batch of 60 Bredele (plus a sufficient amount of dough and Bredele for the quality inspection by each one of the little helpers).

The helpers look up the current ratings (see Fig. 19) to find that the *Spitzbuben* have an average rating of 25 ♡, whereas *Butterbredele* and *Spritzbredele* have an average rating of 20 ♡ and 15 ♡, respectively. The *Kokosbredele* only have an average rating of 10 ♡, because some people love the taste of coconut while others dislike it strongly. However, they are the favourite Bredele of the little helpers, because they are very easy to bake after all.

While the preparation of Bredele gift bags is not only compromised by the little helpers' poor baking skills, it is also pretty late to organise all the necessary ingredients. During the spring, they were able to make exquisite strawberry and sour cherry jam. But now, after many delicious breakfasts, they only have one jar of each left.

Fortunately, there are three suppliers, namely *Lebkuchenhausen*, *Elisenheim*, and *Himmelsberg*, that offer "all-round carefree packages". The three packages are sold at the same price, but include different amounts of ingredients, which are listed in Table 4.

Figure 19: Recipes and average ratings for Butterbredele, Spritzbredele, Spitzbuben, and Kokosbredele.

Supplier	Lebkuchenhausen	Elisenheim	Himmelsberg
Flour (g)	3500	4000	3000
Sugar (g)	2000	2000	2000
Butter (g)	3000	3000	1000
Eggs (#)	30	22	16
Hazelnut (g)	500	1000	500
Coconut (g)	0	0	400

Table 4: Suppliers and the ingredients of their packages.

The little helpers have enough savings to choose exactly one of the three packages. Furthermore, they are quite good at adding and multiplying, but not very confident when it comes to division. Hence, they decide to only bake (non-negative) integer multiples of the given recipes.

Of course, the little helpers would like to assort the "best" Bredele gift bags. More precisely, they want to maximise the *happiness factor* of the gift bags, that is, the sum of the ♡ ratings (according to `www.bredele.alsace`) of every Bredele in the assorted bags. Unfortunately, the helpers disagree on the strategy to achieve this goal:

(A) Annika thinks, "We should choose the package supplied by Lebkuchenhausen."

(B) Bernd thinks, "We should choose the package supplied by Elisenheim."

(C) Chrisleine thinks, "We should choose the package supplied by Himmelsberg."

(D) Daniël is confident that, "No more than two batches of any type of Bredele is baked."

(E) Etienne replies, "One type of Bredele is baked five times."

(F) Fleurianne adds, "At least two types of Bredele are baked at least three times."

(G) Gustav says, "No Spitzbuben are baked."

Which of the given answers is correct?

Artwork: Frauke Jansen

Possible answers:

1. Only Annika and Etienne are right.
2. Only Annika and Fleurianne are right.
3. Only Annika, Etienne, and Gustav are right.
4. Only Annika, Etienne, Fleurianne, and Gustav are right.
5. Only Bernd and Daniël are right.
6. Only Bernd and Fleurianne are right.
7. Only Bernd, Etienne, Fleurianne, and Gustav are right.
8. Only Chrisleine, Daniël, and Etienne are right.
9. Only Chrisleine, Daniël, Fleurianne, and Gustav are right.
10. Only Chrisleine, Etienne, and Fleurianne are right.

Project reference:

Optimisation problems arise in different structures in real world applications. Finding optimal solutions for nested optimisation problems, where one problem (the lower level problem) is embedded within another (the upper level) problem, are called *bilevel optimisation problems*. Bilevel optimisation problems arise e. g. in the modelling of leader-follower games in game theory (so-called *Stackelberg* games), in investment planning, or when minimising costs under some equilibrium constraints (MPECs). The maximisation task at hand involves a linear constrained optimisation problem.

Modeling the problem is also an important subject of optimisation: if we would know more about the preferences of the persons receiving gift bags, for instance on their appreciation of Kokosbredele, we might want to include the possibility to prepare some bags with and some without Kokosbredele.

18.2 Solution

The correct answer is: 6.

First, one observes that is best to bake as many Spitzbuben as possible rather than Butterbredele or Spritzbredele. This stems from the fact that, except for the jam, one always needs more ingredients to bake a batch of Butterbredele or Spitzbredele and that the Spitzbuben have a better rating. Hence, one can always substitute one batch of Butterbredele or Spitzbredele with one batch of Spitzbuben, thereby increasing the rating of the Bredele gift bags. If we use all of the available jam, we are able to bake **two batches of Spitzbuben**. For theses two batches, we additionally need 1000 g of flour, 400 g of sugar, 500 g of butter, and 4 eggs. If one subtracts these quantities from the supplier packages, one is left with:

Supplier	Lebkuchenhausen	Elisenheim	Himmelsberg
Flour (g)	2500	3000	2000
Sugar (g)	1600	1600	1600
Butter (g)	2500	2500	500
Eggs (#)	26	28	12
Hazelnut (g)	500	1000	500
Coconut (g)	0	0	400

Table 5: Ingredients of the packages after baking two batches of Spitzbuben.

Secondly, let us show that Kokosbredele are not part of an optimal solution: The only supplier providing the necessary coconut flakes is Himmelsberg, whose kit is sufficient to bake one batch of Kokosbredele. Furthermore, note that the packages from Lebkuchenhausen and Elisenheim have (besides frome the coconut flakes) at least as many ingredients as the package from Himmelsberg. Hence, if we do not make Kokosbredele, there is no advantage to choose this supplier. Baking one batch of Kokosbredele leaves us with the following ingredients: 2000 g of flour, 1200 g of sugar, 500 g of butter, 12 eggs, and 500 g of hazelnuts.
Note that the amount of butter is the limiting factor: We are able to bake only two more batches of Bredele. Since Butterbredele are rated higher than Spritzbredele, the best choice would be to bake two batches of Butterbredele. Two batches of Spitzbuben, one batch of Kokosbredele, and two batches of Butterbredele would result in a rating of a total of 100 ♡. But this is not

optimal, since (with the packages from Lebkuchenhausen or Elisenheim) we could always bake two batches of Spitzbuben and three batches of Butterbredele, which would give a total rating of 110 ♡.
In conclusion, in an optimal solution, we will bake **no Kokosbredele**.

Finally, we need to determine the number of batches to bake of Butterbredele and Spitzbredele. To this end, we will cast this decision problem as a *Linear Programming* problem, and we will solve it using a graphical method.
Let x_B and x_S be number of batches of Butterbredele and Spritzbredele, respectively. Then, the optimisation problem can be read as:

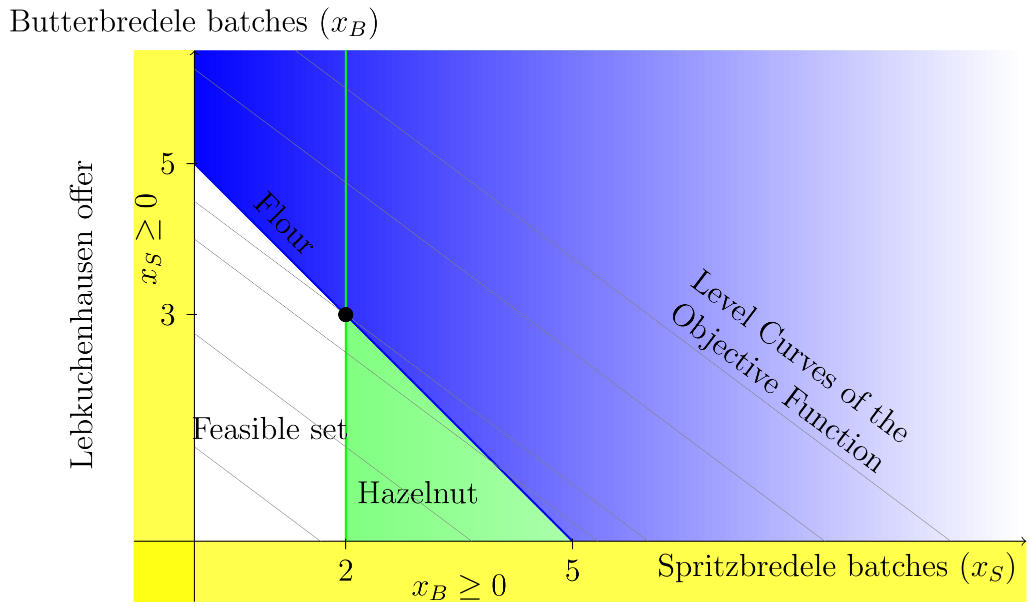
Lebkuchenhausen:

