MATH+ KALENDER

Challenges & Solutions
2020

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Contents

1 Pixy Hat

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Project: 4TU.AMI

1.1 Challenge

Some pixy hat contains exactly 80 euro banknotes. Santa Claus provides the following information on the contents of this hat to Ruprecht: "If you randomly draw 69 banknotes from this hat, then you are *guaranteed* to get

- one 100 € note,
- three 50 € notes,
- five 20 € notes,
- seven 10 € notes, and
- nine 5 € notes."

What is the overall value of all 80 banknotes in this hat?

Artwork: Friederike Hofmann

Possible answers:

1. The overall value of all bank notes is 2500 €.
2. The overall value of all bank notes is 2845 €.
3. The overall value of all bank notes is 2075 €.
4. The overall value of all bank notes is 2465 €.
5. The overall value of all bank notes is 2920 €.
6. The overall value of all bank notes is 2695 €.
7. The overall value of all bank notes is 2150 €.
8. The overall value of all bank notes is 2330 €.
9. The overall value of all bank notes is 2285 €.
10. The overall value of all bank notes is 2710 €.

1.2 Solution

The correct answer is: 1.

We denote the number of 100 €, 50 €, 20 €, 10 €, 5 € notes in the pixy hat by

$$x_{100},\ x_{50},\ x_{20},\ x_{10},\ \text{and } x_5,$$

respectively. Then, we observe the following:

- If $x_{100} \leq 11$, then Ruprecht would be able to draw 69 banknote without drawing a single 100 € note. We deduce that $x_{100} \geq 12$.
- If $x_{50} \leq 13$, then Ruprecht would be able to draw 69 banknote containing only two 50 € notes. We deduce that $x_{50} \geq 14$.
- If $x_{20} \leq 15$, then Ruprecht would be able to draw 69 banknote containing only four 20 € notes. We deduce that $x_{20} \geq 16$.
- If $x_{10} \leq 17$, then Ruprecht would be able to draw 69 banknote containing only six 10 € notes. We deduce that $x_{10} \geq 18$.
- If $x_5 \leq 19$, then Ruprecht would be able to draw 69 banknote containing only eight 5 € notes. We deduce that $x_5 \geq 20$.

Adding the above inequalities, we obtain

$$x_{100} + x_{50} + x_{20} + x_{10} + x_5 \geq 12 + 14 + 16 + 18 + 20 = 80.$$

Since there were only 80 in the pixy hat, every inequality is, in fact, an equation:

$$x_{100} = 12,\ x_{50} = 14,\ x_{20} = 16,\ x_{10} = 18,\ x_5 = 20.$$

Therefore, the total value of all the banknotes in the hat is

$$12 \cdot 100\,€ + 14 \cdot 50\,€ + 16 \cdot 20\,€ + 18 \cdot 10\,€ + 20 \cdot 5\,€ = \mathbf{2500\,€}.$$

2 Snowflake Gas

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Project: 4TU.AMI

2.1 Challenge

Over dinner, physics pixy Philomena tells the following story: “Today, in the physics laboratory, we performed an experiment with snowflake gas. As all of you must know, snowflake gas is an ideal gas that perfectly obeys all the mathematical laws of thermodynamics. In particular, snowflake gas satisfies the so-called *ideal gas law*,

$$p \cdot V = k_B \cdot N \cdot T.$$

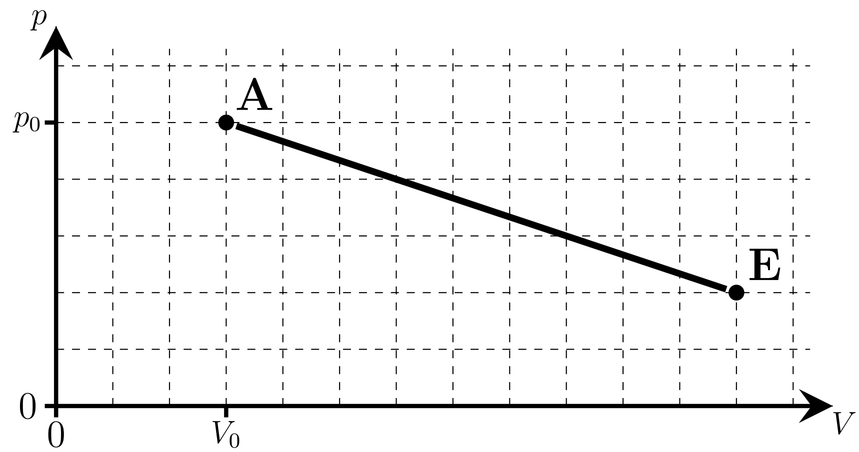
In the ideal gas law,

- p denotes the pressure of the gas,
- V the volume,
- N the number of particles, and
- T the temperature.

Furthermore, the ideal gas law contains the famous *Boltzmann constant*

$$k_B \approx 1.380\,652 \cdot 10^{-23}\, J/K.$$

Throughout the experiment, we managed to keep the amount of substance N constantly at the value ...” At this moment, Philomena chokes on her food and starts coughing, so that nobody at the table gets the value of N. After recovering, Philomena fetches a diagram (as shown in the picture) from her briefcase.



Philomena explains the diagram: "Our experiment went for 8 hours. During the experiment, we continuously measured the pressure, volume, and temperature of the gas. At the beginning of the experiment, the pressure was at $p_0 = 8.0\,\text{bar}$ and the volume was at $V_0 = 800\,\text{liter}$. The diagram shows the relation between the volume V and the pressure p from the beginning (A) to the end (E) of the experiment as a straight line."

Mathematics pixy Mathilda has listened carefully to the story. Now, she says: "Okay, I get it! During the experiment, the temperatur T of the gas first started to increase. Then, after some time, the temperature reached its maximum value. From this point onwards, the temperature decreased until the end of the experiment."

Philomena agrees: "Yes, you are right! That's exactly the way the snowflake gas behaved!"

What was the pressure p^* of the gas at the time when the temperature reached its maximum value?

Artwork: Friederike Hofmann

Possible answers:

1. The pressure of the gas was $p^* \approx 3.6\,\text{bar}$.
2. The pressure of the gas was $p^* \approx 4.0\,\text{bar}$.
3. The pressure of the gas was $p^* \approx 4.4\,\text{bar}$.
4. The pressure of the gas was $p^* \approx 4.8\,\text{bar}$.
5. The pressure of the gas was $p^* \approx 5.2\,\text{bar}$.
6. The pressure of the gas was $p^* \approx 5.6\,\text{bar}$.
7. The pressure of the gas was $p^* \approx 6.0\,\text{bar}$.
8. The pressure of the gas was $p^* \approx 6.4\,\text{bar}$.
9. The pressure of the gas was $p^* \approx 6.8\,\text{bar}$.
10. The pressure of the gas was $p^* \approx 7.2\,\text{bar}$.

2.2 Solution

The correct answer is: 4.

This challenge has a short and simple solution, which ignores most of the information given in the task:

In the given diagram, the pressure p_0 corresponds to five (vertical) unit lengths, whereas the volume V_0 corresponds to three (horizontal) unit lengths. Hence, we scale the volume's unit such that

$$\frac{V_0}{p_0} = \frac{3}{5}.$$

The straight line depicted in the diagram is given by the points

$$A = (V_0, p_0) = \left(\frac{3p_0}{5}, p_0\right),$$

$$E = \left(4V_0, \frac{2p_0}{5}\right) = \left(\frac{12p_0}{5}, \frac{2p_0}{5}\right).$$

Hence, it is given by the following linear equation

$$p(V) = -\frac{V}{3} + \frac{6p_0}{5}$$

or, equivalently, by

$$V(p) = 3\left(\frac{6p_0}{5} - p\right).$$

Since the number of particles N and the Boltzmann constant k_B are constant, the ideal gas law implies that the temperature T and the product of pressure and volume, $p \cdot V$ are proportional, i. e.

$$T = \frac{p \cdot V}{k_B \cdot N}.$$

Consequently, the temperature is maximal if the product

$$p \cdot V = p \cdot V(p) = 3p \cdot \left(\frac{6p_0}{5} - p\right)$$

is maximal. The parabola

$$f(p) = 3p \cdot \left(\frac{6p_0}{5} - p\right)$$

has its vertex (i. e. its maximum) at

$$p^* = \frac{3p_0}{5} = \frac{3}{5} \cdot 8\,\text{bar} = 4.8\,\text{bar}.$$

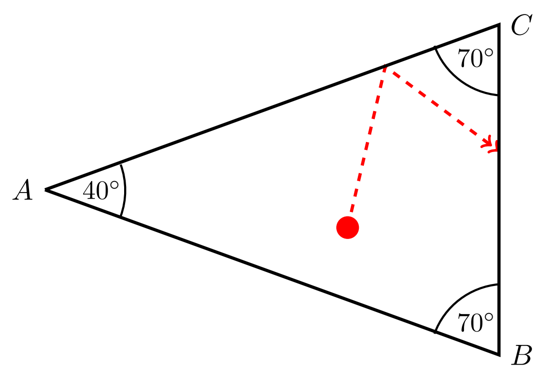
Hence, the temperature is maximal at $p^* = \mathbf{4.8\,bar}$.

3 Billiards Table

Author: Hennie ter Morsche (TU Eindhoven)
Project: 4TU.AMI

3.1 Challenge

There is a large triangular billiards table standing in the pixies' lounge. The angle at corner A is 40°, whereas the two angles at B and C each are 70°. If the ball hits one of the rails AB or AC, it is perfectly reflected so that the angle of reflection is equal to the angle of incidence. However, if the ball hits the sticky rail BC or if it hits one of the three corners A, B, C, it gets stuck and stops moving.



Conveniently, Ruprecht plays with a point-shaped ball that initially is located somewhere in the interior of the triangle and that moves only along straight lines. Ruprecht wants to make a single shot that scores as many rail contacts as possible before the ball gets stuck at some rail or point.

What is the largest possible number of such rail contacts?

Artwork: Julia Nurit Schönnagel

Possible answers:

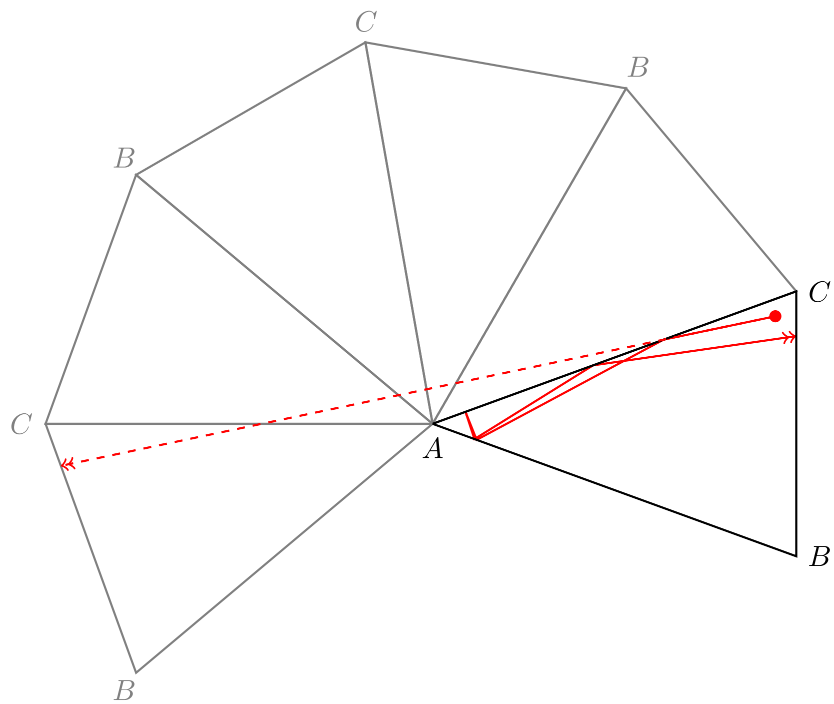
1. The largest possible number of rail contacts is 4.
2. The largest possible number of rail contacts is 5.
3. The largest possible number of rail contacts is 6.
4. The largest possible number of rail contacts is 7.
5. The largest possible number of rail contacts is 8.
6. The largest possible number of rail contacts is 12.
7. The largest possible number of rail contacts is 16.
8. The largest possible number of rail contacts is 24.
9. Ruprecht may score infinitely many rail contacts.
10. The largest possible number of rail contacts depends on the length of BC.

3.2 Solution

The correct answer is: 2.

Since the third rail BC is absorbent, the ball can only run back and forth between the two rails AB and AC. Since this configuration is symmetric, we may assume that the ball touches AC first, then AB, then again AC, and so on.

The following figure shows the billiards table (at the right), together with five symmetric copies:



Since the angle of reflection is equal to the angle of incidence for each reflection at both of the rails AB and AC, the path of the ball corresponds to a straight line running from the starting point and the point of absorption. The dashed line shows such a possible path, which allows for exactly five rail contacts. (The solid line represents the corresponding movement of the ball on our billiards table.)

The rail AC of the original triangle and the rail AB of the fifth copy enclose an angle of $5 \cdot 40^\circ = 200^\circ$. Since this angle is larger than 180°, one cannot find a straight line that runs through more than five copies of the triangle ABC. Hence, six (or more) rail contacts are not possible.

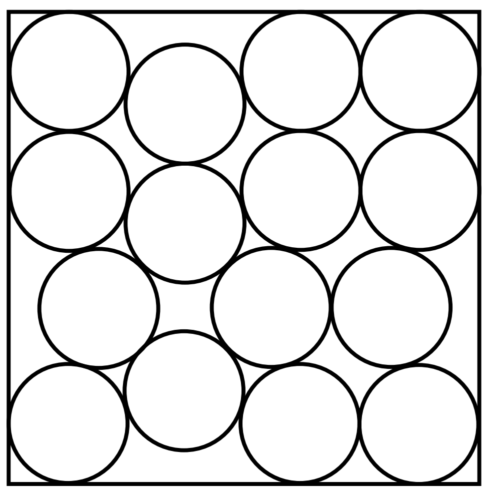
4 Christmas Candles

Authors: Cor Hurkens (TU Eindhoven),
Hennie ter Morsche (TU Eindhoven)

Project: 4TU.AMI

4.1 Challenge

The packer-pixy Paco packs fifteen Christmas candles upright into a box with a square base. Each candle is shaped like a cylinder with a circle of radius 1 cm as its base. The picture shows the base of the box as a square and the bases of the fifteen candles as fifteen circles.



The picture shows which candles touch each other and which candles touch the boundary of the box. The side length s of the square can be written in the form $s = a+\sqrt{b}+\sqrt{c}$ (measured in centimeters) with three integers a, b, c.

What is the value of the sum $a + b + c$ of these three integers?

Artwork: Frauke Jansen

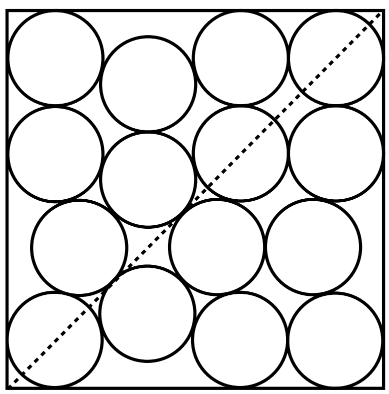
Possible answers:

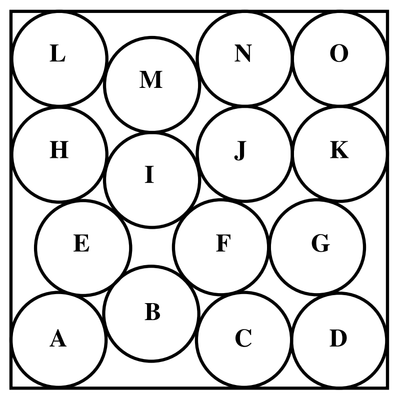
1. The sum of a, b, c is 8.
2. The sum of a, b, c is 9.
3. The sum of a, b, c is 10.
4. The sum of a, b, c is 11.
5. The sum of a, b, c is 12.
6. The sum of a, b, c is 13.
7. The sum of a, b, c is 14.
8. The sum of a, b, c is 15.
9. The sum of a, b, c is 16.
10. The sum of a, b, c is 17.

4.2 Solution

The correct answer is: 5.

The diagonal line, as shown in the left figure, provides a symmetry axis for the given arrangement of 15 circles. This statement can be understood either by looking closely or by making an educated guess. Of course, one can also prove the claim directly. To this end, we denote the circles as indicated in the right figure.





Now, we consider a coordinate system such that its origin coincides with the lower left corner of the square. One unit in the coordinate system corresponds to 1 cm. The centres of the circles are labelled M_A, M_B, M_C etc. If a circle touches a square side, its centre has a distance of exactly 1 cm from that side. If two circles touch each other, their centres are exactly 2 cm apart.

- Since $M_A = (1, 1)$ lies on the diagonal, the circle A is its own mirror image.

  Analogously, since $M_O = (s-1, s-1)$ and $M_J = (s-3, s-3)$ also lie on the diagonal, the circles O and J are their own mirror image too.

- Since $M_D = (s-1, 1)$ and $M_L = (1, s-1)$, the circles D and L are mirror images of each other.

  Analogously, since $M_C = (s-3, 1)$ and $M_H = (1, s-3)$, the circles C and H are mirror images of each other.

  Furthermore, since $M_K = (s-1, s-3)$ and $M_N = (s-3, s-1)$ the circles K and N are mirror images of each other.

- The circles B and E touch. The circle B touches A and C. Furthermore, the circle E touches A (which is it own mirror image) and H (the mirror image of C).

  Hence, the circles B and E are mirror images of each other.

- The circle F touches B and C (which also touch each other). Furthermore, the circle I touches E and H (which also touch each other).

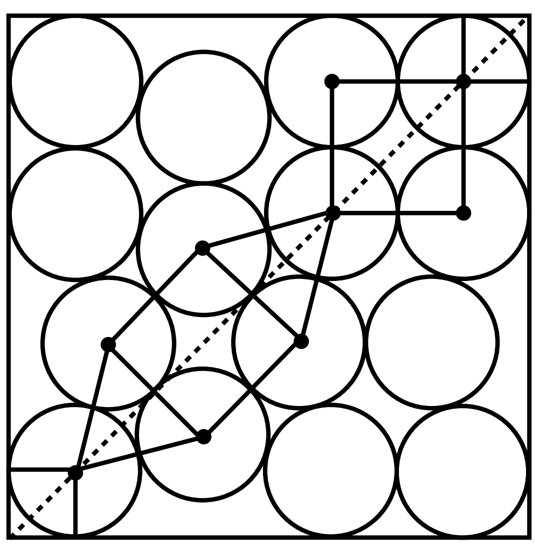
  Since B / C is the mirror image of E / H, the circles F and I are mirror images of each other.

- The circle G touches D, F, and K. In addition, the circle M touches L, I, and N.

  Since D / F / K is the mirror image of L / I / N, the circles G and M are mirror images of each other.

Thus, the claimed symmetry of the given configuration of Christmas candles is proven. In consequence, we are able to observe the following:
The quadrilateral M_B, M_F, M_I, M_E is in fact a rhombus, since its four edges all have length 2. Furthermore, the line segments M_BM_F and M_EM_I are mirror images of each other. Hence, the rhombus M_B, M_F, M_I, M_E is actually a square.

Next, we want to determine the length of the diagonal, which is composed (from bottom left to top right) of the following six line segment:



1. The diagonal of a small square with edges of length 1, i. e. of length $\sqrt{2}$.
2. The altitude of an equilateral triangle with edges of length 2, i. e. of length $\sqrt{3}$.
3. The perpendicular bisector of a small square with edges of length 2, i. e. of length 2.
4. The altitude of a second equilateral triangle with edges of length 2, i. e. of length $\sqrt{3}$.
5. The diagonal of a second large square with edges of length 2, i. e. of length $2\sqrt{2}$.
6. The diagonal of a second small square with edges of length 1, i. e. of length $\sqrt{2}$.

Hence, the length of the diagonal is

$$d = \sqrt{2} + \sqrt{3} + 2 + \sqrt{3} + 2\sqrt{2} + \sqrt{2} = 2 + 4\sqrt{2} + 2\sqrt{3}.$$

Since d is the diagonal of square with edges of length s, one has $d = s\sqrt{2}$. Consequently,

$$s = \sqrt{2} + 4 + \sqrt{6}.$$

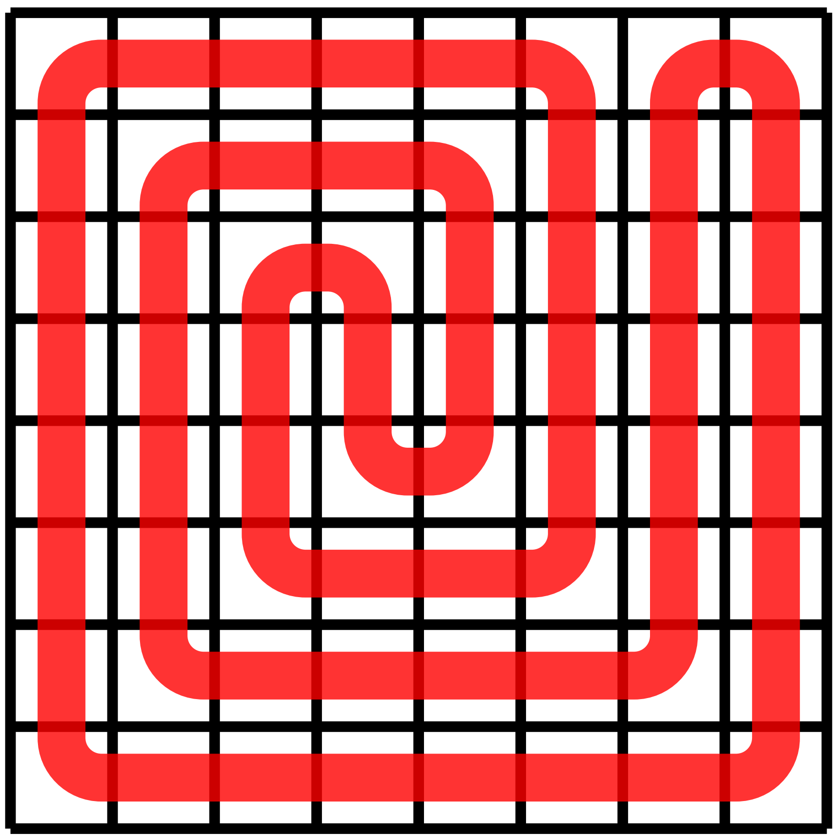
Thus, we obtain $\{a, b, c\} = \{2, 4, 6\}$ and $a + b + c = \mathbf{12}$.

5 Mondrian

Author: Hennie ter Morsche (TU Eindhoven)
Project: 4TU.AMI

5.1 Challenge

Mondrian the painter-pixy has designed a square Christmas card, which he has divided into 64 little square cells, as shown in the picture. Mondrian dips the paintbrush into the red paint pot and draws a long continuous brushstroke that traverses each of the 64 square cells exactly once and that returns to the starting cell in the end.



The picture shows a red brushstroke that traverses 48 cells in a straight line and 16 cells in quarter circles. Since a quarter circle consumes less paint than a straight line, Mondrian would like to find an efficient brushstroke that maximizes the number of quarter circles.

What is the largest possible number M of quarter circles in such a brush-stroke?

Artwork: Julia Nurit Schönnagel

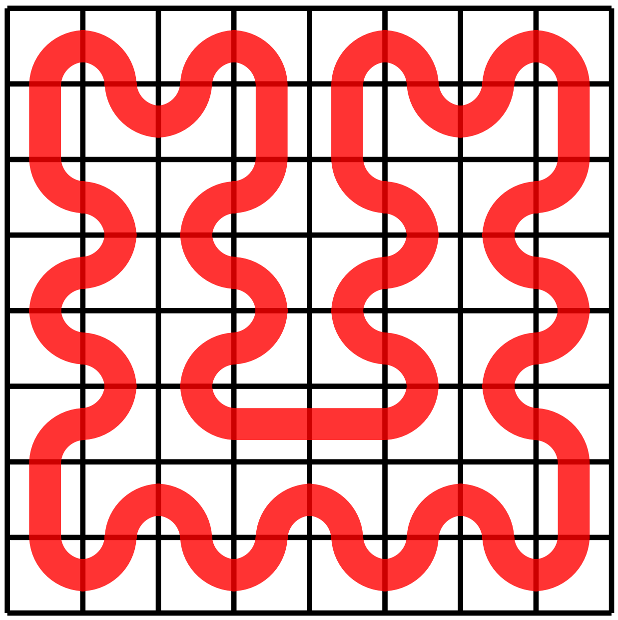
Possible answers:

1. The largest possible number of quarter circles is $M = 42$.
2. The largest possible number of quarter circles is $M = 44$.
3. The largest possible number of quarter circles is $M = 46$.
4. The largest possible number of quarter circles is $M = 48$.
5. The largest possible number of quarter circles is $M = 50$.
6. The largest possible number of quarter circles is $M = 52$.
7. The largest possible number of quarter circles is $M = 54$.
8. The largest possible number of quarter circles is $M = 56$.
9. The largest possible number of quarter circles is $M = 58$.
10. The largest possible number of quarter circles is $M = 60$.

5.2 Solution

The correct answer is: 8.

The following picture shows a possible brush stroke with 56 quarter circles.



We now want to show that the amount of 56 quarter circles cannot be exceeded. To do this, we assume for the sake of contradiction that a brush stroke P with at least 57 quarter circles exists and make the following definitions:

- A square q is called **good** if q contains a quarter circle of P.

  On the other hand, q is called **bad** if it contains a straight line piece of P.

- If a row or column does not contain a single bad square, we call it **clean**.

Thus, according to the assumption, the brush stroke P contains at least 57 good squares and at most 7 bad squares.

We look at a section of the brush stroke P in the i-th row and notice that P enters the i-th row in a square q, passes through some more squares in this row and then leaves the row again in the square q':

- If $q = q'$ holds, then P traverses the square q in the vertical direction and q is therefore bad.
- If $q \neq q'$ holds, then the two squares q and q' are good, while all squares visited in between are bad.

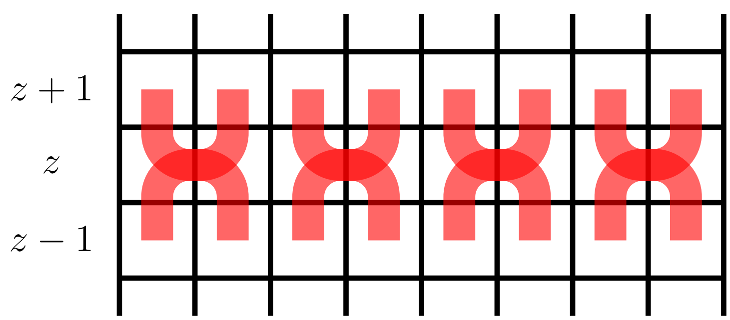
Since each such piece of the brushstroke contributes an even number of good squares (either none or two good squares), the total number of good squares in each row is even.
Since each row contains an even number of good squares, each row also contains an even number of bad squares. And since there are at most seven bad squares, at least five rows are clean.
From this we further conclude that there are at least two *consecutive* clean rows $2a-1$ and $2a$, for $a \in \{1, 2, 3, 4\}$.

Similarly, we can show that there are two consecutive clean columns $2b-1$ and $2b$, $b \in \{1, 2, 3, 4\}$.

Now let us take a closer look at a *clean* row z: the brush stroke P enters the square with coordinates $(z, 1)$ from either $(z-1, 1)$ or $(z+1, 1)$, then continues to $(z, 2)$, and then to $(z-1, 2)$ or $(z+1, 2)$. We see that the brushstroke P has to trespass the four lines

$$(z, 1) - (z, 2), \qquad (z, 3) - (z, 4), \qquad (z, 5) - (z, 6), \qquad (z, 7) - (z, 8)$$

successively each:



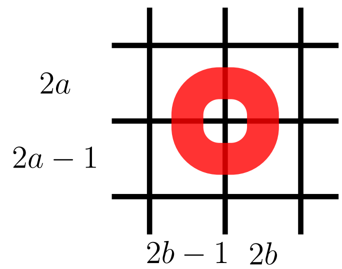
Similarly, we see that the brush stroke P must trespass the four lines of

$$(1, s) - (2, s), \qquad (3, s) - (4, s), \qquad (5, s) - (6, s), \qquad (7, s) - (8, s)$$

in each *clean* column s.

For the two clean rows $z = 2a - 1$ and $z = 2a$ and for the two clean columns $s = 2b - 1$ and $s = 2b$ it follows from this discussion that the brush stroke P must follow the following path:

$$(2a-1, 2b-1) - (2a-1, 2b) - (2a, 2b) - (2a, 2b-1) - (2a-1, 2b-1).$$



However, this path forms a closed circle which is separated from the rest of the brushstroke P—a contradiction to the premise that P can be drawn without lifting the brush.

We summarise that under the given conditions it is impossible to draw a brushstroke with more than **56** quarter circles.

6 Game of Cubes

Authors: Frits Spieksma (TU Eindhoven),
Jesper Nederlof (TU Eindhoven)

Project: 4TU.AMI

6.1 Challenge

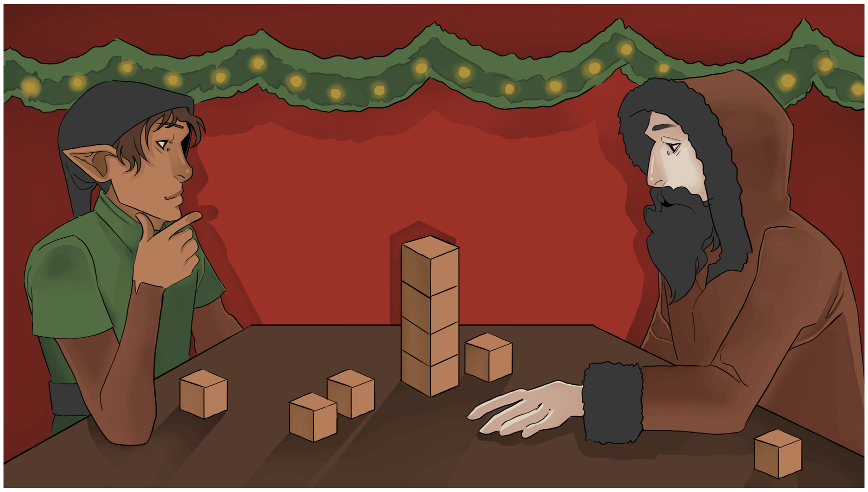
Elf Kubo and Ruprecht play a game with $N \geq 4$ wooden cubes. At the beginning of the game, all six faces of each of these N cubes are empty and unlabeled.

In the **first phase** of the game, the two players label the $6N$ faces of the cubes with integers from the range $1, 2, \ldots, N$. In every move, exactly one face of one cube is labeled. They take turns at moving with Ruprecht making the first move.

In the **second phase** of the game, they build a tower from the N wooden cubes. The first (and bottom-most) cube in the tower must carry the integer 1 on one of its faces, the second one the integer 2, the third cube the integer 3, and so on. Ruprecht and Kubo take turns at choosing a cube with Ruprecht picking the first (and hence bottom-most) cube in the tower. The game ends only if in the k-th move there is no cube with integer k available.

Ruprecht wins the game if at the end of the game the tower consists of all N cubes. Otherwise, Kubo is the winner. During both phases of the game, Kubo and Ruprecht always make the best possible moves.

For which values of N with $4 \leq N \leq 7$ can Ruprecht enforce a win?



Artwork: Julia Nurit Schönnagel

Possible answers:

1. Ruprecht can enforce a win only for $N = 4$.
2. Ruprecht can enforce a win only for $N = 4$, 5.
3. Ruprecht can enforce a win only for $N = 4$, 6.
4. Ruprecht can enforce a win only for $N = 4$, 7.
5. Ruprecht can enforce a win only for $N = 5$, 6.
6. Ruprecht can enforce a win only for $N = 4$, 5, 6.
7. Ruprecht can enforce a win only for $N = 4$, 5, 7.
8. Ruprecht can enforce a win only for $N = 4$, 6, 7.
9. Ruprecht can enforce a win only for $N = 5$, 6, 7.
10. Ruprecht can enforce a win for $N = 4$, 5, 6, 7.

6.2 Solution

The correct answer: 10.

Ruprecht can enforce a victory for all N in range $4 \leq N \leq 7$.

An important observation for Ruprecht's winning strategy is that in the first phase Ruprecht can label three side faces of each cube with numbers of his choice. He manages to do so in the following way: first, Ruprecht labels any one cube with some number. In the following moves, he always puts a number on the cube that Kubo has just labeled before.

The game with N = 4 cubes: In the first phase, Ruprecht labels one side of a cube with 1 and writes the three numbers 2, 3, 4 on each of the other three cubes.
In the second phase, Ruprecht uses the first cube he labeled as the base cube for the tower. In each of the other three moves, there is a cube with the three numbers 2, 3, 4. Therefore, Ruprecht wins the game.

The game with N = 5 cubes: In the first phase, Ruprecht labels

- one cube W_1 with 1,
- a second cube W_2 with the numbers 2, 3, 4,
- and the remaining three cubes W_3, W_4, W_5 *each* with the numbers 3, 4, 5.

In the second phase, Ruprecht first uses W_1 as the base cube. Then, Kubo chooses one cube W_x. If $W_x \neq W_2$, Ruprecht chooses W_2 in the following turn and wins. If, on the other hand, $W_x = W_2$, Ruprecht chooses W_3 on the following turn and wins as well.

The game with N = 6 cubes: In the first phase, Ruprecht labels

- one cube W_1 with 1,
- a second cube W_2 with 2, 3, 4,
- a third cube W_3 with 3, 4, 5,
- and the three remaining cubes W_4, W_5, W_6 *each* with 4, 5, 6.

In the second phase of the game, Ruprecht starts using W_1. Afterwards, it is Kubo's turn, and there are at least

- one cube with the number 2 (W_2),
- two cubes with the number 3 (W_2, W_3),
- three cubes with the number 4 (W_2, W_3, W_4), and
- four cubes with number 5 (W_3, W_4, W_5, W_6) available.

So regardless of Kubo's decisions, the tower will consist of at least five cubes. Ruprecht can also make sure that W_2 is built into the tower as the second cube (by Kubo) or as the third cube (by Ruprecht himself). Furthermore, he can ensure that W_3 is built into the tower as the second, third, fourth or fifth cube. So after the fifth cube is placed, W_1, W_2, W_3 are already a part of the tower. All remaining cubes have the number 6 on one of their faces. The game continues, and the tower contains $N = 6$ cubes in the end. Ruprecht wins.

The game with N = 7 cubes: In the first phase, Ruprecht labels

- one cube W_1 with 1,
- a second cube W_2 with 2, 3, 4,
- a third cube W_3 with 3, 4, 5,
- a fourth cube W_4 with 4, 6, 7,
- and the remaining three cubes W_5, W_6, W_7 *each* with 5, 6, 7.

Now, we can argue as in the case $N = 6$: In the second phase, Ruprecht first uses W_1 as the base cube. Afterwards, it is Kubo's turn, and there are at least

- one cube with the number 2 (W_2),
- two cubes with the number 3 (W_2, W_3),
- three cubes with the number 4 (W_2, W_3, W_4), and
- four cubes with number 5 (W_3, W_5, W_6, W_7) available.

Regardless of Kubo's decisions, the tower will consist of at least five cubes. In addition, Ruprecht can ensure that the cubes W_1, W_2, W_3 are used in the first five turns. All remaining cubes have both the number 6 and the number 7 on one of their faces. Hence, the game continues until the tower consists of $N = 7$ cubes. Ruprecht wins.

7 Tricky Testing Task

Author: Paul Erchinger (MATH+ School Activities)

7.1 Challenge

Oh deer! It's Christmas flu season. Normally, this does not pose a problem for the Christmas preparations, because Christmas elves are immune to this virus. But this year, a particularly insidious Easter mutation seems to circulate. With only four weeks left until Christmas, several of Santa's little helpers are infected, having grown long ears and whiskers. Hence, the Christmas crisis management team meets and decides that the elves need to be tested thoroughly. The plan is to quickly find sick elves, before even more of them become infected.

Of course, the test has to be carried out at the Christmas headquarters at the North Pole too. But testing capacities in the Arctic are quite scarce: due to massive cracks in the ice floe landscape, the sledge, which usually delivers all of the supplies, cannot reach the headquarters. Hence, for Santa and his 14 elves, there are only ten *CE tests*[1] left. Fortunately, they were delivered by mail some time ago.

Santa is at a loss: how can he test the entire staff of 15 people without the sufficient capacity? Furthermore, at this year's Halloween party, all elves could have infected one another. Fortunately, elf Annelie has an idea, "Do you remember our seating arrangements for Halloween? We sat down to dine at five tables. In order to not get bored so easily, we changed seats after the main course. Thus, no one sat at a table twice with the same elves. I should still have the list somewhere..." After a short rummage, she pulls the list out of a box containing fluffy pink Easter Bunny overalls:

	Table 1	Table 2	Table 3	Table 4	Table 5
Main	Annelie	Boris	Carmen	Dirk	Esmeralda
	Frank	Gabi	Hannes	Irene	Jakob
	Klara	Lorenz	Mara	Nathan	Santa

[1] It is known that elf cells are normally strongly attracted to Christmas symbols. Therefore, the CE test involves holding both a Christmas tree ball and an Easter egg in the test sample. If the cells react more strongly to the Easter egg, they are infected by the Easter mutated virus. Unfortunately, a fresh Easter egg is needed for each sample, which is why the tests are very resource-consuming and therefore need to be rationed.

Dessert	Table 1	Table 2	Table 3	Table 4	Table 5
	Frank	Klara	Dirk	Esmeralda	Annelie
	Boris	Nathan	Mara	Hannes	Irene
	Carmen	Jakob	Gabi	Lorenz	Santa

"We just take two samples from each of us and mix the samples of those who shared the same table. This way, we only need to do five tests for the main course and five for the dessert, which amounts to only ten tests! Then, we simply identify those elves as sick who were in groups that tested positive in *both* test runs. Those elves need to isolated from the others."

A lively discussion begins:

Boris states, "With this strategy, we will never find all the infected elves..."

Annelie responds, "On the contrary! We will always find all the infected ones."

Carmen reckons, "That's right! But we also get *false positive* test results. That means: elves who are not infected at all could still be tested positive. This time of the year, we cannot afford them to be absent from work..."

Jakob replies, "But, since we never shared both tables with the same elves, there will be no false positive test results."

Mara has a different opinion, "Unfortunately, your statement is not correct. However, if we knew that at most two people were infected, it would be true."

Lorenz believes, "No! Even then, some elves could get a false positive test result."

Esmeralda has another idea, "We just have to divide the groups in a different way:

Group 1	Group 2	Group 3	Group 4
Annelie	Boris	Carmen	Dirk
Esmeralda	Frank	Gabi	Hannes
Irene	Jakob	Klara	Lorenz
Mara	Nathan	Santa	

Group 5	Group 6	Group 7	Group 8
Annelie	Esmeralda	Irene	Mara
Boris	Frank	Jakob	Nathan
Carmen	Gabi	Klara	Santa
Dirk	Hannes	Lorenz	

> Moreover, if we knew that at most two of us were infected, we could always find out unambiguously who it was."

Santa is as clueless as before: who is right? After all, he has to organise the whole testing...

Artwork: Frauke Jansen

Possible answers:

1. Only Boris.
2. Only Annelie and Carmen.
3. Only Annelie and Jakob.
4. Only Annelie, Carmen, and Mara.
5. Only Annelie, Carmen, and Lorenz.
6. Only Boris und Esmeralda.
7. Only Annelie, Carmena, and Esmeralda.