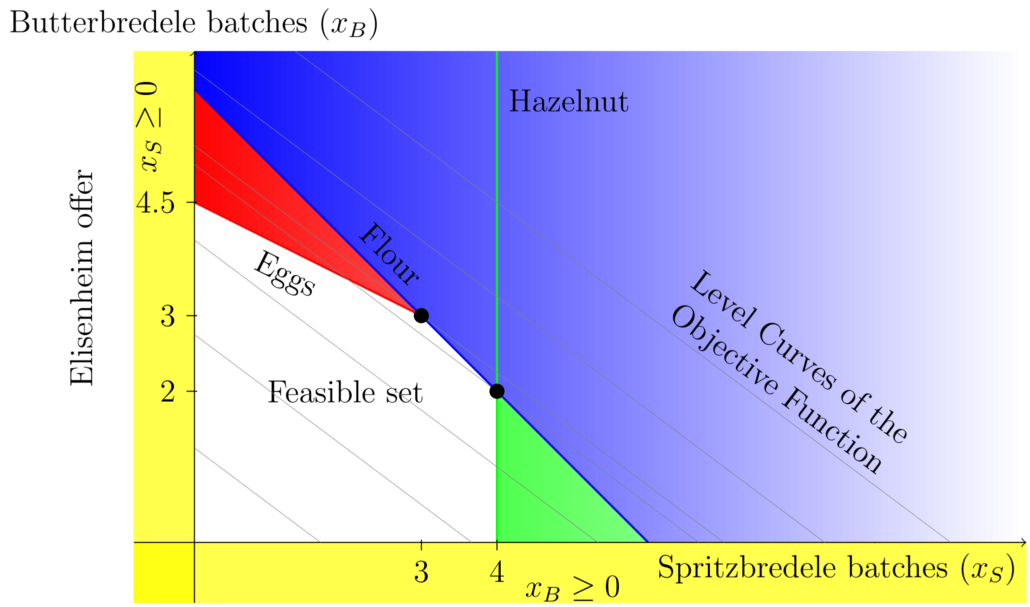
$$\begin{aligned} \text{maximise} \quad & 20x_B + 15x_S \\ \text{such that} \quad & 0.5x_B + 0.5x_S \leq 2.5 \\ & 0.25x_B + 0.2x_S \leq 1.6 \\ & 0.25x_B + 0.25x_S \leq 2.5 \\ & 4x_B + 2x_S \leq 26 \\ & 0x_B + 0.25x_S \leq 0.5 \\ & x_B \geq 0 \text{ and } x_S \geq 0 \end{aligned}$$

Elisenheim:

$$\begin{aligned} \text{maximise} \quad & 20x_B + 15x_S \\ \text{such that} \quad & 0.5x_B + 0.5x_S \leq 3.0 \\ & 0.25x_B + 0.2x_S \leq 1.6 \\ & 0.25x_B + 0.25x_S \leq 2.5 \\ & 4x_B + 2x_S \leq 18 \\ & 0x_B + 0.25x_S \leq 1 \\ & x_B \geq 0 \text{ and } x_S \geq 0 \end{aligned}$$

The set of all inequalities describes a region called *feasible set*. Drawing the feasible region is done in the following way: draw a line corresponding to the equality case of each relation. The half-plane that corresponds to the set described by the inequality is given by the normal to the line along which the value of the left-hand side decreases.
For both optimization problem, we get the following representation:





Finding the optimal choice is done in the following way: The values of x_B and x_S for which the quantity to be maximised is constant, can be represented as lines. The intersection of those lines with the feasible set gives us the value given by the choice of batches. The optimal choice(s) consists in finding the highest value of the objective function such that its associated line still intersects the feasible set.

For the **Elisenheim** offer, such optimal decision is given by baking **three batches of Butterbredele** and **three batches of Spritzbredele**, with an optimal value of 105 ♡ for baking these two types of Bredele. In total (i. e. together with the two batches of Spitzbuben), we get 155 ♡.

Additionally, we would like to present an **alternative solution approach** without employing graphical methods:

As before, we know that an optimal solution consists of baking two batches of Spitzbuben and no Kokosbredele. Furthermore, the package from Himmelsberg is not chosen in an optimal solution.
We extend Table 5 by enumerating over all possible numbers of batches of Spritzbredele (limited by the hazelnut supply). The table displays the number of batches of Spritzbredele (SB) baked, the remaining ingredients after baking of the Spritzbredele, the number of batches of Butterbredele (BB) that can be baked with the remaining ingredients, and total rating of the Bredele gift bags (including the 50 ♡ for the two batches of Spitzbuben):

Supplier	# SB	Flour	Sugar	Butter	Eggs	Hazel	# BB	♡
Lebkuchen-hausen	0	2500	1600	2500	26	500	5	150
	1	2000	1400	2250	24	250	4	145
	2	1500	1200	2000	22	0	3	140
Elisen-heim	0	3000	1600	2500	28	1000	4	130
	1	2500	1400	2250	26	750	4	145
	2	2000	1200	2000	24	500	3	140
	3	1500	1000	1750	22	250	3	**155**
	4	1000	800	1500	20	0	2	150

Finally, we read off the table that is best to choose the **Elisenheim** offer and to bake **two batches of Spitzbuben** and **three batches of Butterbredele**.

In conclusion, we state that

(A) Annika is wrong,

(B) Bernd is correct,

(C) Chrisleine is wrong,

(D) Daniël is wrong,

(E) Etienne is wrong,

(F) Fleurianne is correct, and

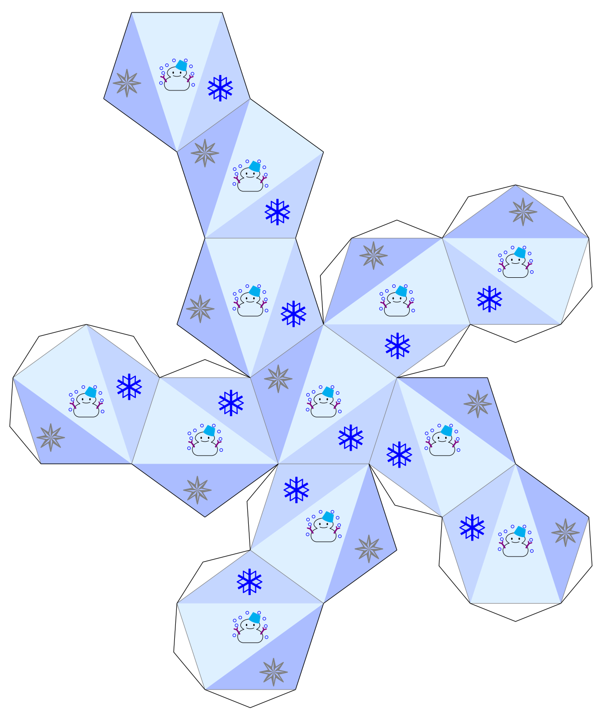
(G) Gustav is wrong.

19 Glittering Gifts

Author: Fleurianne Bertrand (HU Berlin)

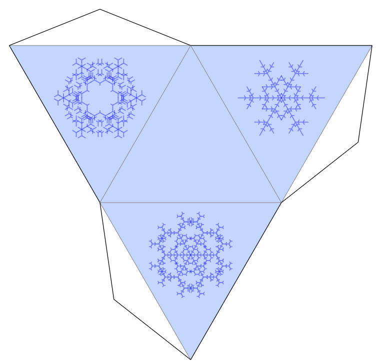
19.1 Challenge

Last year, also the elves received a present—a new decoration machine which can be controlled by a tablet: There, the elves choose a papercraft sheet which is automatically divided into triangles by the software. These triangles can be decorated (where the gluing area is left out, of course). Then, the elves fill sequins into the machine, and, *hop!*, the design is finished. At an internal competition for the design of a gift box in June, elf Nora won with the following design of a dodecahedron:



Since then, elf-in-chief Veronoella has to remind her creative elves again and again that ornaments have to be placed as efficiently as possible. Because 1 g of sequins is used for each ornament. However, for stability reasons, an *additional* amount M of sequins is distributed on the model.

There even was a discussion between Veronella and elf Anna, because Anna claimed that it is not reasonable to place sequins on the bottom side of a pyramid and thus created this design:



But Veronoella explained that, in this case, the machine would use 3 g of sequins for the desired ornaments and additional 6 g for the stability.
"If you had used a fourth ornament on the bottom triangle, there would not have been any more sequins necessary for the stability, and we would have used only 4 g of sequins. So, please be more efficient, such that we can make beautiful ornaments for all children by arranging them more evenly. In order to calculate the amount M of additionally needed sequins, all triangles have to be compared, and I just don't have the time for that", she says.

Now, Ella is very curious, how the amount M is calculated exactly. Therefore, she grabs the decoration machine's user manual and starts to read out loud:

For one papercraft model consisting of n triangles $T_1, T_2, \ldots, T_n$, the **stability amount** M is calculated as follows:

For every triangle T_i ($i = 1, \ldots, n$), we denote by $|T_i|$ the **proportion of the area of this triangle of the total area** of the papercraft model (without the gluing areas), i. e.

$$|T_i| = \frac{area(T_i)}{area(T_1) + area(T_2) + \cdots + area(T_n)}.$$

For every triangle T_i, let n_i be **the number of ornaments** on that triangle. Let $N = n_1 + n_2 + \cdots + n_n$ be the **total number of ornaments** on the model.

For every triangle T_i, let the **stability index** ρ_i of T_i be characterised by

$$\rho_i = \frac{n_i}{|T_i|},$$

i. e. as the number of ornaments on that triangle divided by the proportion of the area of this triangle of the total area.