8. Only Annelie, Jakob, and Esmeralda.
9. Only Annelie, Carmen, Mara, and Esmeralda.
10. Only Annelie, Carmen, Lorenz, and Esmeralda.

7.2 Solution

The correct answer is: 10.

We go through the statements one by one:

Boris' statement is not correct: Each infected elf causes a positive test result in their main course group as well as in their dessert group. Thus, this elf will also be in the subgroup of elves that had positive test in both runs.

We conclude that **Annelie's statement is in fact correct.**

Carmens statement is correct as well: Consider the case where only Annelie and Boris are infected. Then, Frank is part of a group with a positive test result in both runs. Hence, Frank would be identified as sick, although he is not infected.

We see that **Jakob's statement is not correct**.

In general, **Mara's statement is not correct:** Assume that Annelie and Jakob were infected. Then, in the first run, tables 1 and 5 would get a positive test result. In the second run, tables 2 and 5 would get a positive result. The additional information, that at most two elves are sick, does not dissolve the unambiguity: the same test results would have been obtained if Klara and Santa were the infected ones.

In particular, **Lorenz' statement is correct.**

Finally, **Esmeralda's statement is indeed correct** as well: Observing Esmeralda's testing strategy, we note that the suggested division can be depicted in the following two-dimensional array:

Group 1	**Group 2**	**Group 3**	**Group 4**	
Annelie	Boris	Carmen	Dirk	**Group 5**
Esmeralda	Frank	Gabi	Hannes	**Group 6**
Irene	Jakob	Klara	Lorenz	**Group 7**
Mara	Nathan	Santa		**Group 8**

Hence, we need eight tests to identify the rows and columns of the infected elves. If only one elf is infected, Esmeralda's strategy identifies this one without ambiguity. (By the way, this is also true for Annelie's strategy.)

Now, consider the case where two elves are infected. Then, Esmeralda's strategy yields up to four potentially sick candidates.
(Assume, for instance, that Esmeralda and Jakob were infected. Then, groups 1, 2, 6, and 7 had positive test results. This leaves us with the following four candidates: Esmeralda, Frank, Irene, and Jakob.)
Now, we test one of these four candidates, who we denote by K.

- If K has a negative test result, the infected elves are exactly those two that were in one of K's groups as well.

  (Considering the example above: if Frank' test is negative, we know that Esmeralda and Jakob are infected.)

- If K's test is positive, the other infected elf must be in those two groups in which K was not part of.

  (In the example above: If we test Jakob (positive), we know that Esmeralda is the second sick elf.)

Employing Esmeralda's testing strategy, there is even one test left. Hence, we would be able to test Rudolph, the reindeer, as well :)

Research reference:

The basic idea behind the division into the first two test groups goes back to an old combinatorial problem (the "Kirkman Schoolgirl Problem") from 1850. Since then, similar problems and solutions have been applied, for example, in the organisation of sporting events ("Social Golfer Problem"). In particular, the question of our task: "How can many people be tested for a (viral) disease with relatively few tests?" became an important research topic due to the current Covid-19 pandemic. An example of this is the development of the P-BEST algorithm for a less common disease (about 1 % of those tested are positive), which promises a testing efficiency that is eight-times more efficient. This algorithm is based on a multidimensional implementation of Esmeralda's partitioning, i. e. instead of arranging two test instances on a 2-dimensional field, we make n groupings into test instances, such that we can arrange them in an n-dimensional hypercube.

Shental et al., 2020. *Efficient high-throughput SARS-CoV-2 testing to detect asymptomatic carriers*. Sciences Advances, Vol. 6, no. 37, eabc5961, `https://advances.sciencemag.org/content/6/37/eabc5961`

8 Explosion

Author: Cor Hurkens (TU Eindhoven)
Project: 4TU.AMI

8.1 Challenge

Today, yet another explosion occured Santa's chemistry lab. Many beards were burnt, and many eyebrows were scorched. Over dinner, the mathematics pixy Calculus learns the following facts on the explosion from the three chemistry pixies Litmus, Nucleus, and Reductus:

Litmus recounts: "Only a single pixy managed to survive the explosion unharmed. Exactly one pixy got away with only a burnt left eyebrow, exactly one with only a burnt right eyebrow, and exactly one with only a burnt beard."

Nucleus is still very upset: "More than 95 % of the lab pixies suffered a burnt beard as well as a burnt left eyebrow."

Reductus laments: "And more than 95 % of the lab pixies suffered a burnt beard as well as a burnt right eyebrow."

Calculus ponders for a while about these informations and then observes: "Among all the lab pixies with a burnt left *and* a burnt right eyebrow, more than x % also have a burnt beard."

What is the largest integer x for which the statement of Calculus will definitely be true?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. The largest such integer is $x = 90$.
2. The largest such integer is $x = 91$.
3. The largest such integer is $x = 92$.
4. The largest such integer is $x = 93$.
5. The largest such integer is $x = 94$.
6. The largest such integer is $x = 95$.
7. The largest such integer is $x = 96$.
8. The largest such integer is $x = 97$.
9. The largest such integer is $x = 98$.
10. The largest such integer is $x = 99$.

8.2 Solution

The correct answer is: 6.

We denote the number of pixies that have only a burnt left eyebrow by n_L, the number of pixies that have only a burnt right eyebrow by n_R, and the number of pixies that have only a burnt beard by n_B.
Similarly, n_{LR}, n_{BL}, n_{BR}, n_{BLR}, and $n_\emptyset$ denote the number of pixies that have scorched only their two eyebrows, only their left eyebrow and their beard, only their right eyebrow and their beard, both eyebrows and their beard, and nothing at all, respectively.
The statements of Nucleus and Reductus imply

$$\begin{aligned} n_{BL} + n_{BLR} &> \tfrac{95}{100}(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR}) \\ &= \tfrac{19}{20}(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR}), \end{aligned}$$

$$\begin{aligned} n_{BR} + n_{BLR} &> \tfrac{95}{100}(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR}) \\ &= \tfrac{19}{20}(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR}). \end{aligned}$$

We add these two inequalities and get

$$\begin{aligned} 2\, n_{BLR} + n_{BL} + n_{BR} &> \tfrac{38}{20}\,(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR}) \\ &= \tfrac{19}{10}\,(n_\emptyset + n_B + n_L + n_R + n_{LR} + n_{BL} + n_{BR} + n_{BLR})\,. \end{aligned}$$

However, this inequality is equivalent to

$$2\, n_{BLR} - \tfrac{19}{10} n_{BLR} > \tfrac{9}{10}(n_{BL} + n_{BR}) + \tfrac{19}{10}(n_\emptyset + n_B + n_L + n_R + n_{LR}),$$

which can be simplified into

$$n_{BLR} > 9\,(n_{BL} + n_{BR}) + 19\,(n_\emptyset + n_B + n_L + n_R + n_{LR}).$$

Since the six numbers $n_\emptyset, n_B, n_L, n_R, n_{BL}, n_{BR}$ on the right-hand side of the equation are non-negative, this inequality implies the lower bound

$$n_{BLR} > 19\, n_{LR}, \tag{S}$$

which can be equivalently rewritten as

$$\tfrac{5}{100} n_{BLR} > \tfrac{95}{100}\, n_{LR}.$$

However, $\frac{5}{100}n_{BLR} = n_{BLR} - \frac{95}{100}n_{BLR}$, and therefore, inequality (S) is equivalent to

$$n_{BLR} > \tfrac{95}{100}(n_{LR} + n_{BLR}).$$

Hence, among the lab pixies with both a left and a right scorched eyebrow, more than 95 % have a scorched beard. We conclude that for $x = 95$, Calculus' statement is true in any constellation.

Now, we show that Calculus' statement is not necessarily true for $x \geq 96$. To this end, consider the following situation:

$$n_{\emptyset} = n_B = n_L = n_R = n_{BL} = n_{BR} = 1, \qquad n_{LR} = 20, \qquad n_{BLR} = 479.$$

- These figures are obviously compatible with Litmus' statement.
- Out of a total of 505 laboratory pixies, $n_{BL} + n_{BLR} = 480$ scorched both their beard and their left eyebrow. That is about 95,05 per cent. Hence, the situation is also compatible with Nucleus' statement.
- Analogously, one can see that the situation is compatible with Reductus' statement, since the $n_{BR}+n_{BLR} = 480$ pixies with both scorched beard and scorched right eyebrow also account for about 95,05 per cent of all laboratory pixies.
- On the other hand, Calculus' statement is, in this situation, wrong: There are $n_{LR} + n_{BLR} = 499$ lab pixies with two scorched eyebrows, of which exactly $n_{BLR} = 479$ also have a scorch beard—but that is only 95,99 percent.

Therefore, the correct answer is $x = \mathbf{95}$.

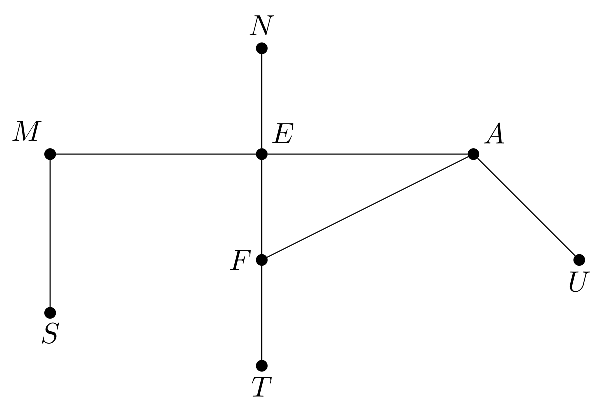
9 Superfluous Presents

Author: Max Klimm (TU Berlin)

Project: Mathematical modelling, simulation and optimization using the example of gas networks (SFB Transregio 154): Combinatorial network flow methods for instationary gas flows and gas market problems (A07)

9.1 Challenge

In order to facilitate the transport of presents from the North Pole, a pipeline network from the North Pole (N) to the continents of Africa (F), Antarctica (T), Asia (A), Australia (U), Europe (E), North America (M) and South America (S) was started to be constructed already in 1547. Before their transport, the presents are transformed into the gaseous phase by a secret process at the North Pole. Then, they are transported via the illustrated pipeline network to the respective continents, where they are condensed to their original state of aggregation.

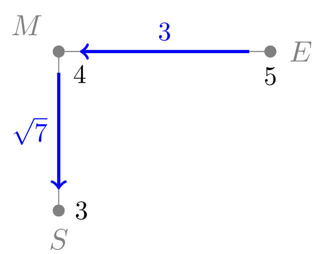


On Christmas Eve, a gift volume of 1 unit is to be discharged from the network at each of the continents (except at the North Pole N). Accordingly, a gift volume of 7 units needs to be fed in at the North Pole.
In pipeline networks, moving gases approximately satisfy the so-called *Weymouth equations*. These equations state that for each pipe between two nodes, the square of the gas flow f is equal to the difference of the squares of the pressures at the respective end nodes, i. e.

$$f^2 = p_+^2 - p_-^2,$$

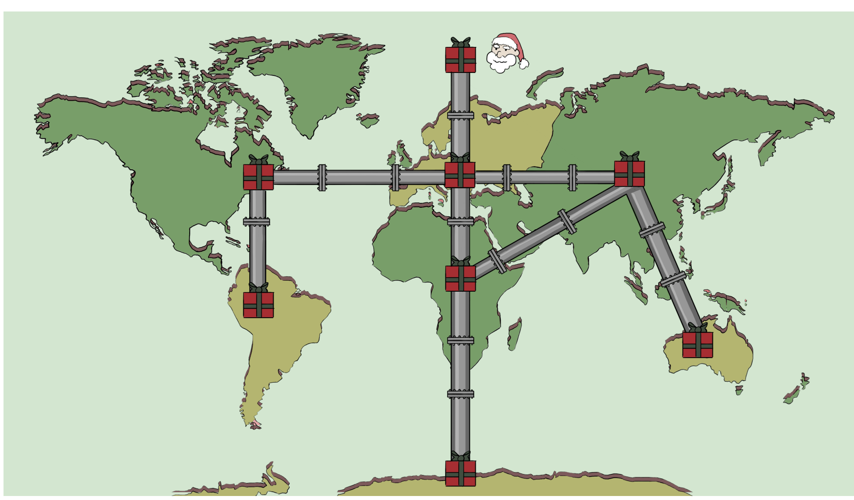
where the gas flows from the node with higher pressure p_+ in the direction of the node with lower pressure p_-.

For example, suppose that there is a pressure of 5 at node E, a pressure of 4 at node M, and a pressure of 3 at node S. Then, $\sqrt{5^2 - 4^2} = 3$ units move from E to M; as well as $\sqrt{4^2 - 3^2} = \sqrt{7} \approx 2.65$ units move from M to S. Consequently, $3 - \sqrt{7}$ units are discharged at point M:



Help Santa Claus to find correct pressures for all nodes A, E, F, M, N, S, T, and U such that at each node (except at N) exactly 1 unit is discharged, at N exactly 7 units are fed in, and all edges of the network satisfy the Weymouth equations. Furthermore, at U, the pressure should be 0.

Which of the following answers is correct?



Artwork: Julia Nurit Schönnagel

Possible answers:

1. The pressure at node E is $\sqrt{1}$.

2. The pressure at node E is $\sqrt{2}$.
3. The pressure at node E is $\sqrt{3}$.
4. The pressure at node E is $\sqrt{4}$.
5. The pressure at node E is $\sqrt{5}$.
6. The pressure at node E is $\sqrt{6}$.
7. The pressure at node E is $\sqrt{7}$.
8. There exists no solution, such that all requirements are fulfilled.
9. Another node in the network has the same pressure as is present at node E.
10. Deleting the edge between A and F and computing a new solution on this altered graph, will yield an increase in pressure at the node E.

Project reference:

The subproject A07 of TRR 154 *Mathematical modelling, simulation and optimisation using the example of gas networks* deals with the control of gas flows in pipeline networks similar to the one in this task. In practice, the problems are much more complex than shown here, since pressure barriers, compressors, as well as more complicated gas flow models are also considered.

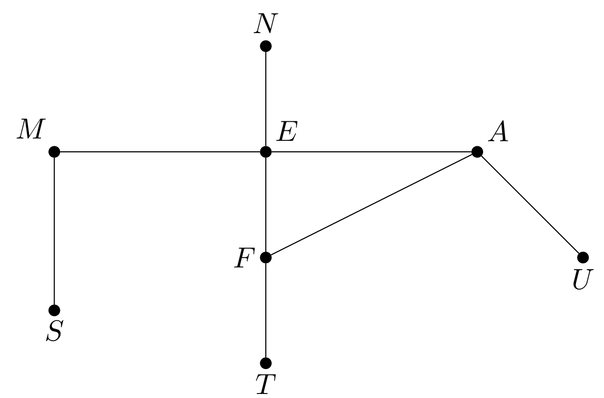
9.2 Solution

The correct answer is: 5.

We want to build up the solution successively. For any node

$$v \in \{A, E, F, M, N, S, T, U\},$$

we will denote the pressure at node v with p_v.



- According to the task, $p_U = 0$ must hold. Since also 1 unit is to be fed out at the node U, exactly 1 unit must flow from A to U, i. e. $p_A^2 - p_U^2 = 1$. By rearranging this equation, we obtain $p_A = 1$.
- Analogously, 1 unit must flow from F to T, which gives us $p_F^2 - p_T^2 = 1$, and
- 1 unit must flow from M to S, yielding $p_M^2 - p_S^2 = 1$.
- Along the edge from E to M the 2 units for M and S have to flow, i. e. $p_E^2 - p_M^2 = 4$ must hold.
- Along the edge from N to E 7 units must flow, so we get $p_N^2 - p_E^2 = 49$.

We observe that all further values are determined by the solutions of p_E, p_F and p_A, respectively.

In order to calculate the values for E, F, and A, we attempt (either by means of a systematic trial and error or because it seems to be intuitive) the approach that the direction of the gas flow in the triangle AEF is such, that gas flows from E to F, from E to A, and from F to A. If we use x to denote the gas flow from F to A, we obtain the following equations:

$$\begin{aligned} p_F^2 - p_A^2 &= x^2, \\ p_E^2 - p_A^2 &= (2-x)^2, \\ p_E^2 - p_F^2 &= (2+x)^2, \\ p_A &= 1. \end{aligned}$$

We can simplify these equations to

$$\begin{aligned} p_F^2 &= x^2 + 1, \\ p_E^2 &= (2-x)^2 + 1, \\ p_E^2 - p_F^2 &= (2+x)^2. \end{aligned}$$

From this the equation

$$(2-x)^2 + 1 - (x^2 + 1) = (2+x)^2$$

follows, and further simplification yields

$$0 = x^2 + 4x = x(4+x).$$

The only non-negative solution of this equation is $x = 0$. By inserting this solution we arrive at

$$p_E = \sqrt{2^2 - 1} = \sqrt{\mathbf{5}}$$

for the pressure at the node E. Obviously, the answer options 1-4 and 6-7 are therefore incorrect.

For the other pressures, we calculate

$$\begin{aligned} p_A &= 1, \\ p_E &= \sqrt{5}, \\ p_F &= \sqrt{p_E^2 - \sqrt{2^2}} = \sqrt{5-4} = 1, \\ p_M &= \sqrt{p_E^2 - 4} = \sqrt{5-4} = 1, \\ p_N &= \sqrt{49 - p_E^2} = \sqrt{44} = 2\sqrt{11}, \\ p_S &= \sqrt{p_M^2 - 1} = \sqrt{1-1} = 0. \end{aligned}$$

Therefore, the answers 8 and 9 are incorrect too.

To see that answer 10 is also not correct, we note that, in the solution calculated above, no gas flows on the edge between A and F (as $p_F^2 - p_A^2 = 0$). If this edge would be removed from the network, the same pressures we aleady calculated would be a solution. The pressure at node E would therefore not change by removing this edge.

10 An Ancient Anniversary Gift

Author: Ariane Beier (MATH+ School Activities)

10.1 Challenge

For the first time, Elf Horton is responsible for the Christmas decoration of the North Pole's festival hall. Since he was waiting a very long time for this opportunity, Horton is quite excited and wants to do something extraordinary. Thus, he buries himself into the stock, where he browses through decorative items from the past several hundred years. After a while, Horton is quite sure that he has found the perfect piece for the top of the Christmas tree: a Möbius strip made of sheet metal ornamented with amethyst, malachite, and tourmaline. Horton wonders how old this exquisite decoration must be. Thankfully, there is a letter enclosed with it:

> My dearest Conny,
>
> on this Monday, our eighth wedding anniversary, I give you this Möbius strip as a symbol of our everlasting love. It is made for you and this special occasion. With endless joy, I remember the weekend day of our wedding on 13 J*** 19**, the beginning of our wonderful journey together.
>
> Love, John.

As you may have noticed, parts of the wedding date are blurred due to the ravages of time. (Of course, the number of *'s does not represent the number of blurred characters...)

Can you still help Horton and figure out how old the Möbius strip is? More precisely, what is the remainder of the digit sum of Conny's and John's wedding date (given as DDMMYYYY), when divided by 10?

> **Example:** If the date would have been 26 December 1937, then the answer would be 1, since
>
> $$2+6 \;+\; 1+2 \;+\; 1+9+3+7 \;=\; 31 \;=\; 3 \cdot 10 + \mathbf{1}.$$

Hints:

- There are 365 days in a non-leap year and 366 days in a leap year.

- If a year is divisible by 4, then it is a leap year, except if it is divisible by 100, except-except it is also divisible by 400.

  Examples:

  - The year 2004 was a leap year, because 2004 is divisible by 4, but not by 100.
  - The year 2100 will *not* be a leap year, since 2100 is divisible by 100, but not by 400.
  - The year 2400 will be a leap year, since 2400 is divisible by 400.

Artwork: Friederike Hofmann

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8

9. 9
10. 0

10.2 Solution

The correct answer is: 5.

You probably noticed that any date moves forward one day of the week after every non-leap year: if your birthday was on a Wednesday in 2018, it was on a Thursday in 2019. The reason for this is that 365 leaves a remainder of 1 when divided by 7:

$$365 = 7 \cdot 52 + 1.$$

Analogously, any date will move two weekdays whenever it passes a leap year, since

$$366 = 7 \cdot 52 + 2.$$

From John's letter, we can extract the following information:

- The wedding date was in January, June, or July.
- They were married on the weekend, that is, either on a Saturday or a Sunday.
- Eight years later, their anniversary was on a Monday.

Over a period of eight years, one passes either only one or two leap years. First, assume that in the given situation two leap years were passed. Then, every date would have advanced one weekday for every non-leap year and two weekdays for every leap year. In total,

$$6 \cdot 1 + 2 \cdot 2 = 10$$

days of the week, which is, of course, the same as advancing three days of the week. But John and Conny were married either on a Saturday or a Sunday. In conclusion, eight years later would have been a Tuesday or a Wednesday—certainly *not* a Monday.

Hence, the eight-year period includes only one leap year, which means that the period must have started in 1900, because every other period of eight years from 1901 to 2000 (a leap year!!!) includes two leap years. Furthermore, Conny and John must have been married before February 28th, as any date thereafter in 1900 would still pass two leap days over an eight-year period.

In conclusion, Conny and John married on January 13th, 1900.

Finally, one calculates the digit sum of the date and its remainder when divided by 10

$$1+3 \;+\; 0+1 \;+\; 1+9+0+0 \;=\; 15 \;=\; 1\cdot 10+\mathbf{5}.$$

Remark:

This challenge was published in honor of John Horton Conway, an English mathematician who died of complications from COVID-19 on 11 April 2020. Conway invented the so-called *Doomsday rule*, a simple algorithm for determinating the day of the week for a given date by mental calculation. For more information, see

https://en.wikipedia.org/wiki/Doomsday_rule.

11 Wormhole

Author: Martin Skutella (TU Berlin)
Project: MATH+ Application Area AA3: Networks

11.1 Challenge

Slightly annoyed, Santa Claus hangs up the phone. Just a moment ago, he was talking to his good-for-nothing brother-in-law, who once again urgently needs Santa's help at the South Pole. But no matter how hard he thinks, Santa does not know how to travel there and still be back in time for Christmas at the North Pole. Helplessly, he stares at the world map on the wall (see Fig. 1), when Elf Einstein passes by. As always, Einstein has his assistant, elf Rosen, in tow.

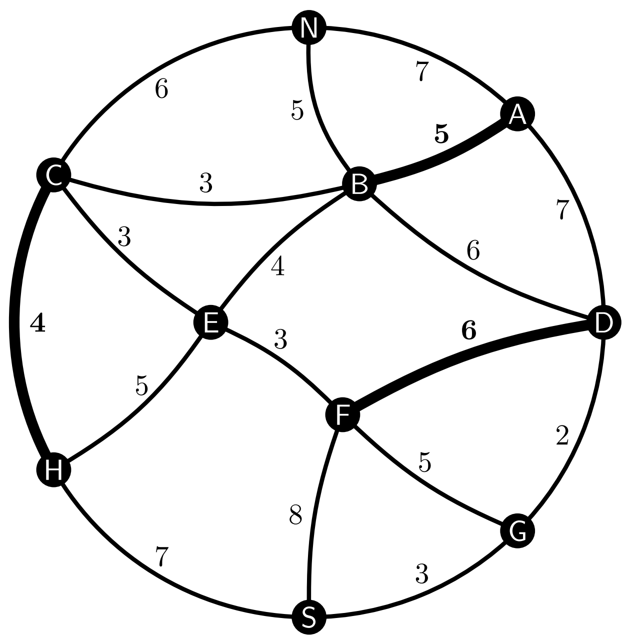


Figure 1: The world map shows the locations A to H, the North Pole N, and South Pole S. Furthermore, the travel times between these locations.

For weeks, the two elves have been talking about nothing but their latest discovery—much to the sorrow of all the other elves. Rumour has it that it has something to do with the field equations of General Relativity. But in the absence of any deeper understanding, everyone just talks about *wormholes.*

"How are your wormholes doing, Einstein?" asks Santa.
Slightly offended, Einstein sticks out his tongue. However, he overcomes his pride and starts lecturing: "Our latest insights into gravitational anomalies point to unexpected connections between certain places; just as a wormhole connects two points on the surface of an apple. These connections may even make time travel possible!"
Santa interrupts musing about his upcoming trip to the South Pole and starts to listen attentively.

Accompanied by Einstein and Rosen, Santa sets off for the South Pole shortly after. For safety reasons, the outward journey to the South Pole is still to be made along the classic connections that are drawn as lines on the world map. The numbers next to the connections indicate the number of hours required. For example, the travel time from the North Pole N along the possible travel routes via C and H to the South Pole S amounts to 6+4+7=17 hours.

Einstein and Rosen want to use the outward journey to the South Pole to clarify final details with regard to the gravitational anomalies of the three wormholes that they have marked in bold on the world map. On the return journey to the North Pole, it will then be possible "to let the clock run backwards" on the three connections marked in bold. Thus, the adventurers are able to travel into the past. For example, the travel time along the route from the South Pole S via H and C to the North Pole N will only be 7+(-4)+6=9 hours.

Santa is very enthusiastic, but Einstein warns against the excessive use of wormholes: "We can use each wormhole at most once, and even that may be risky."
After some thought, Santa states the following objectives:

O1: In order to enable a timely return, the total travel time for the journey from the North Pole to the South Pole and back must not exceed 24 hours.

O2: In order to avoid possible discontinuities in space-time as far as possible, no more than two of the three wormholes are to be used on the journey from the South Pole back to the North Pole.

O3: To make the journey as varied as possible, none of the connections should be used on both the outward and the return journey—not even in the opposite direction.

O4: In order to meet as many friends as possible on the trip, each of the places A to H should be visited at least once during the whole journey.

O5: In order to be a burden to as few friends as possible, at most one of the places A to H should be visited on both the outward and the return journey.

O6: On the other hand, in order to bless at least a few friends with two visits, at least one of the places A to H should be visited on both the outward and the return journey.

Disappointedly, the three travel companions realise that the six objectives are not compatible. Even for certain combinations of three objectives, there seems to be no admissible itinerary for the round trip to the South Pole and back to the North Pole.

For which of the following combinations exists *no* such admissible round trip?

Artwork: Friederike Hofmann

Possible answers:

1. O1, O2, and O3.
2. O1, O2, and O5.
3. O1, O2, and O6.

4. O1, O3, and O5.
5. O1, O3, and O6.
6. O1, O4, and O5.
7. O1, O5, and O6.
8. O1, O4, and O6.
9. O2, O3, and O4.
10. O4, O5, and O6.

Project reference:

Shortest path problems play a fundamental role in many areas of combinatorial optimisation and especially network optimisation. They are essential building blocks in the efficient solution of numerous practical problems, for example in the area of traffic. In the MATH+ research centre, shortest path problems are studied in the following research projects, among others, in the application area *Networks*:

AA3-2: Nash flows over time in transport and evacuation simulation

AA3-3: Discrete-Continuous Shortest Path Problems in Flight Planning

AA3-4: Flow-Preserving Graph Contractions with Applications to Logistics Networks

AA3-5: Tropical Mechanism Design

Obviously, the alleged connection to the theory of the Einstein-Rosen bridge (wormholes) mentioned in the challenge is fictitious.

11.2 Solution

The correct answer is: 6.

First, we observe that there is exactly one shortest path P for the outward journey (see Fig. 2); this has length 16. For the return journey, there are exactly three shortest paths Q_1, Q_2, and Q_3 (see Fig. 3) of length 8.

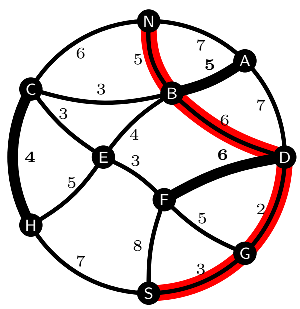


Figure 2: The shortest path P for the outward journey has length 16.

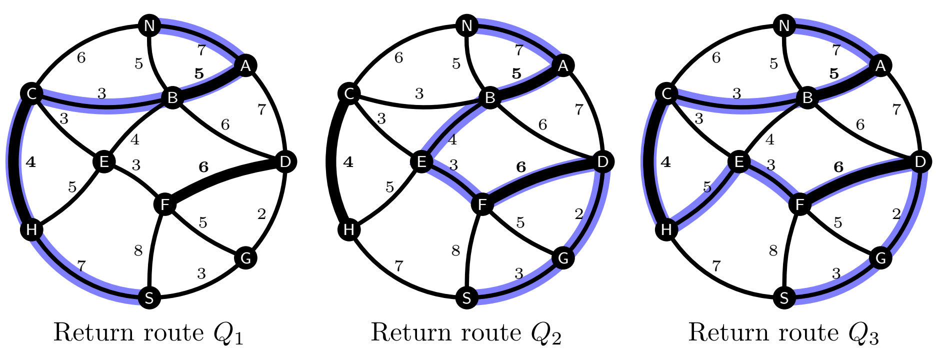


Figure 3: The three shortest paths, Q_1, Q_2, and Q_3, for the return journey each have length 8.

Consequently, there are exactly three possible travel routes that satisfy O1. These are combinations of the path P with one of the paths Q_1, Q_2, or Q_3. However, none of these itineraries simultaneously satisfy O4 and O5:

- (P, Q_1): The places E and F are not visited, i. e. O4 is not satisfied.
- (P, Q_2): The places C and H are not visited, i. e. O4 is not satisfied.
- (P, Q_3): The places B, D, and G are visited on the outward journey as well as on the return journey, i. e. O5 is not satisfied.

Furthermore, we can verify that for each of the given combinations of objectives, there exists an admissible itinerary. To this end, we consider the travel routes (P, Q_1), (P, Q_3), and R (see Fig. 4) and ascertain the following:

1. O1, O2, and O3 are satisfied by (P, Q_1).
2. O1, O2, and O5 are satisfied by (P, Q_1).
3. O1, O2, and O6 are satisfied by (P, Q_1).
4. O1, O3, and O5 are satisfied by (P, Q_1).
5. O1, O3, and O6 are satisfied by (P, Q_1).
6. As we have proven above, it is **not** possible to simultaneously satisfy O1, O4, and O5.
7. O1, O5, and O6 are satisfied by (P, Q_1).
8. O1, O4, and O6 are satisfied by (P, Q_3).
9. O2, O3, and O4 are satisfied by R.
10. O4, O5, and O6 are satisfied by R.

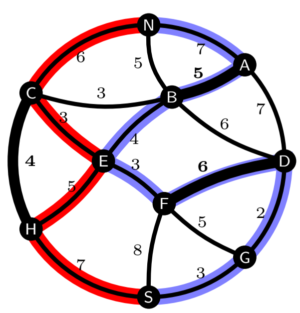


Figure 4: Round trip R with its outward journey in red and return in blue.

Note on the calculation of shortest paths

The main challenge of the task on hand is to calculate shortest paths in a graph with given edge lengths. If all edge lengths are non-negative (as for the outward journey), shortest paths can be calculated quite easily by using Dijkstra's classical algorithm. In particular, the shortest path in Fig. 2 can be found this way.

However, if there are edges of negative length (as for the return), the shortest path problem is generally NP-hard (i. e. there is probably no efficient algorithm that solves the problem). However, in the special case where the graph does not contain any circles of negative total length (as in the graph given here), the shortest path problem can be solved efficiently by reducing it to the problem of determining a so-called T-*join* of minimum total length. The latter problem can in turn be traced back to computing a so-called *perfect matching* of minimal total length in a derived graph. This can also be used to systematically prove that the three paths in Fig. 3 are exactly the shortest paths in the given setting.

12 Frog and Toad

Author: Jesper Nederlof
Project: 4TU.AMI

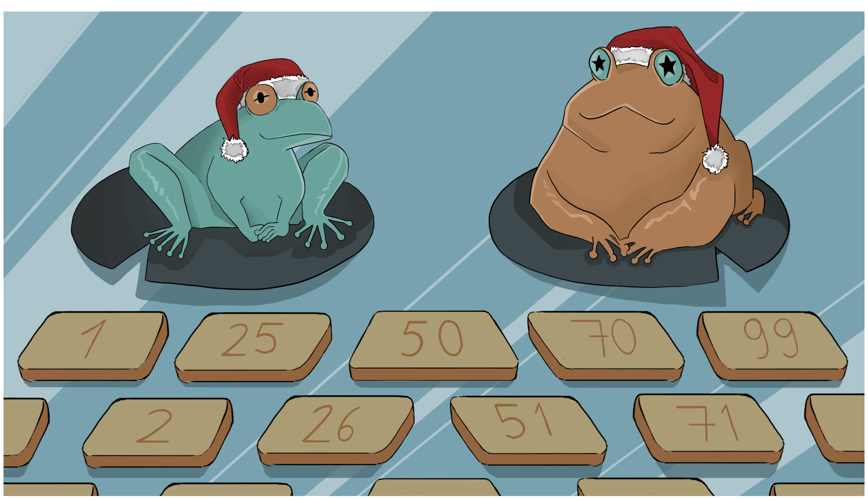
12.1 Challenge

The probability frog Frederick jumps around on the integers. At 8 am, he starts on the number +100 and makes a jump every second. Fredrick prefers to jump to the left (to the smaller numbers) than to the right (to the larger numbers): with probability 1/3 he jumps from his current position f to $f+1$, with probability 2/3 he jumps to $f-1$. In the course of the day, Frederick reaches the number 0 and ends his journey.

The probability toad Topanga also jumps around on the integers. At 8 am she starts on the number −100 and makes a jump every second. Just like Frederick, Topanga prefers to jump to the left rather than to the right: with probability 1/3 she jumps from her current position k to $k+1$ respectively, with probability 2/3 she jumps to $k-1$. In the course of the day (but *not* in the same second as Frederick) Topanga reaches the number 0 and ends her journey.

With p we denote the probability that Frederick reached the number 0 only *after* Topanga.

Which of the following statements about p is true?



Artwork: Julia Nurit Schönnagel

Possible answers:

1. $p \leq 0.002$ holds.
2. $0.002 < p \leq 0.004$ holds.
3. $0.004 < p \leq 0.008$ holds.
4. $0.008 < p \leq 0.016$ holds.
5. $0.016 < p \leq 0.032$ holds.
6. $0.032 < p \leq 0.064$ holds.
7. $0.064 < p \leq 0.128$ holds.
8. $0.128 < p \leq 0.256$ holds.
9. $0.256 < p \leq 0.512$ holds.
10. $0.512 < p$ holds.

12.2 Solution

The correct answer is: 9.

Let F be an arbitrary sequence of jumps that take Frederick from $+100$ to 0, and let T be an arbitrary sequence of jumps that take Topanga from -100 to 0.

When Frederick arrives at 0, he has jumped exactly 100 times as often to the left than to the right. Therefore, the sequence of jumps F consists of a certain number of jumps x to the right and $x + 100$ jumps to left. The probability of occurrence for F is consequently given by

$$P_F = \left(\tfrac{1}{3}\right)^x \cdot \left(\tfrac{2}{3}\right)^{x+100}.$$

Similarly, when Topanga arrives at 0, she has jumped exactly 100 times as often to the right than to the left. Therefore, the sequence of jumps T consists of a certain number of jumps y to the left and $x + 100$ jumps to right. The probability of occurrence for T is consequently given by

$$P_T = \left(\tfrac{2}{3}\right)^y \cdot \left(\tfrac{1}{3}\right)^{y+100}.$$

Hence, the probability for Frederick and Topanga performing the sequence of jumps (F, T) is given by

$$P_{(F,T)} = \left(\tfrac{1}{3}\right)^x \cdot \left(\tfrac{2}{3}\right)^{x+100} \cdot \left(\tfrac{2}{3}\right)^y \cdot \left(\tfrac{1}{3}\right)^{y+100} = \left(\tfrac{2}{9}\right)^{x+y+100}.$$

Now, we define the sequences of jumps F' and T' as follows:

- F': Each jump to the right in F is substituted by a jump to left and each jump to left by a jump to the right.
- T': Analogously, each jump to the right in T is substituted by a jump to left and each jump to left by a jump to the right.

What will happen if Frederick and Topanga perform the jumps according to (T', F') (instead of (K, F))?

The probability for Frederick performing T' is

$$P_{T'} = \left(\tfrac{2}{3}\right)^y \cdot \left(\tfrac{1}{3}\right)^{y+100}.$$

Similarly, Topanga performing F' occurs with probability

$$P_{F'} = \left(\tfrac{1}{3}\right)^x \cdot \left(\tfrac{2}{3}\right)^{x+100}.$$

Consequently, for Frederick and Topanga performing the sequences of jumps (T', F'), the probability is given by

$$P_{(F,K)} = \left(\tfrac{1}{3}\right)^x \cdot \left(\tfrac{2}{3}\right)^{x+100} \cdot \left(\tfrac{2}{3}\right)^y \cdot \left(\tfrac{1}{3}\right)^{y+100} = \left(\tfrac{2}{9}\right)^{x+y+100}.$$

Furthermore, since Frederick and Topanga doe not arrive at 0 at the same time, we have $x \neq y$. Since the given problem is symmetric in x and y, we may assume that $x \leq y$. Then, if Frederick and Topanga perform (F, T), Frederick will arrive at 0 first with $2x + 100$ jumps. However, if Frederick and Topanga perform (T', F'), then Topanga will reach 0 first with $2x+100$ jumps.

We conclude that for every case in which Frederick finishes first, there is symmetric case with the same probability of occurrence in which Topanga finishes first. It follows that they arrive at 0 first with exactly the same probability; and hence, $\mathbf{p} = \mathbf{\frac{1}{2}}$.

13 T-Shirts

Author: Hajo Broersma (Universiteit Twente)
Project: 4TU.AMI

13.1 Challenge

Sixteen pixies are standing in a circle. All of the pixies have one integer printed on their T-shirt. The number printed on every pixie's T-shirt is always (strictly) larger than the sum of the numbers on the T-shirts of the two pixies that are standing to the left of him.

What is the largest possible number of pixies with a positive number on their T-shirts?

Artwork: Frauke Jansen

Possible answers:

1. The largest possible number of positive numbers is 3.
2. The largest possible number of positive numbers is 4.
3. The largest possible number of positive numbers is 5.
4. The largest possible number of positive numbers is 6.
5. The largest possible number of positive numbers is 7.
6. The largest possible number of positive numbers is 8.
7. The largest possible number of positive numbers is 9.
8. The largest possible number of positive numbers is 10.
9. The largest possible number of positive numbers is 11.
10. The largest possible number of positive numbers is 12.

13.2 Solution

The correct answer is: 5.

First of all, we give a possible arrangement of the numbers for 16 T-shirts:

$$1,\ -19,\ 1,\ -21,\ 1,\ -23,\ 1,\ -25,\ 1,\ -27,\ 1,\ -29,\ 1,\ -31,\ -15,\ -17.$$

One can easily check that each number is strictly larger than the sum of both following numbers; in particular,

$$-15 > (-17) + 1 \quad \text{and} \quad -17 > 1 + (-19).$$

Hence, we have found an arrangement such that exactly seven pixies have a positive number on their T-shirts.

Now, we want to show that it is impossible to find an arrangement with eight or more positive integers. To this end, we label the numbers clockwise with $z_1, z_2, \ldots, z_{16}$. In order to handle the indices more easily, we define $z_n = z_{n+16}$ for all $n \in \mathbb{N}$.

Let us consider the largest possible group of, say p, pixies standing next to each other in clockwise order sucht that all of them have a positive integer on their T-shirt.