For two triangles T_i and T_j, let the (i, j)-**stability number** m_{ij} be given by

$$m_{ij} = |T_i|\,|T_j|\,(\rho_i - \rho_j)^2.$$

The **stability amount** M is calculated as the sum of the m_{ij} over all triangle pairs (T_i, T_j), i. e.

$$\begin{aligned} M &= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \qquad \text{(DEF)} \\ &= \quad m_{11} + m_{12} + \cdots + m_{1n} \\ &\quad + m_{21} + m_{22} + \cdots + m_{2n} \\ &\quad + \ldots \\ &\quad + m_{n1} + m_{n2} + \cdots + m_{nn}. \end{aligned}$$

The other elves listen eagerly and have their own thoughts on that matter:

Liah claims: One also could calculate

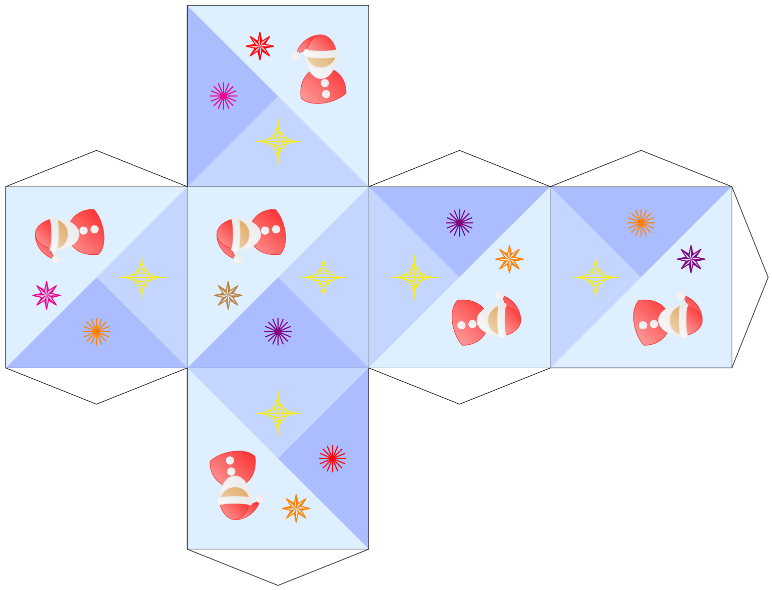
$$\begin{aligned} M &= 2\sum_{i=1}^{n}\sum_{j=1}^{i-1}|T_i|\,|T_j|\,(\rho_i-\rho_j)^2 \qquad \text{(LIAH)} \\ &= 2\Big(\quad |T_2|\,|T_1|\,(\rho_2-\rho_1)^2 \\ &\qquad + |T_3|\,|T_1|\,(\rho_3-\rho_1)^2 + |T_3|\,|T_2|\,(\rho_3-\rho_2)^2 \\ &\qquad + \dots \\ &\qquad + |T_n|\,|T_1|\,(\rho_n-\rho_1)^2 + \dots + |T_n|\,|T_{n-1}|\,(\rho_n-\rho_{n-1})^2\Big). \end{aligned}$$

Sara explains: I will always place only one ornament on each triangle. Then,

$$\begin{aligned} M &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|T_i|-|T_j|)^2 \qquad \text{(SARA)} \\ &= \quad (|T_1|-|T_1|)^2 + (|T_1|-|T_2|)^2 + \dots + (|T_1|-|T_n|)^2 \\ &\quad + (|T_2|-|T_1|)^2 + (|T_2|-|T_2|)^2 + \dots + (|T_2|-|T_n|)^2 \\ &\quad + \dots \\ &\quad + (|T_n|-|T_1|)^2 + (|T_n|-|T_2|)^2 + \dots + (|T_n|-|T_n|)^2. \end{aligned}$$

Nora declares: I would like to divide every face of a cube into one big and two smaller triangles that are half the size of the bigger one. On the bigger triangle, I place two ornaments, and one ornament on each of the two smaller triangles, Then, I will not need any extra ornaments, i. e. $M = 0$.

Anna responds: No, here is a counterexample, where $M \neq 0$ holds:



Elsa mumbles: If all triangles have the same size, then

$$\begin{aligned} M &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\rho_i^2-\rho_j^2) \qquad \text{(ELSA)} \\ &= \quad (\rho_1^2-\rho_1^2)+(\rho_1^2-\rho_2^2)+\dots+(\rho_1^2-\rho_n^2) \\ &\quad + (\rho_2^2-\rho_1^2)+(\rho_2^2-\rho_2^2)+\dots+(\rho_2^2-\rho_n^2) \\ &\quad + \dots \\ &\quad + (\rho_n^2-\rho_1^2)+(\rho_n^2-\rho_2^2)+\dots+(\rho_n^2-\rho_n^2); \end{aligned}$$

that is, one does not need additional sequins for stability if every triangle is decorated with the same number of ornaments.

Ella reckons: It holds

$$\begin{aligned} M &= 2\sum_{i=1}^{n}|T_i|(N-\rho_i)^2 \\ &= 2\Big(|T_1|(N-\rho_1)^2+|T_2|(N-\rho_2)^2+\dots|T_n|(N-\rho_n)^2\Big). \end{aligned}$$

Which elves are correct?

Artwork: Friederike Hofmann

Possible answers:

1. Anna and Elsa.
2. Anna and Ella.
3. Anna and Sara.
4. Anna and Nora.
5. Elsa and Liah.
6. Liah and Sara.
7. Nora and Sara.
8. Anna, Ella, and Elsa.
9. Ella, Liah, and Nora.
10. Ella, Nora, and Sara.

Project reference:

The finite element method (FEM) is a numerical approach to solve partial differential equations, which is used in different physical applications. There, the solution domain is divided into finitely many subdomains. For the HHO-method, arbitrary many polyhedra can be used. For the numerical analysis, it can be helpful to divide these polyhedra into triangles.

19.2 Solution

The correct answer is: 9.

Liah is correct: One has

$$m_{ij} = |T_i|\,|T_j|\,(\rho_i - \rho_j)^2 = |T_j|\,|T_i|\,(\rho_j - \rho_i)^2 = m_{ji},$$

from which

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} m_{ij} = \sum_{i=1}^{n}\sum_{j=i+1}^{n} m_{ij}$$

follows. Furthermore, one observes that

$$m_{ii} = |T_i|\,|T_i|\,(\rho_i - \rho_i)^2 = 0.$$

In total, we get

$$\begin{aligned} M &= \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{i-1} m_{ij} + \sum_{i=1}^{n} m_{ii} + \sum_{i=1}^{n}\sum_{j=i+1}^{n} m_{ij} \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} m_{ij} \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} |T_i|\,|T_j|\,(\rho_i - \rho_j)^2, \end{aligned}$$

where the last equation corresponds to the statement (LIAH).

Sara is wrong: First, we rearrange the equation (DEF) in the definition of

M:

$$\begin{aligned}
M &= \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |T_i|\,|T_j|\,(\rho_i - \rho_j)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |T_i|\,|T_j|\left(\frac{n_i}{|T_i|} - \frac{n_j}{|T_j|}\right)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{|T_i||T_j|}\,(n_i|T_j| - n_j|T_i|)^2
\end{aligned}$$

If we set $n_i = 1$, for all $i = 1, \ldots, n$, we see that Sarah forgot two divide every summand by $|T_i|\,|T_j|$ in her equation (SARA).

Nora is correct: Let every face of the cube be divided into one big and two half as big triangles. That means that each of the big triangles T has a proportion of

$$|T| = \frac{1}{2} : 6 = \frac{1}{12}$$

of the whole area each, whereas the small triangles t each have a proportion of

$$|t| = \frac{1}{4} : 6 = \frac{1}{24}.$$

Similarly, the stability numbers ρ_T and ρ_t of the bigger and the smaller triangles, resp., match:

$$\rho_T = \frac{2}{|T|} = 2 : \frac{1}{12} = 24 = 1 : \frac{1}{24} = \frac{1}{|t|} = \rho_t.$$

Thus, $m_{ij} = 0$ for all triangles, and hence

$$M = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} = 0.$$

Anna is wrong: Above, we just have shown that $m_{ij} = 0$ and $M = 0$ also holds for Anna's dice.

Elsa is also wrong: For Anna's tetrahedron, one has

$$|T_1| = |T_2| = |T_3| = |T_4| = \frac{1}{4}$$

and

$$\rho_1 = \rho_2 = \rho_3 = 4 \qquad \text{und} \qquad \rho_4 = 0.$$

It follows that $m_{ii} = m_{kl} = 0$ for $i = 1, 2, 3, 4$ and $k, l = 1, 2, 3$. Furthermore,

$$m_{k4} = |T_k|\,|T_4|\,(\rho_k - 0)^2 = \frac{1}{4} \cdot \frac{1}{4} \cdot 4^2 = 1$$

for $k = 1, 2, 3$. According to (DEF),

$$M = 2\,(m_{14} + m_{24} + m_{34}) = 2\,(1 + 1 + 1) = 6.$$

Using Elsa's formula (ELSA), one gets

$$M = (\rho_1^2 - 0) + (\rho_2^2 - 0) + (\rho_3^2 - 0) + (0 - \rho_1^2) + (0 - \rho_2^2) + (0 - \rho_2^2) = 0.$$

Finally, elf Ella is also right: The formula

$$2 \sum_{i=1}^{n} |T_i|\,(N - \rho_i)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |T_i|\,|T_j|\,(\rho_i - \rho_j)^2$$

can be proved as follows.