- If $p = 16$, we immediately get the following contradiction: the pixy with the smallest integer on its T-shirt, has a number $z_n > 0$, the next two pixies have the numbers $z_{n+1} \geq z_n$ and $z_{n+2} \geq z_n$. However, then one has

  $$z_n < 2z_n \leq z_{n+1} + z_{n+2},$$

  which is a contradiction to the assumption.

- If $2 \leq p < 16$, consider the pixy standing right next to these p pixies with positive numbers on their T-shirts in the circle. This pixy's T-shirt is labeled with a non-positive number $z_n \leq 0$. However, the assumption

  $$z_n > z_{n+1} + z_{n+2}$$

  implies that at least one of the numbers z_{n+1} or z_{n+2} needs to be negative, which is a contradiction to the assumption.

Therefore $p \leq 1$ has to be true and hence at most eight pixies can have a positive number on their T-shirts.

If now exactly eight pixies had a positive number on their T-shirts, positive and non-positive numbers would always have to alternate in the circle. For symmetry reasons we can assume that the numbers $z_1, z_3, \ldots, z_{15}$ are positive. If we use that every number z_n is larger than the sum of its two following numbers z_{n+1} and z_{n+2}, we get

$$\begin{aligned} z_{16} &> z_1 + z_2 \\ &> z_1 + (z_3 + z_4) \\ &> z_1 + z_3 + (z_5 + z_6) \\ &> z_1 + z_3 + z_5 + (z_7 + z_8) \\ &> z_1 + z_3 + z_5 + z_7 + (z_9 + z_{10}) \\ &> z_1 + z_3 + z_5 + z_7 + z_9 + (z_{11} + z_{12}) \\ &> z_1 + z_3 + z_5 + z_7 + z_9 + z_{11} + (z_{13} + z_{14}) \\ &> z_1 + z_3 + z_5 + z_7 + z_9 + z_{11} + z_{13} + (z_{15} + z_{16}); \end{aligned}$$

or equivalently,

$$z_1 + z_3 + z_5 + z_7 + z_9 + z_{11} + z_{13} + z_{15} < 0.$$

However, this is impossible, since all eight summands were positive.

Hence, we have shown that no more than **seven** pixies can have a positive number on their T-shirts.

14 Clumsy Santa is Coming to Town

Authors: Luise Fehlinger (HU Berlin)
Thilo Steinkrauß (Herder-Gymnasium Berlin)

14.1 Challenge

The elves are nervous. Traditionally, they will personally hand over the Christmas presents to the particularly curious children in a festive ceremony this evening. Each elf has studied the list of children for a long time, and each one memorises precisely which child to give the present to. First, Santa is to hand out the gift to the most curious child in the world. After that, head elf Rebekka will give the present to the second most curious child. It continues with elf Jonathan and the present for the third most curious child, and so on.

Unfortunately, elf Eleonora has observed that Santa has spilled his cocoa on the list. Santa is embarrassed and also afraid that the elves might think he is no longer fit enough for the job. Therefore, he does not want to be helped and pretends that everything is fine. Since it is not the first time something like this has happened, elf Eleonora knows exactly how it will turn out:
Santa will simply give his gift to any child. Then, each elf who is to distribute a present afterwards will choose the right child, provided that child does not yet have a present. Otherwise, they will just randomly choose a child without a present. At the end, elf Eleonora is to hand over the last present.

Now, elf Eleonora and her best friends are puzzling over the probability that she can give her gift to the intended child. But only one elf is right. Which one?

Illustration: Friederike Hofmann

Possible answers:

1. Elf Roland is positive: "As long as Santa doesn't accidentally give his present to the last child on the list, Eleonora has nothing to fear."
2. Elf John says: "If there were only three children on the list, the probability would be more than $\frac{1}{2}$."
3. However, elf Saskia says: "Even if we knew exactly how many children are on the list this year, we can't compute it."
4. Elf Antje suspects: "The more children there are on the list, the smaller the probability that Eleonora can give the present to the right child."
5. Elf Lina replies: "Nonsense! The more children there are on the list, the greater the probability that Eleonora can give the present to the right child."
6. Elf Marek suspects: "We have to calculate exactly two different probabilities. One for the case that the number of children on the list is even. And one for the case that the number is odd."
7. Elf Nadja is pessimistic: "Eleonora should not get her hopes up... Even though the probability is constant (independent of the number of children on the list), it is less than 10 %."
8. Elf Kristina laughs: "I don't know what your problem is. The probability is simply $\frac{1}{2}$."

9. Robert, the head of the elves' school, doesn't understand the excitement: "It's guaranteed to work."
10. Elf Falk is unsure: "All of your answers seem far too simple to my taste. The solution must be different."

Project reference:

The *Berliner Netzwerk mathematisch-naturwissenschaftlich profilierter Schulen* has existed since September 2001 and has set the following goals:

- Creating a city-wide network of Berlin high schools with an existing or future profile and the mathematics in mathematics and science with the institutes of Humboldt-Universität at the campus Adlershof.
- Establishing standards for advanced courses (*Leistungskurse*) in mathematics and the other participating natural science subjects that enable the recognition of *Abitur* achievements as academic credits.

The Senator for Education, Youth and Family Affairs and the university administration of Humboldt-Universität welcome and support this project. For the future, we also envision a high school with a mathematics and natural sciences profile located at the science and business campus Adlershof, which is strongly connected to the institutes of Humboldt-Universität.

Further information:
`http://didaktik.mathematik.hu-berlin.de/de/schule/schulkooperationen`

14.2 Solution

The correct answer is: 8.

Let $n \geq 2$ be the number of children. We claim that the probability for Eleonora giving the present to the right child is always $p(n) = \frac{1}{2}$.

First, consider $n = 2$: Santa either gives his present to the right child or to the wrong one. Both cases occur with the same probability $\frac{1}{2}$. In the first case, Eleonora can also give her present to the right child. In the second case, however, she cannot. Thus, $p(2) = \frac{1}{2}$, as claimed.

Now, let $n \geq 3$. We distinguish the following cases:

Case 1: With a probability $\frac{1}{n}$, Santa gives his present to the correct (i. e. first) child. In this case, Eleonora is also able give her present to the right (i. e. last) child.

Case 2: Again, with a probability $\frac{1}{n}$, Santa gives his present to the last child. In this case, Eleonora definitely has to give her present to another child.

Fall 3: With a probability of $\frac{1}{n}$ each, Santa gives the child k with $1 < k < n$ his present. What happens in these cases?

- For $k = n-1$, the following applies: Child 1 gets the wrong gift (namely that of child $n-1$). Children 2 to $n-2$ get the right gift. Now, child $n-1$ either gets the gift of child 1 (with probability $\frac{1}{2}$) or that of child n (also with probability $\frac{1}{2}$). In the first case, Eleonora can give her gift to the right child. In the second, however, she cannot. Therefore, the probability that Eleonora can give the gift to the right child is $\frac{1}{2}$.
- For $k = n-2$ applies: Child 1 gets the wrong gift. Children 2 to $n-3$ get the right gift. With probability $\frac{1}{3}$ each, child $n-2$ now receives either
  - the gift of child 1, and Eleonora is able to give the present to the right child;
  - the gift of child n, and Eleonora will definitely give the present to the wrong child;
  - the gift of child $n-1$. However, then we know that, with probability $\frac{1}{2}$, Eleonora can still give her present to the right child.

All in all, for $k = n - 2$, Eleonora is able to give the present to the intended child with probability

$$\tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

- For $k = n-3$, the following holds: Child 1 gets the wrong gift. Children 2 to $n - 4$ get the right gift. With probability $\frac{1}{4}$ each, child $n - 3$ now receives either
  - the gift of child 1, and Eleonora is able to give the present to the right child;
  - the gift of child n, and Eleonora will definitely give the present to the wrong child;
  - the gift of child $n - 1$, and we know that, with probability $\frac{1}{2}$, Eleonora can still give her present to the right child;
  - the gift of child $n - 1$, and we know that, with probability $\frac{1}{2}$, Eleonora can still give her present to the right child.

  All in all, for $k = n - 3$, Eleonora is able to give the present to the intended child with probability

$$\tfrac{1}{4} + 2 \cdot \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

- Analogously, for $1 < k < n - 1$, Eleonora is able to give her present to the intended child with probability $\frac{1}{2}$, since we can deduce the case k from the $(n - k - 1)$ cases $k + 1$ bis $n - 1$:

$$\frac{1}{n-k+1} + (n-k-1) \cdot \frac{1}{n-k+1} \cdot \frac{1}{2} = \frac{2 + (n-k-1)}{2(n-k+1)} = \frac{n-k+1}{2(n-k+1)} = \frac{1}{2}$$

Added together, we get the following probability for Eleonara giving the present to the intended (i. e. n-th) child:

$$p(n) = \frac{1}{n} + (n-2) \cdot \frac{1}{n} \cdot \frac{1}{2} = \frac{2 + (n-2)}{2n} = \frac{n}{2n} = \mathbf{\frac{1}{2}}.$$

15 Hat Challenge 2020

Author: Tilman Burghoff (MATH+ School Activities)

15.1 Challenge

As the tradition demands, Santa has once again invited some of his smartest elves for coffee and cake this year. The ten clever elves are already excited when Santa finally shows up.
"Please excuse me for being late," he calls out, panting, "I just had to get a few more hats. All right then, let's get started: My dear clever elves! As every year, I would like to give you the chance to win coffee and cake. All you have to do is guessing the colour of the hat on your head."

"Red!" Egobert shouts.
Santa laughs: "Of course, only *after* I have put the hats on your head. The game is very simple: first, you line up in a row. Please make sure that each of you can only see the people in front of you. This means that if Albert is standing at the back, he will see all the others: Bertha, Claudio and so on up to Julia. But Bertha, who is standing second to last, only sees Claudio up to Julia, but not Albert, understand? And don't you dare cheating! As soon as you've all lined up, you'll each be given either a red or a green hat—such that you can't see them yourself, of course. Then you have to guess the colour of your hat one after another. Albert (who is at the back) starts and says aloud either 'Red!' or 'Green!'. Then it's Bertha's turn, then Claudio's and so on. At the end, I will tell you how many of you have guessed their hat colour correctly. The more you guess correctly, the more pieces of cake you will get."

"Are we allowed to discuss?" Franka asks.
"For now, you are welcome to discuss a strategy. However, as soon as I start putting the hats on your heads, I don't want to hear the slightest peep. You are only allowed to say 'Red!' or 'Green!' once when it's your turn. And please don't try to communicate any more information. Whether it's by how you communicate your guess or how long you wait or something like that. Because then I would have to eat the whole cake all by myself..."
He grievingly glances down at his round Santa belly and asks: "Do you have any further questions?"

"Are the hats chosen at random?" Immanuel asks.

"Yes indeed, I will choose the colour of each hat completely at random."

After some thought, the elves come up with a strategy that will guarantee them as much cake as possible. What is the maximum number of elves who are *guaranteed* to guess their hat colour correctly?

Artwork: Frauke Jansen

Possible answers:

1. None of the elves.
2. One of the elves.
3. Two of the elves.
4. Three of the elves.
5. Four of the elves.
6. Five of the elves.
7. Six of the elves.
8. Seven of the elves.
9. Eight of the elves.
10. Nine of the elves.

15.2 Solution

The correct answer is: 10.

Unfortunately, the elves are not guaranteed to guess all ten hats correctly, as Albert has no information about his hat colour and therefore only guesses his cap colour correctly with probability 1/2. However, with the following strategy, they manage to guess all nine other caps correctly:
Albert starts and says either,

- "**Green!**" if he sees an *even number of green hats* on the heads of the other nine elves, or,
- "**Red!**" if he sees an *odd number of green hats* on the heads of the other nine elves.

Suppose that he says, "Green!" Then, next in line is Bertha, who knows that Albert saw an even number of green hats. If she also sees an even number of green hats on the heads of the remaining eight elves, she herself must wear a red hat. Otherwise, her hat is green. Now, the next elf, Claudio, knows Bertha's hat colour and sees how many of the seven elves that are standing in front of him wear green hats. accordingly, he can deduce the colour of his own hat. In the same way, the remaining elves are able to figure out their hats' colour.

Similarly, the elves know that Albert sees an odd number of green hats in front of him, if he says, "Red!" Again, they can deduce the colour of their hats. Consequently, all **nine elves** but Albert are able to correctly "guess" their hats' colour.

In fact, we are able do give an explicit formula for the hat colour of an arbitrary elf. To this end, let x be the number of times the elves before uttered, "Green!" and y be the number of green hats the elf in question sees. Then:

$$\text{Elf is wearing a green hat} \quad \Leftrightarrow \quad x + y \equiv 0 \mod 2.$$

16 Head Protecting Hats

Author: Paul Erchinger (MATH+ School Activities)

16.1 Challenge

It's Christmas time. In Elf Town, the anticipation of the coming festive season is almost unstoppable. There is only one worry: since November, there has been a dramatic increase in cases of Halloween Frightening Fleas. This is a particularly sneaky species of fleas that settles on the heads of its hosts and infects them with the *spook*. Elves affected by the spook give others a proverbial heart attack (without warning and at random times). To ensure the well-being of the population, elf mayor Alva and her team of experts must develop a strategy to contain this threat to the spirit of Christmas.

A flea infection happens as follows: if a flea-infested person and a "healthy" person meet, the cunning fleas will definitely hop over to the other person. Afterwards both persons will be infected. It is courtesy that every inhabitant of Elf Town wears a hat (as soon as she leaves the house). Although it is very fashionable, this *normal* hat does not protect the elves from a flea infection: the fleas simply crawl out of the hat of an infected elf and jump onto the hat of another elf. There, they crawl underneath the hat to reach her head.

Therefore, researchers of Elf Town have developed the novel hat *IBeanie*, which drastically reduces the risk of infection:

- In 50 % of the cases, the *IBeanie* prevents the fleas from crawling out of the hat.
- In 95 % of the cases, the fleas are not able to get through the hat (to the head) from the outside.

Hence, if two elves wearing *IBeanies* meet, one of whom is infected with fleas, the risk of infection for the other one is only 2.5 %.

From experience, we know that some of the inhabitants of Elf Town will refuse to exchange their stylish *normal* hat for the somewhat clumsier new *IBeanie*. However, if two people meet, only one of whom is wearing the *IBeanie*, the risk of infection is correspondingly higher.

Alva wants that the risk of infection for when any healthy person meets any infected person is at most 13 %.

Let p be the percentage of Elf Town residents who *do* wear the *IBeanie*. What is the smallest integer p such that Alva's condition is satisfied? More precisely, what is the digit sum of that p?

Artwork: Frauke Jansen

Possible answers:

1. The digit sum of p is 1.
2. The digit sum of p is 3.
3. The digit sum of p is 5.
4. The digit sum of p is 7.
5. The digit sum of p is 9.
6. The digit sum of p is 11.
7. The digit sum of p is 13.
8. The digit sum of p is 15.
9. The digit sum of p is 17.
10. There exists no such p.

16.2 Solution

The correct answer is: 6.

To solve this problem, we first examine the probabilities of infection for the different types of encounters.

1. **Both the infected and the healthy person wear a *IBeanie*.**

   This case was already mentioned in the problem definition, but we want to convince ourselves of the correctness of the statement: Since the *IBeanie* prevents the 50 % of the fleas from crawling out of the hat and 95 % of them from getting in, the probability of infection is

   $$p_1 = (1 - 0.5) \cdot (1 - 0.95) = 0.5 \cdot 0.05 = 0.025,$$

   that is 2.5 %, as claimed.

2. **The infected person wears *IBeanie*; whereas the healthy person does not.**

   In this case, we know that only the *IBeanie* of the infected person prevents the fleas from crawling out of the hat. However, the second person has no protection at all. Accordingly, the probability of fleas jumping over is

   $$p_2 = (1 - 0.5) \cdot (1 - 0) = 0.5 \cdot 1 = 0.5.$$

3. **The infected person does not wear a *IBeanie*; whereas the healthy person does.**

   Here, the *IBeanie* of the healthy person protects against the fleas entering the head area, but the fleas can escape from the head of the infected person without hindrance. Therefore, for the probability of a new infection, we get

   $$p_3 = (1 - 0) \cdot (1 - 0.95) = 1 \cdot 0.05 = 0.05.$$

4. **Neither the infected nor the healthy person is wearing a *IBeanie*.**

   In this case, no protective measures are present and the probability of infection is $p_4 = 1$.

Now, in order to be able to consider Alva's condition, we find that, with different values of p, the cases considered above are not always equally probable. For example, if only 25 % of the inhabitants of Elf City wear a *IBeanie*, case 1 only occurs with a probability of $0.25 \cdot 0.25 = 0.0625 = 6.25\,\%$, whereas case 4 occurs much more often with probability $0.75 \cdot 0.75 = 0.5625 = 56.25\,\%$. Now, if p is the percentage of Elf City inhabitants who wear a *IBeanie*, the probabilities of occurrence of the four cases are

1. $h_1 = p \cdot p = p^2$,
2. $h_2 = p \cdot (1 - p)$,
3. $h_3 = (1 - p) \cdot p$, and
4. $h_4 = (1 - p) \cdot (1 - p) = (1 - p)^2$.

Now, if we combine these values the previously calculated infection probabilities, we obtain a function

$$\begin{aligned} f(p) &= h_1 \cdot p_1 + h_2 \cdot p_2 + h_3 \cdot p_3 + h_4 \cdot p_4 \\ &= p^2 \cdot 0.025 + p(1 - p) \cdot 0.5 + (1 - p)p \cdot 0.05 + (1 - p)^2 \cdot 1 \\ &= 0.025p^2 - 0.5p^2 + 0.5p - 0.05p^2 + 0.05p + p^2 - 2p + 1 \\ &= 0.475p^2 - 1.45p + 1, \end{aligned}$$

which gives us the contagion rate for every p. Alva wants that $f(p) \leq 0.13$, or equivalently,

$$f(p) - 0.13 \leq 0.$$

Hence, we obtain the following quadratic inequality:

$$\begin{aligned} 0 &\geq 0.475p^2 - 1.45p + 1 - 0.13 \\ &= 0.475p^2 - 1.45p + 0.87 \\ &= \tfrac{19}{40} p^2 - \tfrac{29}{20} p + \tfrac{87}{100}, \end{aligned}$$

where the zeros of the right hand side are given by

$$\begin{aligned} p_- &= \frac{1}{95} \left(145 - \sqrt{4495} \right) \approx 0.8206, \\ p_+ &= \frac{1}{95} \left(145 + \sqrt{4495} \right) \approx 2.2321. \end{aligned}$$

Since p is a probability, $p \leq 1$ holds. Additionally, $f(p)$ decreases for $0.8206 \leq p \leq 1$. Thus, we have $f(p) \leq 0.13$ for all $p \geq 0.8206$. The problem asks for the smallest integer p (in percent), for which the above inequality is satisfied. Since $p = 0.83 = 83\,\%$ is the smallest such number, the correct answer is $8 + 3 = \mathbf{11}$.

17 Elf Routing

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Project: Algebraic and Tropical Methods for Periodic Timetabling
(MATH+ Incubator Project IN-A1)

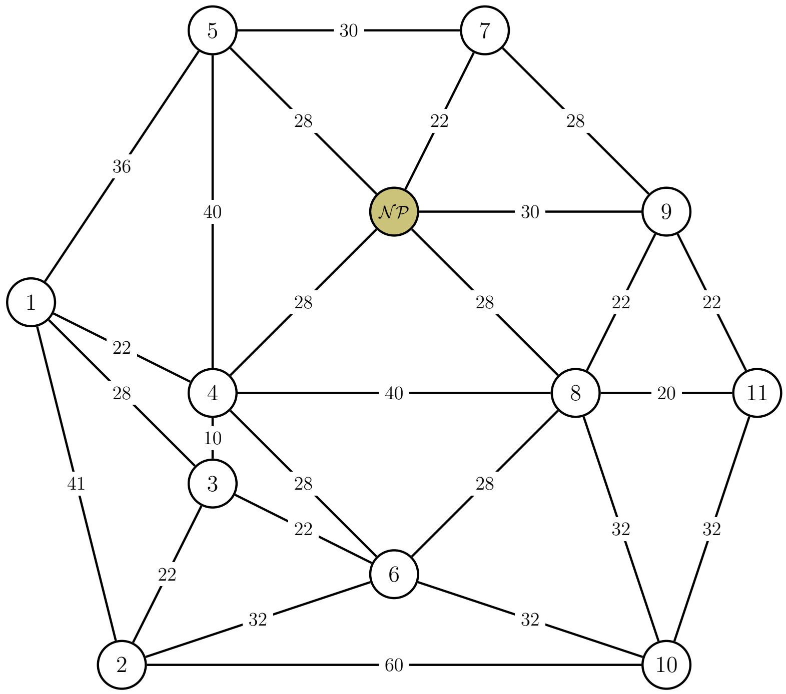
17.1 Challenge

Oh, how many children there are on Earth! And come Christmas, all of them write letters to Santa. In recent years, the elves had to build more and more post offices to sort all the letters. By now there are twelve of them scattered all over the North Pole: the main building, called $\mathcal{NP}$, and eleven additional offices, numbered from 1 to 11.

Ralph, the elf, sighs. He and his little brother Steffan were instructed to get the next batch of letters from the post offices to the $\mathcal{NP}$-building where they would be processed. Although they were given reindeers, riding through the snow for hours is still not a nice task. Ralph looks at the map that shows the post offices, the usable trails between them, and the corresponding travel times in minutes. Together with the map he was also given a list of the offices, specifying how many letters need to be picked up at each of them (see Fig. 5).
"Each of our backpacks can hold exactly 143 letters. I am pretty sure we can do this!" Ralph claims. "But what would be the best way to go, to be done as quickly as possible?"

Ralph and Steffan wish to finish their task as soon as possible. The task starts when the two elves leave the $\mathcal{NP}$-building, and it is done when both elves have returned to the $\mathcal{NP}$-building, having collected all of the letters. They leave at the same time. Every time one of them reaches a post office, he either picks up *all* of the letters there or leaves *all* of them in the office. Of course, to pick up letters they need to have sufficient space in their backpack. Being enthusiastic sightseers, neither Ralph nor Steffan want to visit the same post office twice (except for the $\mathcal{NP}$-building, to which they return only at the very end). Ralph, who is stronger, will take the longer route.

The two elves think about their task for a while, but are not able to find the best routes to take. Thankfully, their friend Petra, who is very good at



Post office	1	2	3	4	5	6	7	8	9	10	11
Letters	38	76	4	61	37	19	2	3	5	18	23

Figure 5: The map of the post offices. The circles are the numbered offices. The lines are the trails between them, with numbers indicating the time needed to ride the trail (in minutes).

List of the post offices with the number of letters that have to be picked up there.

math, passes by and is able to help them. She quickly finds the best tours for each of them so that the elves' task will be completed within as little time as possible.
"This way," she says, "Ralph will be back after 3 hours and 24 minutes."

Discussing Petra's solution, the elves say:

(a) Steffan: "Even though Ralph will pick up the letters at office 3, I have enough time to pass by there too, visit our cousin Luise, and still be back at the $\mathcal{NP}$-building before Ralph."

(b) Ralph: "Nice! We can ride together to the first post office."

(c) Ralph: "I will not visit office 4."

(d) Steffan: "The one who picks up the letters at office 3 will also get those at office 9."

(e) Steffan: "I can manage to be back exactly half an hour before Ralph and prepare a hot chocolate for him. He will love that!"

Which of the statements are true?

Artwork: Friederike Hofmann

Possible answers:

1. (a), (b), and (c).
2. (a), (b), and (d).
3. (a), (b), and (e).
4. (a), (c), and (d).
5. (a), (c), and (e).
6. (a), (d), and (e).
7. (b), (c), and (d).
8. (b), (c), and (e).

9. (b), (d), and (e).
10. (c), (d), and (e).

Project reference:

The MATH+ Incubator Project *Algebraic and Tropical Methods for Periodic Timetabling* explores connections between the mathematical optimization of timetables in public transport and pure mathematics disciplines. Routing of passengers and vehicles is an important problem in public transport. The problem presented here is a so-called *Capacitated Vehicle Routing Problem*, which belongs to the class of NP-hard optimization problems. Until now, no efficient algorithm solving this kind of problems to optimality has been found, and it is an open *Millenium Problem* to prove whether such a method exists at all.

17.2 Solution

The correct answer is: 3.

First, we take a closer look at the list of the post offices and the number of letters that have to be picked up there:

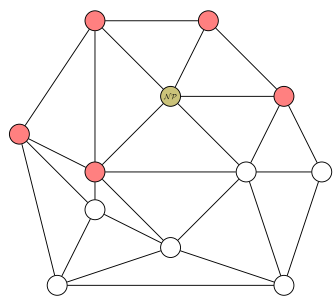
Post office	1	2	3	4	5	6	7	8	9	10	11
Letters	38	76	4	61	37	19	2	3	5	18	23

There are 286 letters in total, which is exactly $143+143$. Since the backpacks can contain at most 143 letters each, both of them have to be filled completely. There are only ten possible ways to divide the post offices into two subsets such that the sum of their letters is exactly 143. These pairs of subsets are as follows:

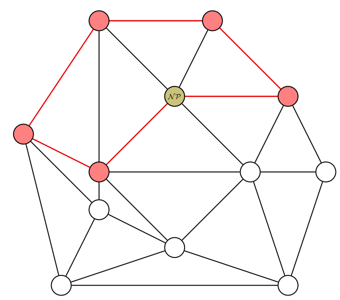
$$\begin{array}{rcl}
76, 61, 4, 2 & \text{and} & 38, 37, 23, 19, 18, 5, 3 \\
76, 38, 23, 4, 2 & \text{and} & 61, 37, 19, 18, 5, 3 \\
76, 38, 19, 5, 3, 2 & \text{and} & 61, 37, 23, 18, 4 \\
76, 38, 18, 5, 4, 2 & \text{and} & 61, 37, 23, 19, 3 \\
76, 37, 23, 5, 2 & \text{and} & 61, 38, 19, 18, 4, 3 \\
76, 37, 23, 4, 3 & \text{and} & 61, 38, 19, 18, 5, 2 \\
76, 37, 19, 5, 4, 2 & \text{and} & 61, 38, 23, 18, 3 \\
76, 37, 18, 5, 4, 3 & \text{and} & 61, 38, 23, 19, 2 \\
76, 23, 19, 18, 5, 2 & \text{and} & 61, 38, 37, 4, 3 \\
76, 23, 19, 18, 4, 3 & \text{and} & 61, 38, 37, 5, 2
\end{array}$$

They correspond to the following subsets of post offices:

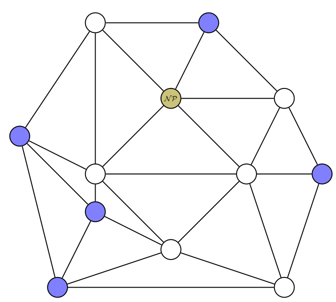
$$2, 3, 4, 7 \quad \text{and} \quad 1, 5, 6, 8, 9, 10, 11 \tag{1}$$
$$1, 2, 3, 7, 11 \quad \text{and} \quad 4, 5, 6, 8, 9, 10 \tag{2}$$
$$1, 2, 6, 7, 8, 9 \quad \text{and} \quad 3, 4, 5, 10, 11 \tag{3}$$
$$1, 2, 3, 7, 9, 10 \quad \text{and} \quad 4, 5, 6, 8, 11 \tag{4}$$
$$2, 5, 7, 9, 11 \quad \text{and} \quad 1, 3, 4, 6, 8, 10 \tag{5}$$
$$2, 3, 5, 8, 11 \quad \text{and} \quad 1, 4, 6, 7, 9, 10 \tag{6}$$
$$2, 3, 5, 6, 7, 9 \quad \text{and} \quad 1, 4, 8, 10, 11 \tag{7}$$
$$2, 3, 7, 8, 9, 10 \quad \text{and} \quad 1, 4, 5, 6, 11 \tag{8}$$
$$2, 6, 7, 9, 10, 11 \quad \text{and} \quad 1, 3, 4, 5, 8 \tag{9}$$
$$2, 3, 6, 8, 10, 11 \quad \text{and} \quad 1, 4, 5, 7, 9 \tag{10}$$



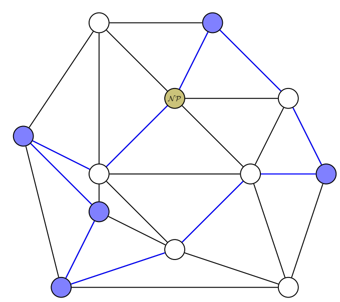
(a) Nodes containing 143 letters, close together



(b) Shortest tour with duration 174 min



(c) Nodes containing 143 letters, scattered



(d) Shortest tour with duration 252 min

Figure 6: Node subsets and their shortest tours

A simple way of solving the problem would now be to start at the top of the list and compute the two shortest tours for each pair. In a small graph like the one given in the exercise, finding them can be done quickly using intuition. In larger graphs, this is very hard and it is still subject to current research how to shorten the computation time. Since Petra told us that the optimal duration of the task is 3 hours and 24 minutes, we know that if we find a pair of subsets where the longer tour takes exactly that amount of time, we will have found an optimal solution.

However, the problem on hand may be solved in a more efficient way. Since the given graph is a map, one idea is to inspect the locations of the offices of two associated subsets. If the offices of a subset are scattered around the map, the resulting tour will likely be long (see Fig. 6(c), (d)). In contrast, if they are close together, the tour will likely be short (see Fig. 6(a), (b)). Instead of mindlessly going through all pairs of subsets one by one, we can check first how close the offices of each subset are. Then, the pair with the most promising subsets will be inspected first.

Among the list of pairs the most promising one is number 10. Having found the two shortest tours, it is easy to calculate their durations. The travel times of the trails contained in the red tour sum up to 174 minutes, those of the blue tour sum up to 204 minutes. Ralph, who takes the longer tour, would therefore return after 3 hours and 24 minutes. Since we already know that this is the optimal duration, we found an optimal solution (see Fig. 7).

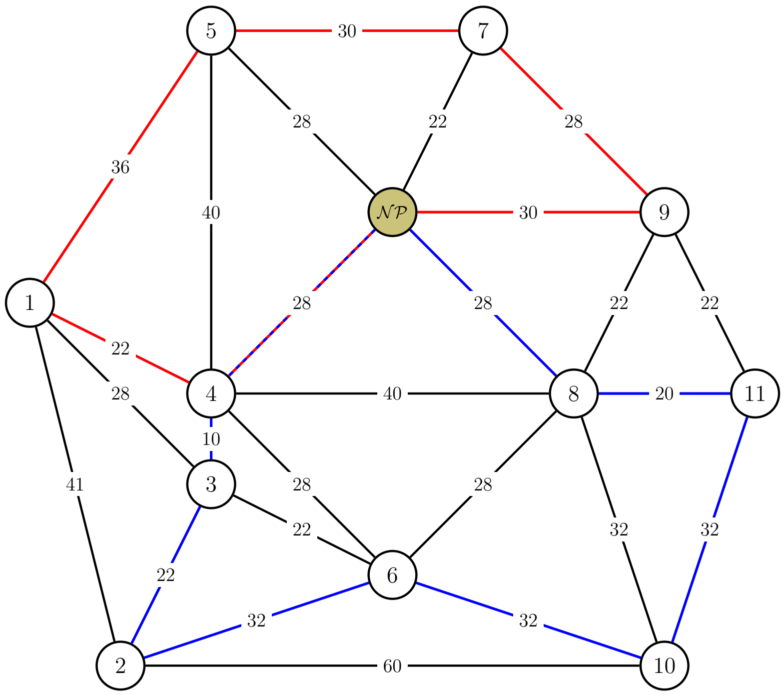


Figure 7: An optimal solution

Knowing this, the five given statements can easily be verified or contradicted:

(a) **True.** Since Steffan takes the shorter tour, he has some time to spare. As long as he does not return later than Ralph, he can use that extra time to visit Luise without affecting the total time needed to fulfill the task. If, instead of riding from office 4 directly to office 1 (or the other way around), he stops at office 3 in between, his tour will take 16 minutes longer. In this scenario, he can spend 14 minutes with Luise.

(b) True. Using the assignment induced by pair number (10), we find two tours as shown in Figure 2. Since there is no restriction on the direction of the tours, Ralph and Steffan can decide to ride from the $\mathcal{NP}$-building to office 4 together.

(c) False. Although Steffan will be the one picking up the letters in office 4, Ralph still needs to pass by there. Otherwise, his tour would be longer than 3 hours and 24 minutes.

(d) False. In office 3 there are 4 letters, in office 9 there are 5 letters. The only pairs where the same elf picks up letters in those two offices are pairs (4), (7), and (8). However, choosing these routes would lead to travel times that are longer than 204 minutes:

Nr.	Tour	Dauer
(4)	$\mathcal{NP} \to 7 \to 9 \to 10 \to 2 \to 3 \to 1 \to \mathcal{NP}$	264 min
(7)	$\mathcal{NP} \to 8 \to 11 \to 10 \to 1 \to 4 \to \mathcal{NP}$	212 min
(8)	$\mathcal{NP} \to 5 \to 1 \to 4 \to 6 \to 11 \to \mathcal{NP}$	214 min

(e) True. The shortest possible tour for Steffan is depicted in red in Fig. 7. It takes 2 hours and 54 minutes. If Steffan takes this tour, he will arrive back at the $\mathcal{NP}$-building half an hour before Ralph.

18 Magic Glue

Authors: Anieza Maltsi (WIAS),
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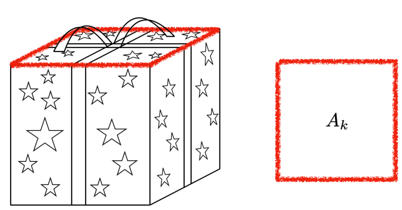
Project: Model-based geometry reconstruction from TEM images (MATH+ Emerging Field Project EF3-1)

18.1 Challenge

Last Christmas a truly terrible thing happened. Like every year, Santa brought presents to children all over the world. However, some of them were stolen by the Grinch. The mean Grinch unwrapped the presents, stole the gifts, and left the empty packaging behind. Consequently, many children did not get their presents—leaving not only them, but also Santa, quite miserable. In fact, Santa was shocked by what had happened and vowed that this would never happen again. Thus, Santa asked his elves to find a solution.

After a lot of pondering, the elves came up with an idea: they want to create boxes that can be opened only by precisely the child the present is addressed to. Of course, this scheme will require magic, but luckily they can consult their fairy friend. Indeed, the fairy agrees to help them and provides them with some magic glue. The elves fill the gift boxes with toys and seal them with the fairy's magic glue. However, after some boxing and gluing, the elves realize that the magic glue will be used up before they are finished with all the presents. Therefore, they ask their fairy friend for help once again. The fairy is eager to help, but only has exactly 2020 g of the magic glue left. Instead of panicking, the elves decide to use their mathematical superpowers and calculate how many boxes they will manage to seal with this amount of magic glue.



The boxes are cubes of different sizes. The elves need to apply the glue only at the boundary of the top face (a square). Let us denote these squares by $A_1, A_2, A_3, \ldots$. The elves know that the area of the square A_k of the k-th box is given by

$$\operatorname{area}(A_k) = 625\left(1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}\right)\,\text{cm}^2.$$

For efficiency reasons, the boxes are stacked into one another, and the elves have to process the boxes in decreasing order of their size. From earlier packaging experiences, the elves know that they need exactly 0.15 g of glue for a line of 1 cm.

How many boxes will the elves be able to seal with the 2020 g of magic glue?

Artwork: Friederike Hofmann

Possible answers:

1. 127 boxes.
2. 128 boxes.
3. 129 boxes.
4. 130 boxes.
5. 131 boxes.
6. 132 boxes.

7. 133 boxes.
8. 134 boxes.
9. 135 boxes.
10. 136 boxes.

Project reference:

The project MATH+ EF3-1 *Model-based geometry reconstruction from TEM images* analyzes images from transmission electron microscopy (TEM) obtained by experimental physicists at TU Berlin. The mathematical task is to reduce the electronic Schrödinger equation by a finite-dimensional approximation. To evaluate the approximation error one uses similar estimates as in this puzzle.

18.2 Solution

The correct answer is: 7.

The elves have enough glue for exactly 133 boxes. We give the solution in two steps:

1. For a given number of boxes n, we compute the total line length $l(n)$ on which the glue is applied and the amount of glue $g(n) = 0.15\,\frac{\mathrm{g}}{\mathrm{cm}} \cdot l(n)$ needed for this length.

2. Then, we equate the available amount of glue (2020 g) with $g(n)$ and solve the equation for n.

Let us start by computing the length $l(n)$ for a given number n of boxes: We know that the area of the k-th square is given by

$$\mathrm{area}(A_k) = 625\left(1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}\right)\mathrm{cm}^2.$$

Therefore, each edge of A_k has length

$$a_k := 25\sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}\ \mathrm{cm}.$$

Since the top side of the box has four edges, glue is needed for a total line length of

$$l(n) = 4\sum_{k=1}^{n} a_k.$$

We calculate the total length by simplifying the sum. At first, we observe that the term under the square root can be written as

$$\begin{aligned}1+\frac{1}{k^2}+\frac{1}{(k+1)^2} &= \frac{k^2(k+1)^2+(k+1)^2+k^2}{k^2(k+1)^2}\\ &= \frac{k^2(k+1)^2+2k^2+2k+1}{k^2(k+1)^2}\\ &= \frac{k^2(k+1)^2+2k(k+1)+1}{k^2(k+1)^2}\\ &= \frac{(k(k+1)+1)^2}{k^2(k+1)^2}\\ &= \frac{(k^2+k+1)^2}{k^2(k+1)^2}.\end{aligned}$$

It follows that

$$\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \frac{k^2+k+1}{k(k+1)}.$$

Further on, we use the partial fraction decomposition to find

$$\frac{1}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

and get

$$\frac{k^2+k+1}{k(k+1)} = \frac{k(k+1)+1}{k(k+1)} = 1+\frac{1}{k(k+1)} = 1+\frac{1}{k}-\frac{1}{k+1}.$$

Hence, the total length is given by

$$
\begin{aligned}
l(n) &= 4\sum_{k=1}^{n} a_k \\
&= 4\sum_{k=1}^{n} 25\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\ \text{cm} \\
&= 100\,\text{cm}\cdot\sum_{k=1}^{n}\left(1+\frac{1}{k}-\frac{1}{k+1}\right) \\
&= 100\,\text{cm}\cdot\left(n+\sum_{k=1}^{n}\left\{\frac{1}{k}-\frac{1}{k+1}\right\}\right) \\
&= 100\,\text{cm}\cdot\left(n+\left\{1-\tfrac{1}{2}\right\}+\left\{\tfrac{1}{2}-\tfrac{1}{3}\right\}+\cdots+\left\{\tfrac{1}{n-1}-\tfrac{1}{n}\right\}+\left\{\tfrac{1}{n}-\tfrac{1}{n+1}\right\}\right) \\
&= 100\,\text{cm}\cdot\left(n+1+\left\{-\tfrac{1}{2}+\tfrac{1}{2}\right\}+\left\{-\tfrac{1}{3}+\tfrac{1}{3}\right\}+\cdots+\left\{-\tfrac{1}{n}+\tfrac{1}{n}\right\}-\tfrac{1}{n+1}\right) \\
&= 100\,\text{cm}\left(n+1-\frac{1}{n+1}\right)
\end{aligned}
$$

As mentioned before, the total amount of glue needed is given by the total length $l(n)$ multiplied by the glue density of 0.15 gram per cm:

$$g(n) = 0{,}15\,\tfrac{\text{g}}{\text{cm}}\cdot l(n).$$

Hence, the wanted total number of boxes is given by the largest number n satisfying

$$
\begin{aligned}
& g(n) \le 2020\,\text{g} \\
\Leftrightarrow\quad & 0{,}15\,\tfrac{\text{g}}{\text{cm}}\cdot l(n) \le 2020\,\text{g} \\
\Leftrightarrow\quad & 0{,}15\,\tfrac{\text{g}}{\text{cm}}\cdot 100\,\text{cm}\cdot\left(n+1-\frac{1}{n+1}\right) \le 2020\,\text{g} \\
\Leftrightarrow\quad & \left(n+1-\frac{1}{n+1}\right) \le \frac{2020}{15} = \frac{404}{3} = 134.\bar{6}.
\end{aligned}
$$

Since $0 < 1-\frac{1}{n+1} < 1$, the largest possible n is either $n = 133$ or $n = 134$. One easily checks that

$$133+1-\frac{1}{133+1} = \frac{17955}{134} \approx 133.99 \le 134.\bar{6},$$

but

$$134 + 1 - \frac{1}{134 + 1} = \frac{18225}{135} \approx 134.99 > 134.\bar{6}.$$

Thus, the wanted number of boxes is $\mathbf{n} = \mathbf{133}$.