$$2 \sum_{i=1}^{n} |T_i|\,(N - \rho_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} |T_i|\,|T_j|\,(\rho_i - \rho_j)^2$$

$$= \sum_{i=1}^{n} \left[2\,|T_i|\,(N^2 - 2N\rho_i + \rho_i^2) - \sum_{j=1}^{n} |T_i|\,|T_j|(\rho_i^2 - 2\rho_i\rho_j + \rho_j^2) \right]$$

Since $\rho_i = \frac{n_i}{|T_i|}$, we obtain:

$$= \sum_{i=1}^{n} \left[2\,|T_i| \left\{ \left(\sum_{j=1}^{n} n_j \right)^2 - 2 \left(\sum_{j=1}^{n} n_j \right) \frac{n_i}{|T_i|} + \frac{n_i^2}{|T_i|^2} \right\} - \sum_{j=1}^{n} |T_i|\,|T_j| \left\{ \frac{n_i^2}{|T_i|^2} - 2 \frac{n_i\,n_j}{|T_i|\,|T_j|} + \frac{n_j^2}{|T_j|^2} \right\} \right]$$

$$= \sum_{i=1}^{n} \left[2\,|T_i| \left(\sum_{j=1}^{n} n_j \right)^2 - 4 \left(\sum_{j=1}^{n} n_j \right) n_i + 2 \frac{n_i^2}{|T_i|} - \sum_{j=1}^{n} \frac{n_i^2\,|T_j|}{|T_i|} + 2 \left(\sum_{j=1}^{n} n_j \right) n_i - \sum_{j=1}^{n} \frac{n_j^2\,|T_i|}{|T_j|} \right]$$

$$= \sum_{i=1}^{n} \left[2\,|T_i| \left(\sum_{j=1}^{n} n_j \right)^2 - 2\,n_i \left(\sum_{j=1}^{n} n_j \right) \right] + 2 \sum_{i=1}^{n} \frac{n_i^2}{|T_i|} - \sum_{i=1}^{n} \frac{n_i^2}{|T_i|} \left(\sum_{j=1}^{n} |T_j| \right) - \sum_{j=1}^{n} \frac{n_j^2}{|T_j|} \left(\sum_{i=1}^{n} |T_i| \right)$$

By renaming the indices $(j \to i)$ in the last summand and because $\sum_{j=1}^{n} |T_j|$, it follows that

$$= \sum_{i=1}^{n} \left[2\,|T_i| \left(\sum_{j=1}^{n} n_j \right)^2 - 2\,n_i \left(\sum_{j=1}^{n} n_j \right) \right] + 2 \sum_{i=1}^{n} \frac{n_i^2}{|T_i|} - 2 \sum_{i=1}^{n} \frac{n_i^2}{|T_i|}$$

$$= \sum_{i=1}^{n} \left[2\,|T_i| \left(\sum_{j=1}^{n} n_j \right)^2 - 2\,n_i \left(\sum_{j=1}^{n} n_j \right) \right]$$

$$= 2 \left(\sum_{i=1}^{n} |T_i| \right) \left(\sum_{j=1}^{n} n_j \right)^2 - 2 \sum_{i=1}^{n} n_i \left(\sum_{j=1}^{n} n_j \right)$$

Again, we use the identity $\sum_{j=1}^{n} |T_j| = 1$:

$$= 2 \left(\sum_{j=1}^{n} n_j \right)^2 - 2 \sum_{i=1}^{n} n_i \left(\sum_{j=1}^{n} n_j \right)$$

$$= 0$$

Summarizing, we see that **Liah, Nora, and Ella are correct**, whereas Sara, Anna, and Elsa are wrong.

20 Gift Ribbon

Authors: Christian Hercher (Europa-Universität Flensburg)
Michael Schmitz (Europa-Universität Flensburg)

20.1 Challenge

One would think that Elias Elf, who works in the Christmas gift packaging factory, has a lot to do this time of the year. But far from it! Right now, he is taking a break and ponders about an interesting problem:

In the middle of his hand, he holds three indistinguishable strings of gift ribbon. If he closes his hand to a fist, six loose ends dangle out of it (see Figure 20).

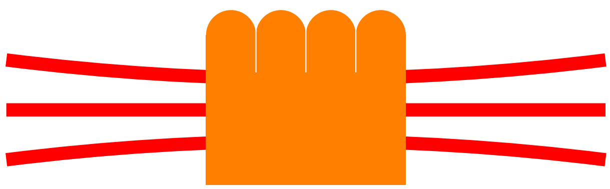


Figure 20: Elias' fist with the three cords or six loose ends, resp.

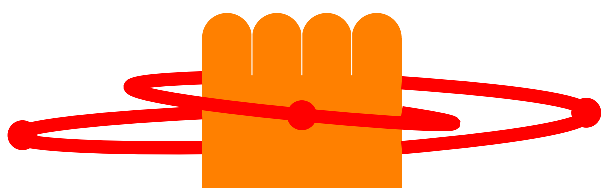
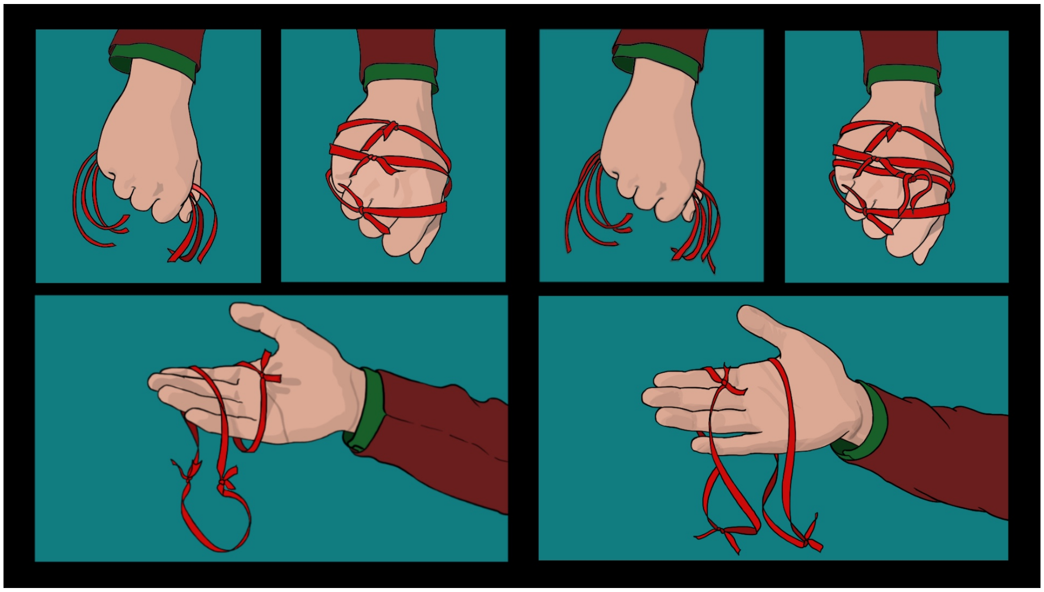


Figure 21: An example for a possible outcome after knotting. In this case, only one ring is produced.

Now, Adam Adjunct-Elf randomly knots two of these six string ends until there are no loose ends left—in doing so, Adam makes a total of three knots. Elias opens his hand and looks at one, two, or three closed rings (see Figure 21). By the way, an old elf wisdom says that if only a single ring is produced in this way, one has the day after Christmas off :)

(a) What is the probability p_3 that Elias gets exactly two rings after the knotting?

(b) Now, Elias is curious: If he increases the number of strings taken into his hand at the beginning from three to four, what is the probability p_4 of getting exactly two rings after the knotting?



Artwork: Frauke Jansen

Possible answers:

1. (a) $p_3 = 30\,\%$ and (b) $p_4 \approx 41\,\%$.
2. (a) $p_3 = 30\,\%$ and (b) $p_4 \approx 42\,\%$.
3. (a) $p_3 = 30\,\%$ and (b) $p_4 \approx 43\,\%$.
4. (a) $p_3 = 30\,\%$ and (b) $p_4 \approx 44\,\%$.
5. (a) $p_3 = 30\,\%$ and (b) $p_4 \approx 45\,\%$.
6. (a) $p_3 = 40\,\%$ and (b) $p_4 \approx 41\,\%$.

7. (a) $p_3 = 40\,\%$ and (b) $p_4 \approx 42\,\%$.
8. (a) $p_3 = 40\,\%$ and (b) $p_4 \approx 43\,\%$.
9. (a) $p_3 = 40\,\%$ and (b) $p_4 \approx 44\,\%$.
10. (a) $p_3 = 40\,\%$ and (b) $p_4 \approx 45\,\%$.

20.2 Solution:

The correct answer is: 7.

(a) First Adam selects any of the six string ends. Now we distinguish two cases:

1. With probability $\frac{1}{5}$ Adam chooses the other end of *the same* string, whereby the first ring is already created and two single strings remain. Again Adam picks one of the four remaining string ends. In order to create *two* rings, he has to knot this with one end of the *other* string, which happens with a probability of $\frac{2}{3}$. Now, two string ends remain, which will be knotted in any case (i. e. with probability 1).

   In total, this case occurs with a probability of

   $$\frac{1}{5} \cdot \frac{2}{3} = \frac{2}{15}.$$

2. In the other case, Adam knots the first string end with one end of *another* string, which happens with a probability of $\frac{4}{5}$. Elias now holds a short string and a long string (consisting of the two strings just knotted) in his fist. (We do not distinguish between the two strings in the following, as Adam makes his choice completely randomly). To make two rings out of these two strings, their ends must be knotted together. Again Adam takes one of the remaing four string ends. With a $\frac{1}{3}$ chance, he knots this end with the other end of *the same* string and receives the first ring. The second ring is now created in any case (i. e. with probability 1).

   This amounts to a probability of

   $$\frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15}$$

   in this case

Finally, the wanted probability is the sum of the probabilities determined above:

$$p_3 = \frac{2}{15} + \frac{4}{15} = \frac{6}{15} = \frac{2}{5} = 40\,\%.$$

(b) One can cleverly trace this part of the task back to (a). For this purpose, we introduce the following notation: Let us assume $P(X_n = k)$ is the probability of obtaining exactly k rings when knotting n strings. In part (a), we calculated the probability $P(X_3 = 2) = p_3 = \frac{2}{5}$. Now, we are looking for $p_4 = P(X_4 = 2)$.

Let us now calculate the probability of obtaining exactly two rings from four cords by knotting them: Adam first grabs one of the eight string ends. Again, we distinguish two cases:

1. With a probability of $\frac{1}{7}$, Adam knots the first end with the other end *the same* cord, forming the first ring. With the remaining three cords he now has to form another ring. This happens with the probability $P(X_3 = 1)$—which we still have to calculate. Altogether, this results in a probability of
$$\tfrac{1}{7} \cdot P(X_3 = 1).$$
2. In the second case, Adam knots the first end of the cord with one end of *another* cord, which happens with a probability of $\frac{6}{7}$. Thus, in Elias fist there is a long cord (consisting of the two just knotted) and two short ones. Since we again can neglect the length of the strings here, Elias now holds three strings in his fist, which Adma is supposed to connect to two rings, which happens with a probability of $P(X_3 = 2)$ (which was calculated in (a)). Altogether, this results in a probability of
$$\tfrac{6}{7} \cdot P(X_3 = 2).$$

Again, the wanted probability is the sum of the two probabilities calculated above:

$$P(X_4 = 2) = \tfrac{1}{7} \cdot P(X_3 = 1) + \tfrac{6}{7} \cdot P(X_3 = 2).$$

We have already determined the probability $P(X_3 = 2) = p_3 = \frac{2}{5}$ in (a). Thus, it remains to calculate $P(X_3 = 1)$, i. e. the probability of forming only one ring from three cords. To this end, Adam has to connect the first of the six loose string ends with the end of another string, which happens with the probability $\frac{4}{5}$. This results in two cords, from which now with probability $P(X_2 = 1)$ a single ring is formed. One has $P(X_2 = 1) = \frac{2}{3}$, since Adam has to knot one of the four loose ends (of the two cords) with one end of *another* cord to obtain a ring. Hence, one obtains

$$P(X_3 = 1) = \tfrac{4}{5} \cdot P(X_2 = 1) = \tfrac{4}{5} \cdot \tfrac{2}{3} = \tfrac{8}{15}.$$

Finally, one computes

$$\begin{aligned} p_4 = P(X_4 = 2) &= \tfrac{1}{7} \cdot P(X_3 = 1) + \tfrac{6}{7} \cdot P(X_3 = 2) \\ &= \tfrac{1}{7} \cdot \tfrac{8}{15} + \tfrac{6}{7} \cdot \tfrac{2}{5} \\ &= \tfrac{1}{7} \cdot \left(\tfrac{8}{15} + \tfrac{36}{15}\right) \\ &= \tfrac{1}{7} \cdot \tfrac{44}{45} = \tfrac{44}{105} \\ &\approx 0{,}419 \\ &\approx 42\,\%. \end{aligned}$$

21 Everything must go!

Author: Max Klimm (HU Berlin)

Project: MATH+ Application Area AA3-5: *Tropical Mechanism Design*

21.1 Challenge

After Christmas, Santa Claus tries to get rid of all the presents that have been rejected or were returned. To this end, he auctions off whatever is still there. The only currency accepted at the North Pole are fir cones, which can be divided arbitrarily. Santa Claus always uses *second price auctions with reserve price*. In such an auction every bidder hands in a sealed bid. The highest bid wins as long as it is at least as high as the reserve price. In this case, the **profit** is equal to the maximum of the second highest bid and the reserve price. The profit is kept by Santa Claus for buying presents the next Christmas season.

(i) The first article considered for auction is a tacky vase in which only Cato shows interest. From previous years' experience, it is known that Cato will bid either one or two cones, each with probability $\frac{1}{2}$.

(ii) The second article to be auctioned off is a sock sorting device. Only Luca is willing to bid for it. Luca will bid either one, two or three fir cones, each with probability $\frac{1}{3}$.

(iii) The third article in the auction is a solar powered paperweight. Elisa and Frana will hand in bids for this item. Elisa bids either one or three fir cones, each with a probability of $\frac{1}{2}$, and Frana bids either two or four fir cones, each with a probability of $\frac{1}{2}$. Both elves bid independently of each other.

(iv) The fourth article that is called up is a pottery flower bouquet in which only Cato is interested. Cato will either bid one or two cones, each with probability $\frac{1}{2}$. The bid and the interest of Cato in the flower bouquet is independent from the first auction.

Now, Santa contemplates on how to choose the reserve price for each auction in order to maximise the expected profit. The **expected profit** of an auction with reserve price is the sum of all possible profits, *each* multiplied with the probability of obtaining this profit. Santa wrote down the following conjectures concerning the above auctions:

(A) For the first auction, a reserve price of 2 gives an expected profit of 1.

(B) For the first auction, a reserve price of 2 is optimal, i. e., there is no other reserve price that yields a strictly higher expected profit.

(C) For the third auction, a reserve price of 4 is optimal.

(D) The expected profit of the second auction with optimal reserve price is smaller than the expected profit in the third auction with the reserve price of 0.

(E) Instead of selling the vase and the bouquet in two separate auctions, one may also sell them together as bundle. Assuming that Cato will bid for the bundle the sum of the bids for the items within, a reserve price of 3 for the bundle gives an expected profit of $\frac{9}{4}$.

(F) The reserve price of 3 for the bundle is optimal.

(G) Assume now that Cato will bid one or four cones in the first *and* fourth auction, each with probability $\frac{1}{2}$. Then, selling both items in a bundle gives more profit than selling both items separately (where we investigate the expected profit corresponding to the optimal reserve price for each auction).

Which of the statements (A) to (G) are correct?

Illustration: Friederike Hofmann

Possible answers:

1. None of the statements (A) to (G) is correct.
2. Only the statements (A) and (B) are correct.
3. Only the statements (A), (B), and (C) are correct.
4. Only the statements (A), (B), and (D) are correct.
5. Only the statements (A), (B), (D), and (E) are correct.
6. Only the statements (A), (B), (C), (D), and (E) are correct.
7. Only the statements (A), (B), (D), (E), and (F) are correct.
8. Only the statements (A), (B), (E), (F), and (G) are correct.
9. Only the statements (A), (B), (D), (E), (F), and (G) are correct.
10. All of the statements (A) to (G) are correct.

Project reference:

The MATH+ project *Tropical Mechanism Design* studies the design of revenue-optimal auctions for multiple goods. A central difficulty when designing these auctions is to decide which goods should be bundled. In this project, these decisions are made with the help of tools from tropical geometry.

21.2 Solution

The correct answer is: 7.

(A) **The statement is correct:** With a reserve price of 2, the tacky vase will only be sold if Cato offers two fir cones, which happens with probability $\frac{1}{2}$. Since there are no other bidders, the profit in this case is just the reserve price, i. e. 2. Therefore, the expected profit is $2 \cdot \frac{1}{2} = 1$.

(B) **The statement is correct:** Since Cato only bids either one or two fir cones, it is only reasonable to consider the reserve prices 1 and 2.

For a reserve price of 2, we have already calculated in (A) the expected profit $E(2) = 1$.

For a reserve price of 1, the vase will be sold with probability 1, for the reserve price 1. In this case, the expected profit is also $E(1) = 1 = E(2)$ and thus not strictly greater than $E(2)$. Hence, a reserve price of 2 is optimal.