19 Cherry Wine

Authors: Aart Blokhuis (TU Eindhoven),
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Project: 4TU.AMI

19.1 Challenge

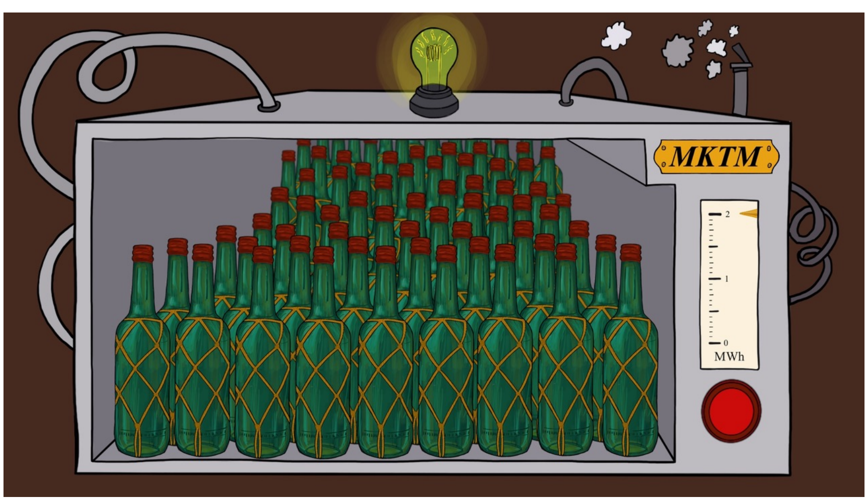
Ruprecht has 377 indistinguishable sealed bottles in his wine cellar. One of these bottles contains delicious cherry wine, whereas all the remaining 376 bottles contain highly toxic belladonna juice. Ruprecht's Magical Cherry Wine Testing Machine (MCWTM) has a huge compartment (that can accomodate up to 377 bottles), a huge red button, and a light bulb.

If Ruprecht puts some bottles into the compartments and then presses the red button, the MCWTM wakes up, starts to work, and consumes a lot of energy.

- If one of the bottles in the compartment does contain cherry wine, the MCWTM consumes 2 MWh of energy and the bulb lights up green.
- If none of the bottles in the compartment does contain cherry wine, the MCWTM consumes only 1 MWh of energy and bulb lights up red.

Ruprecht would like to give a bottle of cherry wine as a present to Santa Claus.

How much energy does Ruprecht have to use in the worst case under an optimal strategy if he wants to identify the bottle of cherry wine?



Artwork: Frauke Jansen

Possible answers:

1. In the worst case, Ruprecht has to use 11 MWh of energy.
2. In the worst case, Ruprecht has to use 12 MWh of energy.
3. In the worst case, Ruprecht has to use 13 MWh of energy.
4. In the worst case, Ruprecht has to use 14 MWh of energy.
5. In the worst case, Ruprecht has to use 15 MWh of energy.
6. In the worst case, Ruprecht has to use 16 MWh of energy.
7. In the worst case, Ruprecht has to use 17 MWh of energy.
8. In the worst case, Ruprecht has to use 18 MWh of energy.
9. In the worst case, Ruprecht has to use 19 MWh of energy.
10. In the worst case, Ruprecht has to use 20 MWh of energy.

19.2 Solution

The correct answer is: 3.

We will work with the sequence of Fibonacci numbers that starts with

$$F_0 = F_1 = 1$$

and is defined recursively by

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$. In the following table, the first Fibonacci numbers are stated:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
1	2	3	5	8	13	21	34	55	89	144	233	377

In the following discussion, we consider n bottles of which $n-1$ contain belladonna juice and only one cherry wine. Let $M(n)$ denote the amount of energy in MWh that Ruprecht needs to identify the bottle of cherry wine in the worst case.

For $n = 1$, one has $M(1) = 0$, since Ruprecht does not need to employ the MCWTM. By induction, we will show that $M(F_k) = k$ holds for all $k \geq 2$.

First, consider the case $k = 2$ with $n = F_2 = 2$ bottles:
Ruprecht needs to test only one of the bottles. However, this test can have a positive result, and we have $M(2) = 2$ as claimed.

Next, let $k = 3$; i. e. we consider $n = F_3 = 3$ bottles:

1. If Ruprecht tests one bottle first, this test may have a negative result; i. e. the MCWTM consumes 1 MWh. Additionally, at most $M(2) = 2$ MWh are needed to test the remaining two bottles.
2. However, if Ruprecht tests two bottles at first, this test may have a positive result; i. e. the MCWTM consumes 2 MWh. Again, further $M(2) = 2$ may be needed to test these two bottles. All in all, 4 MWh may be consumed.

For $n = F(3) = 3$ bottles, Ruprecht should pick the first strategy, where—as claimed—at most $M(3) = M(F_3) = 3$ MWh are consumed.

For the induction step, consider $k \geq 4$ with $n = F_k$ bottles.
The statement $M(F_k) = k$ will be proved by employing the induction hypothesis

$$M(F_{k-2}) = k - 2 \quad \text{und} \quad M(F_{k-1}) = k - 1.$$

To this end, we will show first that there is a strategy which needs at most k MWh. Afterwards, we prove that for every other strategy the MCWTM consumes at least k MWh in the worst case.

Suppose that Ruprecht first tests F_{k-2} of the the F_k bottles. Then, we have either one of the following cases:

+ The test of F_{k-2} bottles yields a positive result, and the MCWTM consumes 2 MWh.

  Then, Ruprecht has to test F_{k-2} bottles. By the induction hypothesis, this test consumes at most $M(F_{k-2}) = k - 2$ MWh.

  In total, at most $2 + k - 2 = k$ MWh are needed.

– The test of F_{k-2} bottles yields a negative result, and the MCWTM consumes 1 MWh.

  Then, Ruprecht has to test F_{k-1} bottles. By the induction hypothesis, this test consumes at most $M(F_{k-1}) = k - 1$ MWh.

  Again, at most $1 + k - 1 = k$ MWh are needed.

As promised, we will show that there is no strategy for which the MCWTM consumes less energy (in the worst case). To this end, we distinguish the following two cases:

1. Ruprecht first tests $m \geq F_{k-2}$ bottles.

   Since this test can yield a positive result, the MCWTM may consume 2 MWh. Then, in the worst case further $M(m)$ MWh are needed to test the m bottles.

   Since $m \geq F_{k-2}$, one has $M(m) \geq M(F_{k-2}) = k - 2$.

   In total, Ruprecht may need more than $2 + k - 2 = k$ MWh.

2. Ruprecht first tests $m \leq F_{k-2}$ bottles.

   This test can yield a negative result, and the MCWTM consumes 1 MWh. Then, in the worst case further $M(l)$ MWh are needed to test the remaining $l = F_k - m$ bottles.

And $l = F_k - m \geq F_k - F_{k-2} = F_{k-1}$ gives $M(l) \geq M(F_{k-1}) = k - 1$. Again, Ruprecht may need more than $1 + k - 1 = k$ MWh.

Hence, the strategy suggested above is indeed optimal and for testing F_k bottles at most $M(F_k) = k$ MWh are needed.

Finally, since $F_{13} = 377$, this formula immediately gives $M(377) = 13$. Thus, Ruprecht has to use **13 MWh** in the worst case.

20 Another Tricky Hat Challenge

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Project: 4TU.AMI

20.1 Challenge

Santa Claus addresses the seventeen super-smart elves Alba, Bilbo, Carla, Dondo, Edda, Frodo, Greta, Harpo, Izzy, Jacco, Kira, Loco, Mila, Nemmo, Olga, Puzzo, and Quibo: "My dear super-smart elves! Once again, we want to have a tricky puzzle with coloured hats in the Mathematical Advent Calendar. For this reason, I will invite you over to my place for coffee and cake tomorrow afternoon."
"Great, we are happy to accept your invitation!", the seventeen elves shout in chorus.

Santa Claus continues: "Tonight I am going to prepare 16 red and 16 blue hats. Tomorrow's game will then consist of two phases. During the **first phase**, Quibo has to wait outside alone. The other sixteen elves come to my vestibule, where the 32 hats are lying on top of the dresser. I will randomly point at one of the elves. Then, this elf has to choose a red or a blue hat to put on the head. Afterwards, I point at a second elf, who also has to choose a hat. Then I point at a third, fourth, fifth elf, and so on, until the fifteenth elf has chosen a hat and put it on the head. At the end, I myself will choose a hat for the remaining sixteenth elf. Since the sixteenth elf does play a special role in the game, this elf will be called *Elf X*. Are there any questions about this first phase of the game?"

Alba wants to know: "In which order are you going to point at the elves?"
"The order is arbitrary, just as it comes to my head," answers Santa Claus.

Then, Bilbo asks: "Does the second elf know the hat colour of the first elf, at the moment when he or she has to pick the own hat colour?"
"Yes!" answers Santa Claus. "Everything in this game is transparent. With the sole exception of Quibo waiting outside, every elf is able to see all the hats on all the heads of all the other elves at any time."

Finally, Carla wants to know: "Which colour are you going to pick for Elf X?"

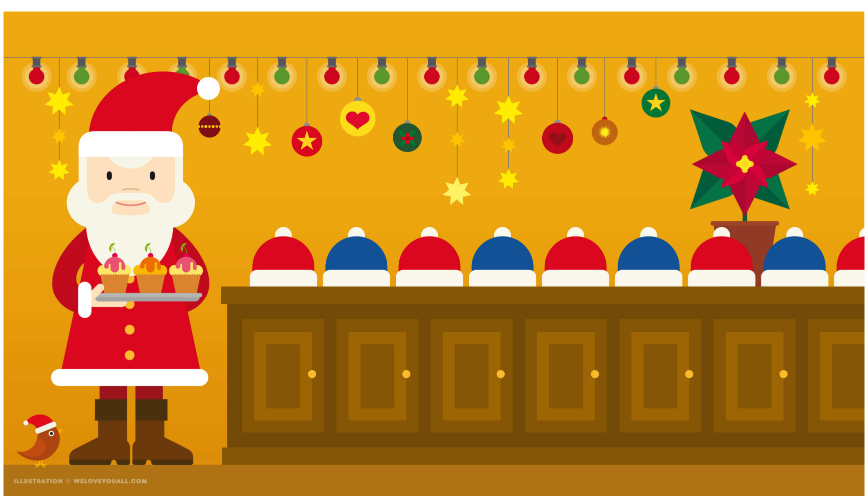
"I am going to pick the colour that I find appropriate," says Santa Claus.

Santa Claus continues: "In the game's **second phase**, Quibo is finally allowed into the vestibule. Quibo's task is to arrange the sixteen other elves into a long row from left to right. Every elf ending up to the left of Elf X must leave and stay hungry. However, every elf standing to the right of Elf X receives a piece of apple pie and a cup of coffee. Also Elf X and Quibo receive pie and coffee. Are there any questions about this second phase?"

Dondo asks: "Are we allowed to help and advise Quibo in arranging the row of sixteen elves?"
"No!", answers Santa Claus. "You are not allowed to communicate any information to Quibo. And you are also not allowed to cheat! But of course you already know all that from hat puzzles in the preceding editions of the Mathematical Advent Calendar."

The seventeen elves start to ponder. They discuss and they think. They think and they discuss. Then, they discuss some more, and then they think some more. Eventually they manage to develop an amazing strategy that maximizes the number M of elves who receive pie and coffee—independently of the decisions and actions of Santa Claus in the first phase of the game.

Which of the following statements is true for this number M?



Artwork: Friederike Hofmann

Possible answers:

1. $M \leq 5$.
2. $M = 6$.
3. $M = 7$.
4. $M = 8$.
5. $M = 9$.
6. $M = 10$.
7. $M = 11$.
8. $M = 12$.
9. $M = 13$.
10. $M \geq 14$.

20.2 Solution

The correct answer is: 10.

We describe a possible strategy that divides the sixteen elves into four groups of four:

Group 1:	Alba, Bilbo, Chico, Dondo
Group 2:	Edda, Frodo, Greta, Harpo
Group 3:	Izzy, Jacco, Kira, Loco
Group 4:	Mila, Nemmo, Olga, Puzzo

During the first phase of the game, the members of each group behave as follows: The first three elves (of each group) chosen by Santa each take a **blue** hat. The fourth elf in the group is either Elf X (and is assigned his hat color by Santa) or otherwise picks a **red** hat.
In the second phase of the game, Quibo does the following:

- If Quibo sees only three elves with **red** hats, Elf X must be in the (only) group with four **blue** hats.

  Quibo places the four pixies of this group to the left of the other groups.

- If Quibo sees four elves with **red** hats, Elf X must be among these four elves.

  Quibo places the four elves with **red** hats to the left of the other 12 elves.

In both cases, Elf X is among the first four elves on the far left. Therefore, at most three elves are to the left of Elf X and are sent home, while **at least 14** elves receive a piece of apple pie and a cup of coffee.

21 Xmasium

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Projekt: 4TU.AMI

21.1 Challenge

In one of Santa Claus' research laboratories the scientists have discovered a new chemical element and named it *Xmasium* (in analogy with the famous elements Rubidium, Caesium and Francium). Every Xmasium atom consists of several fermions, bosons, and other elementary particles. Cutting-edge breakthrough research has shown that Xmasium atoms show up in two fundamentally different types: there are Xmasium atoms *with* a Higgs boson, and there are Xmasium atoms *without* a Higgs boson. With the currently available lab equipment, it is not possible to distinguish the two types from each other.

Xmasium atoms possess some rather interesting properties:

- If you throw an Xmasium atom A with Higgs-boson against an Xmasium atom B without Higgs boson, the Higgs boson is separated from atom A and snuggles up to atom B. In other words, both Xmasium atoms change their type.
- In the other three cases (A and B with Higgs boson; A and B without Higgs boson; A without and B with Higgs-boson) such a throw remains without consequences, and both atoms keep their old type.

The transmission of a Higgs boson occurs lightning-fast and cannot be observed with the available lab equipment.

Ruprecht has six Xmasium atoms X_1, X_2, X_3, X_4, X_5, and X_6 of unknown type in his cooking pot. Ruprecht would like to brightly polish two atoms of the same type, put them into a gift box, and give them to Santa Claus as a present. (The other four atoms will remain in the cooking pot.)

What is the least number of throws that Ruprecht has to perform in the worst case, so that he ends up with at least two atoms of the same type?

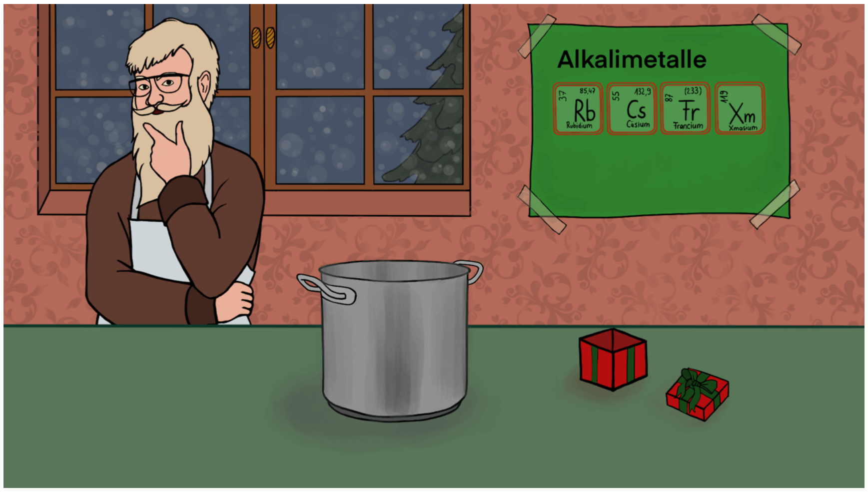


Illustration: Frauke Jansen

Possible answers:

1. In the worst case, Ruprecht needs 2 throws.
2. In the worst case, Ruprecht needs 3 throws.
3. In the worst case, Ruprecht needs 4 throws.
4. In the worst case, Ruprecht needs 5 throws.
5. In the worst case, Ruprecht needs 6 throws.
6. In the worst case, Ruprecht needs 8 throws.
7. In the worst case, Ruprecht needs 10 throws.
8. In the worst case, Ruprecht needs 12 throws.
9. In the worst case, Ruprecht needs 15 throws.
10. In the worst case, Ruprecht is not able to achieve this goal.

21.2 Solution

The correct answer is: 10.

First, we arrange the six atoms $X_1, \ldots, X_6$ (while we do not know their types) in a row from left to right:

X_1	X_2	X_3	X_4	X_5	X_6

With each throw, we want to rearrange the atoms. We distinguish the following two cases:

Case 1: If Ruprecht throws an atom X_a at an atom X_b that is to the left of X_a, we do nothing. Before and after the throw, we have the following arrangement:

...	X_b	...	X_a	...

Case 2: However, if Ruprecht throws an atom X_a at an atom X_b that is to the right of X_a, we swap the two atoms X_a and X_b:

...	X_a	...	X_b	...	$\rightarrow$	...	X_b	...	X_a	...

We now consider the five situations $S(k)$, $k = 1, \ldots, 5$ that are defined as follows:

> In the situation $S(k)$, exactly the first k of the six atoms in the row have a Higgs boson (1), while the remaining $6 - k$ atoms have no Higgs boson (0):

	X_1	X_2	X_3	X_4	X_5	X_6
$S(1)$	1	0	0	0	0	0
$S(2)$	1	1	0	0	0	0
$S(3)$	1	1	1	0	0	0
$S(4)$	1	1	1	1	0	0
$S(5)$	1	1	1	1	1	0

Since Knecht Ruprecht does not know the type of $X_1, \ldots, X_6$, he cannot distinguish these five situations.

Now, we claim the following: If the atoms $X_1, \dots, X_6$ were in the situation $S(k)$ for a $k \in \{1, 2, 3, 4, 5\}$ at the beginning, then they are also in the situation $S(k)$ after any number of throws, provided that we rearrange the atoms at each throw according to 1) or 2). For the proof, we distinguish the following cases:

Case 1: Ruprecht throws X_a at X_b that is to the left of X_a. Then

a) either X_a and X_b have the same type:

...	X_b	...	X_a	...
1	1	1	1	...

bzw.

...	X_b	...	X_a	...
...	0	0	0	0

b) or X_a has no Higgs boson but X_b has:

...	X_b	...	X_a	...
1	1	...	0	0

According to the prerequisite, no Higgs boson is transferred in any of the cases a) or b). Moreover, we do not rearrange the atoms either, since we are in case 1). Thus, if the atoms were in situation $S(k)$ for a $k \in \{1, 2, 3, 4, 5\}$ before the throw, then this is also true after the throw.