(C) **The statement is false:** Elisa and Frana bid either one or three and two or four pine cones, respectively—each with a probability of $\frac{1}{2}$. thus, for this auction there are four possible pairs of biddings,

$$(1,2),\ (1,4),\ (3,2),\ (3,4),$$

each of them with a probability of $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Let $\pi(m,(i,j))$ be the profit of the auction corresponding to the reserve price $m \in [0,\infty)$ and the bidding (i,j), where $i \in \{1,3\}$ and $j \in \{2,4\}$. Hence, the expected profit is for this auction with reserve prize m is

$$E(m) = \tfrac{1}{4}\big(\pi(m,(1,2)) + \pi(m,(1,4)) + \pi(m,(3,2)) + \pi(m,(3,4))\big).$$

One has

$$\pi(m,(1,2)) = \begin{cases} 1, & \text{for } 0 \le m \le 1, \\ m, & \text{for } 1 \le m \le 2, \\ 0, & \text{for } 2 < m, \end{cases}$$

$$\pi(m,(1,4)) = \begin{cases} 1, & \text{for } 0 \le m \le 1, \\ m, & \text{for } 1 \le m \le 4, \\ 0, & \text{for } 4 < m, \end{cases}$$

$$\pi(m,(3,2)) = \begin{cases} 2, & \text{for } 0 \le m \le 2, \\ m, & \text{for } 2 \le m \le 3, \\ 0, & \text{for } 3 < m, \end{cases}$$

$$\pi(m,(3,4)) = \begin{cases} 3, & \text{for } 0 \le m \le 3, \\ m, & \text{for } 3 \le m \le 4, \\ 0, & \text{for } 4 < m. \end{cases}$$

Consequently,

$$\begin{aligned} E(0) &= \tfrac{1}{4}(1+1+2+3) = \tfrac{7}{4} = 1\,\tfrac{3}{4}, \\ E(1) &= \tfrac{1}{4}(1+1+2+3) = \tfrac{7}{4} = 1\,\tfrac{3}{4}, \\ E(2) &= \tfrac{1}{4}(2+2+2+3) = \tfrac{9}{4} = 2\,\tfrac{1}{4}, \\ E(3) &= \tfrac{1}{4}(0+3+3+3) = \tfrac{9}{4} = 2\,\tfrac{1}{4}, \\ E(4) &= \tfrac{1}{4}(0+4+0+4) = 2. \end{aligned}$$

Since $E(4) = 2 < 2\,\frac{1}{4} = E(3) = E(2)$, a reserve prize of 4 is not optimal for the third auction.

(D) **The statement is correct:** First, we determine the optimal reserve price and the corresponding expected profit of the second auction: Since Luca only bids either one, two, or three fir cones for the sock sorting device, it only makes sense to consider the reserve prices 1, 2, and 3. For a reserve price of 1, the sock sorting device is sold with probability 1 at the reserve price 1. Thus, the expected profit in this case is also

$E(1) = 1$. For a reserve price of 2, the sock sorting device is sold if Luca offers two or three pine cones, i. e. with probability $\frac{2}{3}$. Since there are no other bidders, the profit in this case is just the reserve price, i. e. 2. Hence the expected profit in this case is $E(2) = 2\frac{2}{3} = \frac{4}{3}$. For a reserve price of 3, the sock sorting device is only sold if Luca bids three fir cones—this happens with a probability of $\frac{1}{3}$. In this case, the profit correspond to the reserve prize 3, and the expected profit is $3 \cdot \frac{1}{3} = 1$.

Consequently, the optimal reserve price for the second auction is 2, and the corresponding expected profit is $\frac{4}{3}$.

For the third auction, we already determined in (C) that a reserve prize of 0 gives an expected profit of $\frac{7}{4} > \frac{4}{3}$.

(E) **The statement is correct:** Since Cato bids with probability $\frac{1}{2}$ either one or two fir cones for each of the two items contained in the bundle, the following probabilities result for the bids on the bundle:

- With a $\frac{1}{4}$ chance he bids two;
- with a $\frac{1}{2}$ chance he bids three, and
- with a $\frac{1}{4}$ chance he bids four fir cones.

With a reserve price of 3, the selling probability is exactly $\frac{3}{4}$, and the expected profit is$3 \cdot \frac{3}{4} = \frac{9}{4}$.

(F) **The statement is correct:** Since Cato only bids two, three, or four fir cones, it only makes sense to consider the reserve prices 2, 3, and 4.

With a reserve price of 2, the bundle is sold in any event and the expected profit is precisely the reserve price, i. e. $E(2) = 2$.

According to (F), we have $E(3) = \frac{9}{4}$.

With a reserve price of 4 the bundle is sold with a probability of $\frac{1}{4}$, and the expected profit is $E(4) = 4 \cdot \frac{1}{4} = 1$.

Hence, $E(4) < E(2) < E(3)$, and the reserve prize 3 is optimal.

(G) **The statement is false:** First, consider the individual auctions of the two articles: With a reserve price of 1, the items are sold in any case and the expected profit is 1. With a reserve price of 4, the items are sold (each) with a probability of $\frac{1}{2}$, and the expected profit is 2. If both items are sold individually at the optimal reserve price (4), then the expected profit for both items together is $2 + 2 = 4$.

If both items are sold in a bundle, Cato bids

- with a $\frac{1}{4}$ chance two,
- with a $\frac{1}{2}$chance five, and
- with a $\frac{1}{4}$ chance eight fir cones.

Therefore, only the reserve prices 2, 5, and 8 are therefore eligible. For a reserve price of 2, the expected profit for the bundle is $E(2) = 2 < 4$. For a reserve price of 5, the expected profit for the bundle is $E(5) = 5 \cdot \frac{3}{4} = \frac{15}{4} = 3\,\frac{3}{4} < 4$. For a reserve price of 8, the expected profit for the bundle is $E(8) = 8\frac{1}{4} = 2 < 4$. Hence, we deduce it is better to sell the items separately.

22 Christmas Baubles

Author: Jacques Resing (TU Eindhoven)
Project: 4TU.AMI

22.1 Challenge

Santa Clause tells Ruprecht, “This box contains many blue and many red Christmas baubles. There are more blue baubles than red baubles.”
Ruprecht draws two baubles from the box, one of which is blue and one of which is red.
Santa Clause says, “Interesting! The probability to draw two baubles with different colors is 1/2.”

Ruprecht puts the two baubles back into the box, and then draws three baubles from the box. All three drawn baubles have the same color.
Santa Clause says, “Very interesting! The probability to draw three baubles with the same color is 1/4.”

Ruprecht puts the three baubles back into the box. Then he draws a red bauble, then a blue bauble, and finally another red bauble.
Santa Clause says, “Very, very interesting! The probability to draw red-blue-red is 0.123456, when rounded to six digits after the decimal point.”

Question: How many red and blue Christmas baubles are in the box (before the drawings)?

Artwork: Frauke Jansen

Possible answers:

1. There are 5,929 Christmas baubles in the box.
2. There are 6,084 Christmas baubles in the box.
3. There are 6,241 Christmas baubles in the box.
4. There are 6,561 Christmas baubles in the box.
5. There are 6,724 Christmas baubles in the box.
6. There are 6,889 Christmas baubles in the box.
7. There are 7,056 Christmas baubles in the box.
8. There are 7,396 Christmas baubles in the box.
9. There are 7,569 Christmas baubles in the box.
10. There are 8,281 Christmas baubles in the box.

22.2 Solution

The correct answer is: 4.

This challenge is a classical *urn problem*, where the balls are placed back after each drawing and the order of drawn balls is important.

We assume that the box contains b blue and r red baubles. In **Ruprecht's first drawing**, there are $2br$ possibilities to draw two differently colored baubles (blue-red and red-blue). Furthermore, there are $b(b-1)$ possibilities to draw two blue baubles and $r(r-1)$ possibilities, to draw two red baubles. Since the probability for drawing two differently colored baubles is $1/2$, the probability for drawing two equally colored baubles is also $1/2$. Accordingly, we obtain

$$\begin{aligned} & 2br = b(b-1) + r(r-1) \\ \iff \quad & 2br = b^2 - b + r^2 - r \\ \iff \quad & b + r = b^2 - 2br + r^r \\ \iff \quad & b + r = (b-r)^2. \end{aligned}$$

Hence, the total number $b + r$ of all baubles is a square number.

We set

$$x := b - r \tag{5}$$

and get

$$b + r = x^2. \tag{6}$$

Since $b, r \geq 1$, one has $x \geq 2$. Equations (5) and (6) are equivalent to

$$r = b - x, \qquad r = x^2 - b, \qquad \text{resp.}$$

Hence, it follows that

$$b = \frac{1}{2}(x^2 + x) \tag{7}$$

Analogously, one proves that

$$r = \frac{1}{2}(x^2 - x). \tag{8}$$

In **Ruprecht's second drawing**, there are $b(b-1)(b-2)$ possibilities for drawing three blue baubles and $r(r-1)(r-2)$ possibilities for drawing three red baubles. In this case, the number of favorable events is

$$
\begin{aligned}
&b(b-1)(b-2)+r(r-1)(r-2)\\
&=\frac{1}{2}(x^2+x)\left(\frac{1}{2}(x^2+x)-1\right)\left(\frac{1}{2}(x^2+x)-2\right)\\
&\quad+\frac{1}{2}(x^2-x)\left(\frac{1}{2}(x^2-x)-1\right)\left(\frac{1}{2}(x^2-x)-2\right)\\
&=\frac{1}{8}(x^2+x)(x^2+x-2)(x^2+x-4)+\frac{1}{8}(x^2-x)(x^2-x-2)(x^2-x-4)\\
&=\frac{1}{8}\left[x(x+1)(x-1)(x+2)(x^2+x-4)+x(x-1)(x-2)(x+1)(x^2-x-4)\right]\\
&=\frac{1}{8}\,x(x+1)(x-1)\left[(x+2)(x^2+x-4)+(x-2)(x^2-x-4)\right]\\
&=\frac{1}{8}\,x(x^2-1)(x^3+x^2-4x+2x^2+2x-8+x^3-x^2-4x-2x^2+2x+8)\\
&=\frac{1}{8}\,x(x^2-1)(2x^3-4x)\\
&=\frac{1}{4}\,x^2(x^2-1)(x^2-2). \qquad (9)
\end{aligned}
$$

Whereas, the number of possible outcomes in the second drawing is

$$(b+r)(b+r-1)(b+r-2)=x^2(x^2-1)(x^2-2). \qquad (10)$$

Therefore, the quotient of the number of favorable events (9) and the number of all possible outcomes (10) is $1/4$. Hence, Santa's second statement does not provide new information.