Case 2: Ruprecht throws X_a at X_b that is to the right of X_a. Then

a) either X_a and X_b have the same type

b) or X_a has a Higgs boson but X_b has not.

According to the prerequisite, no Higgs boson is transferred in case a). But since the atoms are in case 2), we swap the atoms X_a and X_b. However, they have the same type, and the arrangement of the Higgs bosons in the row does not change:

...	X_a	...	X_b	...
1	1	1	1	...

$\rightarrow$

...	X_a	...	X_b	...
1	1	1	1	...

...	X_a	...	X_b	...
...	0	0	0	0

$\rightarrow$

...	X_a	...	X_b	...
...	0	0	0	0

In case b), the Higgs boson is transferred from X_a to X_b. But since we then swap the atoms, the arrangement of the Higgs bosons in the row does not change here either:

...	X_a	...	X_b	...
1	1	...	0	0

$\rightarrow$

...	X_b	...	X_a	...
1	1	...	0	0

Therefore, also in case 2) it holds that if the atoms were in situation $S(k)$ for a $k \in \{1, 2, 3, 4, 5\}$ before the throw, then this is also true after the throw.

Finally, we reason as follows: Suppose that, after a few throws, Ruprecht claims he knows that two concrete atoms have the same type. One of the two atoms is further to the left, say at position ℓ with $\ell \in \{1, 2, 3, 4, 5\}$, and one further to the right, say at position r with $r \in \{2, 3, 4, 5, 6\}$. However, if the six atoms $X_1, \ldots, X_6$ were in the situation $S(\ell)$ at the beginning, then they are still in this situation. Hence, Ruprecht's statement is wrong, because, in this case, the ℓ-th atom has a Higgs boson whereas the r-th does not. In particular, they do *not* have the same type.

In summary, Knecht Ruprecht can never be sure that two atoms have the same type. Therefore, **he is not able to achieve his goal**.

22 Midoku

Author: Ariane Beier (MATH+ School Activities)

22.1 Challenge

Last year, directly after Christmas, elf Sasha was given a Sudoku puzzle by her friend Alex. The puzzle was intended to entertain her on the long train ride from the North Pole to her home town. But, as soon as she sat down in the comfortable train seat and the coach started moving soothingly, she fell asleep—after all, she just had worked her way through a very exhausting Christmas season—and woke up only shortly before arriving at her destination. Thus, Sasha did not even attempt to solve the puzzle.

Back home, Sasha forgot to empty her pants' pockets before putting them into the washing machine. When she hung the pants up to dry, she pulled out a piece of paper with only the following on it:

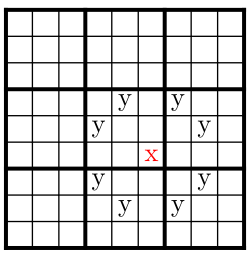
		1						
						2		

Although Sasha was pretty sure that there were more digits given on the Sudoku grid before its involuntary bath and roller coaster ride in the washing machine, she was not able to recollect them. However, she did remember the rules of this special Sudoku:

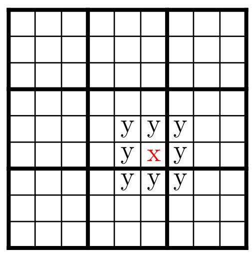
(i) The normal Sudoku rules apply, i. e. every row, every column, and every 3 by 3 subgrid contains the digits from 1 to 9 exactly once.

(ii) Any two cells separated by a knight's (see Fig. 8(a)) or king's (chess) move (see Fig. 8(b)) cannot contain the same digit.

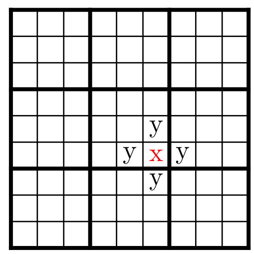
(iii) Any two orthogonally adjacent cells (see Fig. 8(c)) *cannot* contain consecutive digits (e. g. not 1 & 2 or 5 & 6).



(a) Each y is separated from x by a knight's move.



(b) Each y is separated from x by a king's move.



(c) Every y is orthogonally adjacent to the x.

Figure 8: Special rules of the given Sudoku.

Always tempted by a tricky challenge, Sasha whipped out her pencil and started to ponder. After 25 minutes of heavy thinking, she was able to finish the Sudoku.

What digit did Sasha write down in the uppermost right corner?

Artwork: Friederike Hofmann

Possible answers:

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. Apparently, Sasha made a mistake. The given Sudoku cannot be solved without ambiguity.

22.2 Solution

The correct answer is: 9.

The given Sudoku is known as the *The Miracle Sudoku*. It was invented by Mitchell Lee, and it is indeed unambiguously solvable. The solution is:

4	8	3	7	2	6	1	5	**9**
7	2	6	1	5	9	4	8	3
1	5	9	4	8	3	7	2	6
8	3	7	2	6	1	5	9	4
2	6	1	5	9	4	8	3	7
5	9	4	8	3	7	2	6	1
3	7	2	6	1	5	9	4	8
6	1	5	9	4	8	3	7	2
9	4	8	3	7	2	6	1	5

A complete solution is presented in the mesmerising video on the Youtube Channel *Cracking the Cryptic*:

`https://www.youtube.com/watch?v=yKf9aUIxdb4`.

23 Sea of Lights

Author: Ariane Beier (MATH+ School Activities)

23.1 Challenge

Finally! Tomorrow is Christmas Eve. Accordingly, the elves have almost finished their Christmas preparations: they baked biscuits, made presents, and put up and decorated the Christmas tree in the festival hall. Now, all that's left to be done is decorating Main Street in Elf Town. To do this, head elf Omega instructs elf Epsilon to put up 2020 lights along the street.

Epsilon sets off and completes her task in almost no time. All of the 2020 lights shine brightly. After her work is done, she is about to treat herself to a delicious speculoos and a nice hot cocoa when Gamma approaches her frantically: "Epsilon, you forgot to label the switches on the big control panel in the basement. If we don't know which switch belongs to which light, we won't be able to perform our choreography of lights tomorrow evening. So, hurry up! Go down to the basement and label the control panel!"

In her mind, Epsilon can already see herself running back and forth between the street and the control panel in the basement for the next few hours, in order to assign all 2020 lights to all 2020 switches without ambiguity—while her cocoa is slowly but surely getting cold. However, while she takes the last motivating sip from her cup, she has an idea how to minimise the number m of ways to the basement that are required to fulfil her task.

Which statement about this minimal number m is true?

Artwork: Frauke Jansen

Possible answers:

1. $m \leq 12$.
2. $12 < m \leq 42$.
3. $42 < m \leq 63$.
4. $63 < m \leq 173$.
5. $173 < m \leq 196$.
6. $196 < m \leq 308$.
7. $308 < m \leq 712$.
8. $712 < m \leq 1608$.
9. $1608 < m \leq 1984$.
10. $1984 < m \leq 2019$.

23.2 Solution

The correct answer is: 1.

Of course, Epsilon can complete her task by running down the basement 2019 times, flipping and labelling the switches one by one. However, she rather likes to use her head than her heels and derives the following strategy:

Before she starts, Epsilon puts a sticky note (or something similar) on each light bulb and each switch on the control panel.

On her first trip to the basement, Epsilon turns 1010 switches on, labels them with a "1". Then, she flips the other 1010 switches off and labels them with a "0". After that, she goes upstairs and marks the glowing light bulbs with a "1" and the black ones with a "0" accordingly.

On her second trip to the basement, she keeps off half (i. e. 550) of the 1010 switches marked with a "0" and turns off half (i. e. 550) of the 1010 switches marked with a "1". These switches are marked with an additional "0". Then, all switches which are currently marked with only one digit ("0" or "1") are turned on (if there not already) and marked with an additional "1". Now, all switches are labelled with two digits, "00", "01", "10", or "11". Afterwards, Epsilon goes back on the street and labels all glowing light bulbs with an additional "1" and all black ones with an additional "0". Similarly, all light bulbs are now labelled with two digits.

Now, you might guess where this strategy is headed:

- With *one trip* to the basement, Epsilon has divided the set of 2020 switches and corresponding light bulbs into $2^1 = 2$ subsets, one where all switches and light bulbs are marked with a "0" and one subset with switches and light bulbs marked with a "1".

- After *two trips*, she has further divided the 2020 switches and corresponding light bulbs into $2^2 = 4$ subgroups: 550 labelled with "00", 550 labelled with "01", 550 labelled with "10", and 550 labelled with "11".

- After the *third trip*, Epsilon has $2^3 = 8$ subgroups marked with a distinct 3-digit binary number, i. e. either with "000", "001", "010", "100", "011", "110", "101", or "111".

- ...
- Penultimately, after *ten trips*, she has $2^{10} = 1024$ subgroups, each marked with a distinct 10-digit binary number.
- Finally, with *eleven trips*, Epsilon can label up to $2^{11} = 2048$ switches and corresponding light bulbs with a 11-digit binary number without ambiguity.

In particular, she only needs **eleven** trips to the basement to assign all of the 2020 switches to the 2020 light bulbs.

24 Meeting Point

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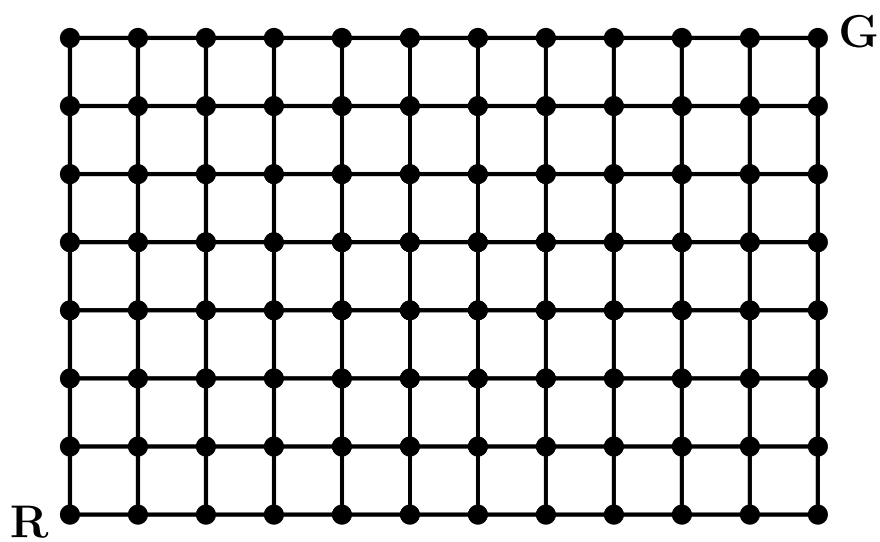
Project: 4TU.AMI

24.1 Challenge

Ruprecht and the Grinch play a game on the depicted board. Ruprecht starts at the point R. In every round, he makes a step either to the adjacent point to the north or the adjacent point to the east of his current position. The Grinch starts at the point G. In every round, he makes a step either to the adjacent point to the south or the adjacent point to the west of his current position.



At the beginning of *every* round, Ruprecht chooses a probability p_R and either moves towords the north (with probability p_R) or towards the east (with probability $1-p_R$). *At the same time*, the Grinch chooses a probability p_G for *this* round and either moves towards the south (with probability p_G) or towards the west (with probability $1-p_G$).

Ruprecht wins the game, if he manages to reach the point G without ever occupying the same point as the Grinch. The Grinch wins the game, if he meets Ruprecht in a point or on an edge of the grid.

In every round, Ruprecht and the Grinch take the best possible decisions that maximize their respective probabilities of winning the game.

What is the winning probability p for the Grinch?

Artwork: Frauke Jansen

Possible answers:

1. $p \leq 0.001$.
2. $0.001 < p \leq 0.002$.
3. $0.002 < p \leq 0.004$.
4. $0.004 < p \leq 0.008$.
5. $0.008 < p \leq 0.016$.
6. $0.016 < p \leq 0.032$.
7. $0.032 < p \leq 0.064$.
8. $0.064 < p \leq 0.128$.
9. $0.128 < p \leq 0.256$.
10. $0.256 < p$.

24.2 Solution

The correct answer is: 4.

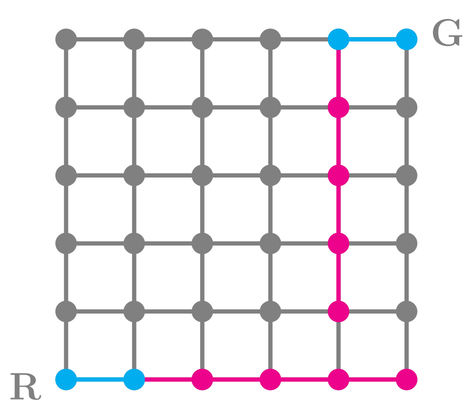
We analyse a generalisation of the game where the playing field is a $x \times y$ grid with $x \geq y$ and $x \equiv y \bmod 2$. Here we start counting at 0, so the figure above shows a grid of size 11×7. Ruprecht starts in the bottom left corner (at $(0, 0)$) and the Grinch starts in the top right corner (at (x, y)). We will prove by method of induction via y that the Grinch in this generalised variant wins with probability $p_y = 2^{-y}$.

For $y = 0$ the grid is a line with $x + 1$ points:

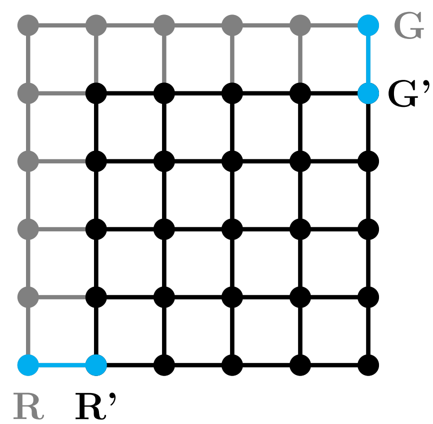
Since the players have no choices for their movement, they meet in the middle after $\frac{x+1}{2}$ rounds and the Grinch wins with the claimed probability of $p_0 = 2^{-0} = 1$.

Next we consider the case $x = y \geq 1$:

- If in the first round (marked blue in the following figure) Ruprecht and the Grinch both move in the x direction or both move in the y direction, Ruprecht wins the game, as Ruprecht can simply keep moving in the direction of his first move (in pink) for the following $y - 1$ rounds and thus avoid meeting the Grinch:



- If Ruprecht moves in the x direction in the first round (blue in the following figure) and the Grinch moves in the y direction (or vice versa), they reach the game on a grid of size $(y-1)\times(y-1)$ (in black):



  By induction hypothesis the Grinch will win this game on the smaller board with probability $p_{y-1} = 2^{-(y-1)}$.

On the grid of size $y \times y$ the Grinch can now force a winning probability of *at least* 2^{-y} by going south with probability $1/2$ and west with probability $1/2$. His probability of winning will then amount to

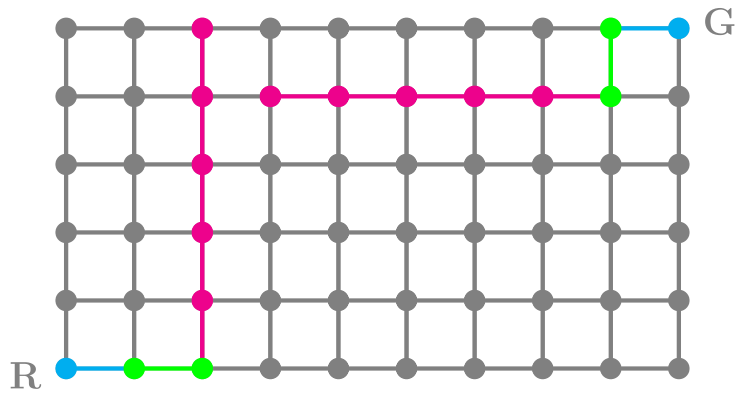
$$p = \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 2^{-(y-1)} = 2^{-y}.$$

On the other hand, Ruprecht can keep the winning probability of the Grinch at *at most* 2^{-y} by also going north with probability $1/2$ and east with probability $1/2$. Then the probability of the Grinch winning will be

$$\begin{aligned} p &= 1 - \left(\tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot \left(1 - 2^{-(y-1)}\right)\right) \\ &= 1 - \left(\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} \cdot 2^{-(y-1)}\right) \\ &= 1 - \left(1 - 2^{-y}\right) \\ &= 2^{-y}. \end{aligned}$$

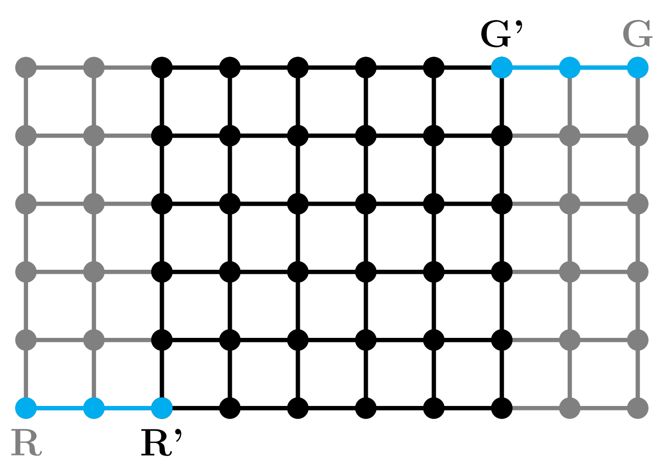
Thus, the case $x = y \geq 1$ is settled, and one has $p_y = 2^{-y}$, as claimed.

Finally, we consider the remaining cases with $x > y \geq 1$: to do this, first note that $x \geq y + 2$ holds, since $x \equiv y \mod 2$.
If the Grinch goes south only once in his first $\frac{x-y}{2}$ ($\in \mathbb{N}$) moves, Ruprecht can escape the Grinch in any case by making y moves towards the north (in pink):



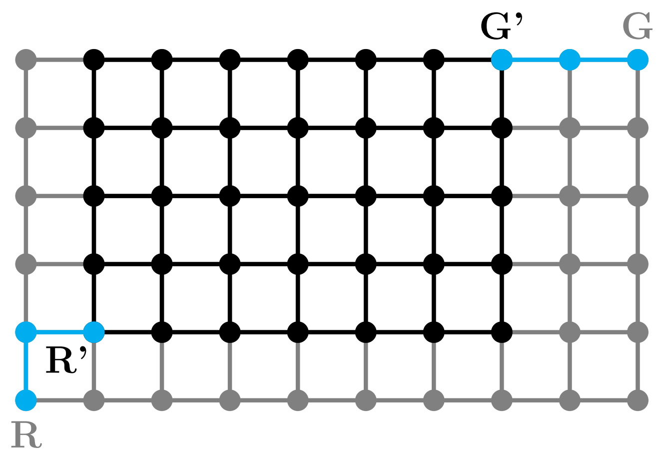
Ruprecht therefore knows that the Grinch will make his first $\frac{x-y}{2}$ moves towards the west.

- If Ruprecht makes the first $\frac{x-y}{2}$ moves towards the east (blue in the following figure), both players will reach the game on a grid of size $y \times y$ (in black):



There, as already shown, the Grinch wins with a probability of 2^{-y}.

- If for $1 \leq k \leq \frac{x-y}{2}$ Ruprecht moves towards the north k-times in his first $\frac{x-y}{2}$ moves, they reach the game on a grid of size $(y+k) \times (y-k)$:



But then, according to the induction hypothesis, the Grinch would win with probability $2^{-(y-k)} = 2^k \cdot 2^{-y} > 2^{-y}$.

Therefore, Ruprecht will make his first $\frac{x-y}{2}$ moves towards the east, and the Grinch wins the game on the grid of size $x \times y$ with probability 2^{-y}.

This completes the analysis of the generalisation of the game and shows that the Grinch always wins with a probability of

$$p_y = 2^{-y}.$$

Hence, the given game on a grid of size 11×7 will be won by the Grinch with probability

$$p = 2^{-7} = \frac{1}{128} = \mathbf{0.0078125}.$$