Finally, in **Ruprecht's third drawing**, there are $rb(r-1)$ possibilities for drawing red-blue-red. With (7) and (8), one obtains

$$\begin{aligned} rb(r-1) &= \frac{1}{2}(x^2-x)\,\frac{1}{2}(x^2+x)\left(\frac{1}{2}(x^2-x)-1\right) \\ &= \frac{1}{8}\,x^2(x-1)(x+1)(x^2-x-2) \\ &= \frac{1}{8}\,x^2(x-1)(x+1)(x+1)(x-2) \\ &= \frac{1}{8}(x-2)(x-1)x^2(x+1)^2. \end{aligned} \tag{11}$$

Since the number of all possible outcomes is again given by (10), the quotient of the number of favorable events (11) and the number of all possible outcomes (10) is

$$\begin{aligned} \frac{\frac{1}{8}(x-2)(x-1)x^2(x+1)^2}{x^2(x^2-1)(x^2-2)} &= \frac{1}{8}\,\frac{(x-2)(x-1)x^2(x+1)(x+1)}{x^2(x-1)(x+1)(x^2-2)} \\ &= \frac{1}{8}\,\frac{(x-2)(x+1)}{x^2-2} \\ &= \frac{1}{8}\,\frac{x^2-x-2}{x^2-2} \\ &= \frac{1}{8}\left[1-\frac{x}{x^2-2}\right]. \end{aligned}$$

The function $p(x) = \frac{1}{8}\left[1-\frac{x}{x^2-2}\right]$ is strictly increasing for $x > \sqrt{2}$. Hence, one has

$$p(x) \le p(80) = \frac{3159}{25592} \approx 0{,}1234370$$

for $2 \le x \le 80$. Analogously,

$$p(x) \ge p(82) = \frac{415}{3361} \approx 0{,}1234752.$$

holds for $x \ge 82$. For $x = 81$, finally one obtains

$$p(81) = \frac{3239}{26236} \approx 0{,}1234563.$$

In conclusion, $x = 81$ must hold and the number of baubles in the box is given by

$$b + r = x^2 = 81^2 = 6{,}561.$$

23 Covering Matchsticks with Paper

Authors: Nicolas Grelier (ETH Zürich)
Saeed Gh. Ilchi (ETH Zürich)
Tillmann Miltzow (Utrecht University)
Shakhar Smorodinsky (Ben-Gurion University)

23.1 Challenge

It is Christmas season, and Isabel is taking part in the annual mandatory candle lighting workshop "Light 'em up!" at *Santa's Academy*. She is very bored during her class, stops listening to the teacher, and places three matchsticks on her table. Now, she takes a very large piece of paper and notices that she is able to arrange the matchsticks such that she can cover any single matchstick with the paper without completely covering the other two. Furthermore, she is able to cover any two of them without completely covering the third.
"That was easy!", Isabel thinks and tries the same with four matchsticks. After a short time, she finds a configuration of the four matchsticks that allows her to cover every single, pair, and triple without completely covering the remaining ones (see Figure 22).

On the following weekend, Isabel travels from Wolfsburg to Leipzig to visit her cousin Jesko. Eagerly, she demonstrates her observations. Jesko is intrigued and positive about finding a similar configuration for five matchsticks. After an hour, he presents his beautiful solution: an arrangement of five matchsticks, where any single, pair, triple, and quadruple can be covered by a piece of paper, whereas the respective remaining matchsticks are at least partially visible. Motivated by Jeskos success, Isabel seeks a solutions for six matchsticks as well. But after a while, she gives up.

Of course, it is not allowed to use the corners of the paper to cover some matchsticks while leaving others uncovered. We rather think of the piece of paper as an infinite halfplane. Hence, we are looking for ways to arrange a number of matchsticks $M = \{m_1, \ldots, m_n\}$, $n \in \mathbb{N}$, such that every subset $S \subset M$ can be *covered*. We say that S is **covered** if every matchsticks of S is completely below the paper (i. e. the infinite halfplane), whereas every matchstick not in S is at most partially covered, but not completely.

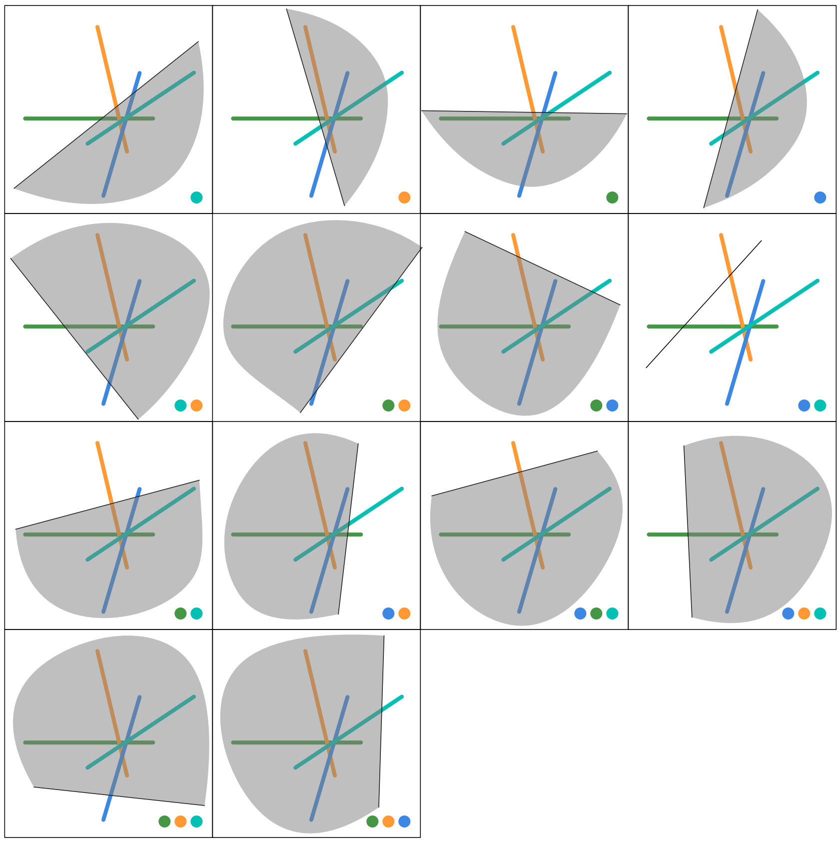


Figure 22: A configuration of four matchsticks such that each subset of matchsticks can be covered by a large piece of paper (gray).

Question: Which of the following statements are true?

(A) Isabel can improve her results: It is possible to arrange four matchsticks in the manner described above, with only two of the four overlapping.

(B) Jesko made a mistake: It is impossible to arrange five matches in the manner described above.

(C) Isabel and Jesko should have tried a little longer: It is even possible to arrange ten matchsticks in the manner described above.

(D) It is even possible to find such an arrangement for any number of matches.

Artwork: Frauke Jansen

Possible answers:

1. None of the statements is correct.
2. Only statement (A) is correct.
3. Only statement (B) is correct.
4. Only statement (C) is correct.
5. Only statements (A) and (B) are correct.
6. Only statements (A) and (C) are correct.
7. Only statements (B) and (C) are correct.
8. Only statements (A), (B), and (C) are correct.
9. Only statements (A), (C), and (D) are correct.
10. All of the above statements are correct.

23.2 Solution

The correct answer is: 2.

A detailed solution and more results on covering matchsticks can be found in the authors' manuscript https://arxiv.org/abs/1907.01241.

From now on, we call matchsticks *segments*. Covering all of subsets $S \subset M$ is called *shattering*.

First, we show that **statement (A) is correct:** Figure 23 shows a possible configuration of four shattered segments, where only two of the segments intersect.

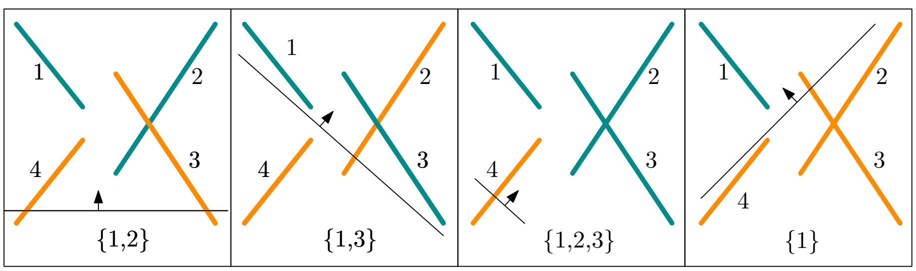


Figure 23: Shattering a set of 4 segments with only one intersection. We indicate how to shatter the subsets $\{1, 2\}, \{1, 3\}, \{1, 2, 3\}$, and $\{1\}$. All other subsets follow the same principle.

Next, we show that Jesko did not make a mistake and that **statement (B) is false:** Indeed, it is possible to shatter five segments, as shown in Figure 24.

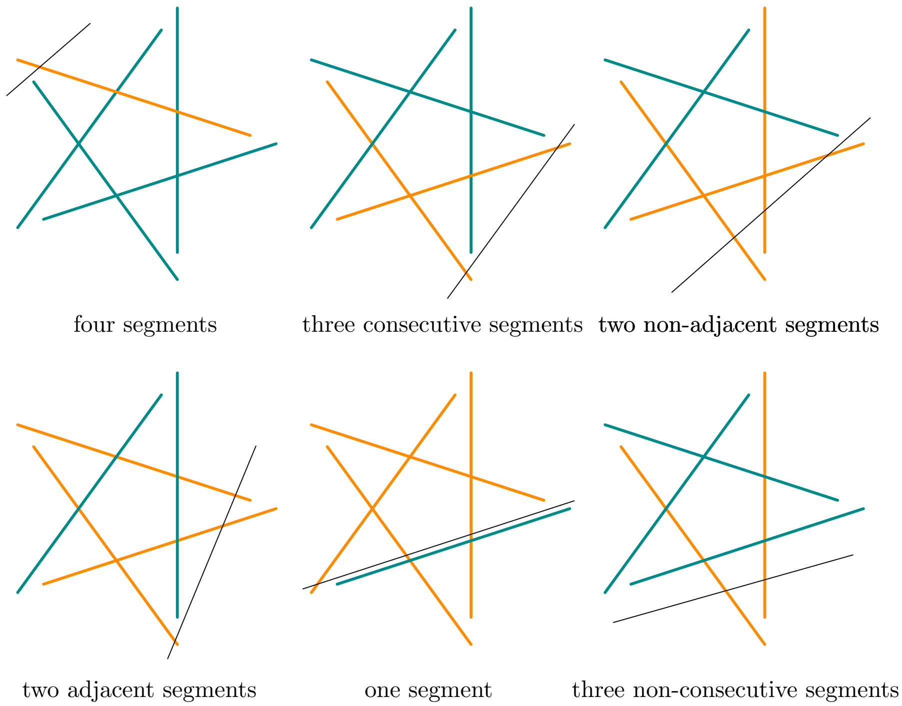


Figure 24: Shattering five segments. The shattered set is marked turquoise, the other segments are coloured orange. Due to the symmetry of the set of segments, we have shown that all hyperedges can be realized and thus the set can be shattered.

Finally, we show that **statements (C) and (D) are false:** To this end, let n be the number of segments that can be shattered. Note that there are $2^n - 2$ different sets of segments that need to be covered. Let S be a set of segments that is covered by the paper. Let ℓ be the line that represents the edge of the paper. We can move the paper while preserving the property that it covers S. At some point, ℓ will touch two endpoints of some segments, see Figure 25.

This implies that there are at most $4\binom{2n}{2} = 4n(2n-1)$ possible different sets of S that we can cover as there are only $\binom{2n}{2}$ many pairs of segment endpoints, and each pair of segment endpoints gives rise to at most 4 different sets (see Figure 26).

Note that $4n(2n-1) \geq 2^n - 2$ holds true only for $n \leq 9$ (see Table 6).

Hence, there is no way to shatter a set of ten (or more) segments.

One can actually show that it is not possible to shatter more than five segments; see the authors' manusscript on

https://arxiv.org/abs/1907.01241.

So Isabel tried in vain to cover the six matchsticks.

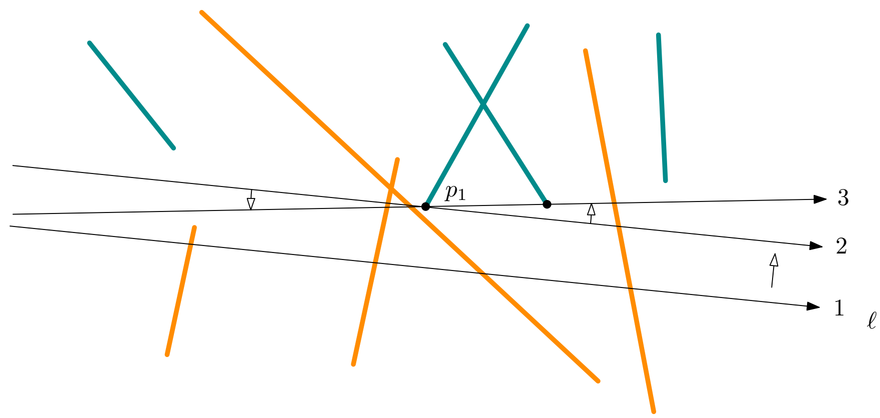


Figure 25: The line ℓ is shifted and rotated until it touches two segment endpoints. This is always possible.

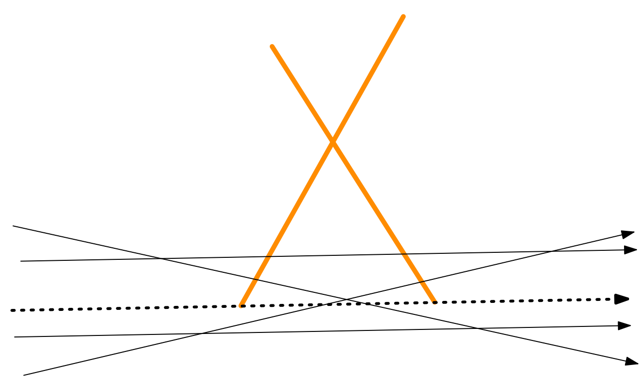


Figure 26: Each pair of segment endpoints gives rise to at most four sets to be covered.

n	$4n(2n-1)$	$2^n - 2$
1	4	0
2	24	2
3	60	6
4	112	14
5	180	30
6	264	62
7	364	126
8	480	254
9	612	510
10	760	1022

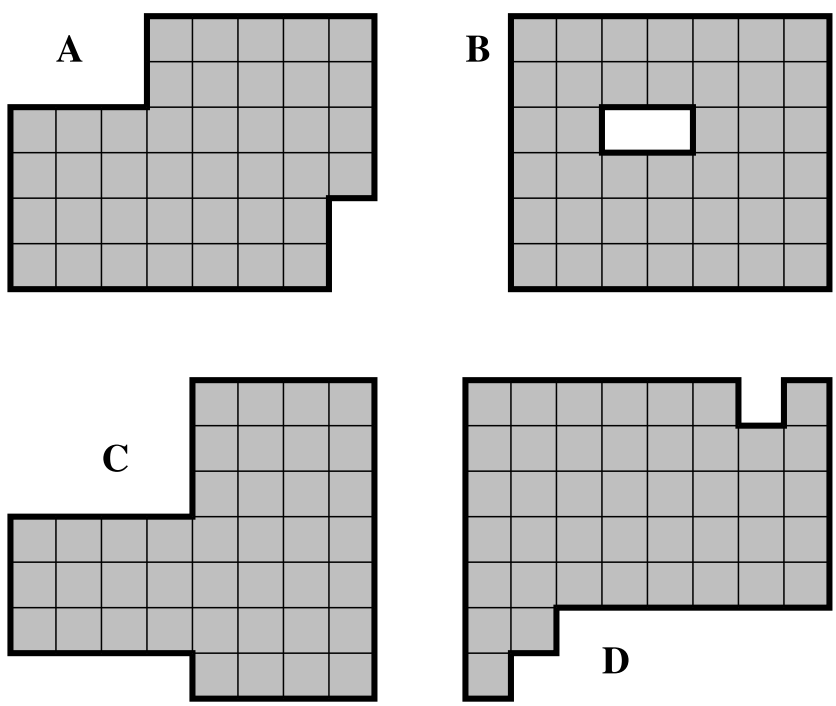
Table 6: Values of $4n(2n-1)$ and $2^n - 2$ for $1 \leq n \leq 10$.

24 Sawmill

Author: Hennie ter Morsche (TU Eindhoven)
Project: 4TU.AMI

24.1 Challenge

The sawing elves Zick and Zack are working in the sawmill of Santa Clause. Today's work schedule asks them to cut the wooden panel A into exactly two congruent pieces. All cuts must be made vertically or horizontally (along the lines in the picture) and must not go through the interior of the 40 small squares. Afterwards, the three wooden panels B, C, D have to be cut into exactly two congruent pieces each subject to analogous rules.



Zick scratches his head and complains, "This work schedule is horribly imprecise. It does not tell us the desired shape of these congruent pieces."
Also Zack scratches his head and laments, "Perhaps the central administration is once again sending us on an impossible mission. This has happened

before!"

Can you help Zick and Zack?

Illustration: Friederike Hofmann

Possible answers:

1. Only panels A and B can be cut into exactly two congruent pieces each.
2. Only panels A and C can be cut into exactly two congruent pieces each.
3. Only panels A and D can be cut into exactly two congruent pieces each.
4. Only panels B and C can be cut into exactly two congruent pieces each.
5. Only panels B and D can be cut into exactly two congruent pieces each.
6. Only panels A, B, C can be cut into exactly two congruent pieces each.
7. Only panels A, B, D can be cut into exactly two congruent pieces each.
8. Only panels A, C, D can be cut into exactly two congruent pieces each.

9. Only panels B, C, D can be cut into exactly two congruent pieces each.
10. All four panels can be cut into exactly two congruent pieces each.

24.2 Solution

The correct answer is: 10.

All four panels can be cut into exactly two congruent pieces. Hence, the central administration has done a good job.