Challenges & Solutions

2018

MATHEON KALENDER

Forschungszentrum MATHEON
Mathematik für Schlüsseltechnologien

4TU.AMI

Contents

1 Perfect Christmas Pudding

Authors: Catriona Shearer, Ariane Beier (Matheon)

1.1 Challenge

The famous cooking gnome Oliver James made three of his delicious Christmas puddings for the elves' Christmas party.

The stuffy elf Symmetrix is pretty annoyed at the sight of the three puddings and notes grumpily: "Was is not possible for this bungler to produce three perfectly alike puddings of the same size?!"

Elf Geometria is however very delighted: "Look how beautifully arranged these Christmas puddings are: all three are shaped exactly like hemispheres. Each one of them is placed onto its flat side with its vault directed towards the ceiling. The lower two puddings sit side by side on a flat plane, and they touch at exactly one point. They are of different size; however, the upper pudding, which is placed onto the lower two, has exactly the right size such that its diameter conincides with the distance of the contact points of the lower puddings with their mutual tangent plane." See Figure 1.

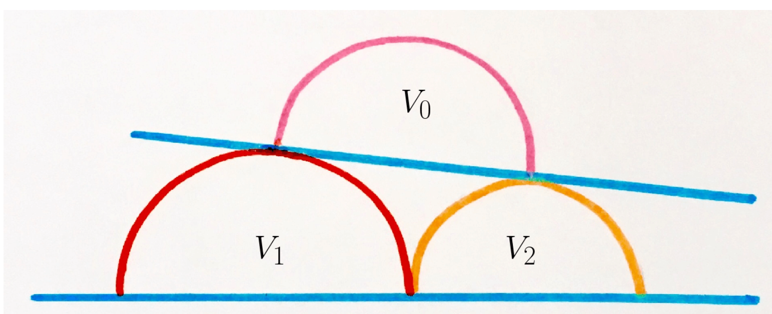


Figure 1: The diagram shows the cross section of the three Christmas puddings.

We want to know: If the bigger of the two lower Christmas puddings is of volume $V_1 = 15\,l$, and the smaller one is of volume $V_2 = 10\,l$, what is the first digit of the fractional part of the volume V_0 in l of the upper Christmas pudding?

Artwork: Frauke Jansen

Possible answers:

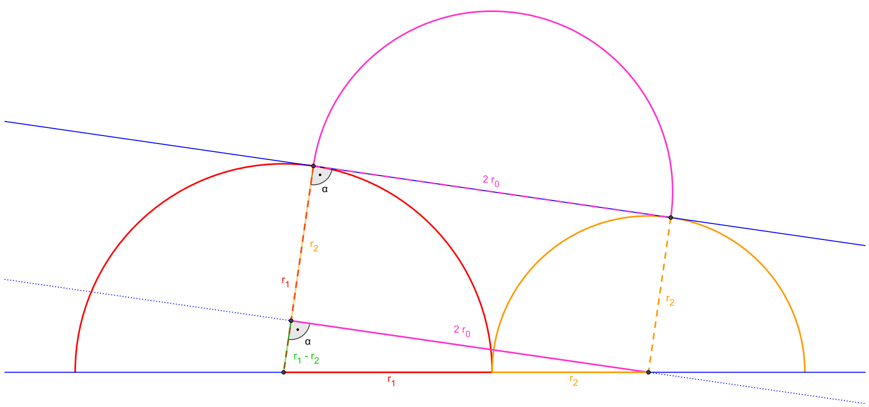
1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 0

About the author:

Catriona Shearer is a math teacher at a secondary school in Cambridge, UK. As `@Cshearer41`, she posts a new geometric puzzle every day on Twitter. Warning: Highly addictive!

1.2 Solution

The correct answer is: 2.



Let r_1 be the radius of the bigger of the lower hemispheres K_1 with volume $V_1 = 15\,l$, and let r_2 be the radius of the smaller hemisphere K_2 with volume $V_2 = 10\,l$. We want to determine the volume V_0 of the upper hemisphere K_0.

Since the diameter of the upper hemisphere K_0 coincides with the distance of the contact points of lower hemispheres, K_1 and K_2, with their mututal tangent plane, we have $\alpha = 90°$. Employing *Pythagorean theorem*, we deduce r_0:

$$\begin{aligned}(r_1 - r_2)^2 + (2r_0)^2 &= (r_1 + r_2)^2,\\ r_1^2 - 2r_1r_2 + r^2 + 4r_0^2 &= r_1^2 + 2r_1r_2 + r_2^2,\\ 4r_0^2 &= 4r_1r_2,\\ r_0 &= \sqrt{r_1r_2},\end{aligned}$$

i. e. the radius of the upper hemisphere equals the *geometric mean* of the radii of the lower two hemispheres.

Furthermore, we observe that $V_1 = \frac{2}{3}\pi r_1^3$, $V_2 = \frac{2}{3}\pi r_2^3$, and

$$\begin{aligned} V_0 &= \frac{2}{3}\pi r_0^3 \\ &= \frac{2}{3}\pi \sqrt{r_1 r_2}^3 \\ &= \sqrt{\frac{2}{3}\pi r_1^3 \cdot \frac{2}{3}\pi r_2^3} \\ &= \sqrt{V_1 V_2}. \end{aligned}$$

Again, the volume of the upper hemisphere is the geometric mean of the volumes of the lower hemispheres. Thus,

$$V_0 = \sqrt{V_1 V_2} = \sqrt{15 \cdot 10}\, l = 5\sqrt{6}\, l \approx 12, \mathbf{2}47\, l$$

2 Apple Wine

Authors: Aart Blokhuis (TU Eindhoven), Cor Hurkens (TU Eindhoven)

2.1 Challenge

Ruprecht has six indistinguishable sealed bottles in his wine cellar. Three of these bottles contain delicious apple wine, whereas the other three bottles contain highly toxic datura juice. The MAGICAL APPLE WINE TESTING MACHINE (MAWTM) has three compartments, a huge red button, and a light bulb. If Ruprecht puts one bottle into each of the compartments and then presses the red button, the MAWTM wakes up and starts to work. After one hour, the bulb lights up in green or in red: If the bulb lights up in green, then all three bottles contain delicious apple wine. If the bulb lights up in red, then at most two of the three bottles do contain apple wine.

Ruprecht wants to give a bottle with apple wine to Santa Clause. How often does he have to use the MAWTM (in the worst case) in order to identify a bottle with apple wine?

Artwork: Friederike Hofmann

Possible answers:

1. Eight times.
2. Nine times.
3. Ten times.
4. Eleven times.
5. Twelve times.
6. Thirteen times.
7. Sixteen times.
8. Eighteen times.
9. Nineteen times.
10. Twenty times.

2.2 Solution

The correct answer is: 3.

Ruprecht puts a bottle aside and tests every group of three bottles of the remaining five. If the bulb lights up green at one of the combinations, than he found three bottles of apple wine. If all combinations result in a red light, than the bottle he put aside in the beginning is apple wine. Thus, Ruprecht is able to identify one bottle with at most **ten** tests.

Now, take a further look at an arbitrary testing strategy. Assume that the first **nine** tests with MAWTM were negative, then Ruprecht cannot be sure if one fixed bottle F contains apple wine: besides F there are five other bottles. Out of the ten possible triplets of these five bottles, at most nine have been tested. Thus, it is possible that such an untested triplet would have a positive test result. In this case, these three bottles contain apple wine, and the bottle F contains toxic datura juice. In conclusion, nine tests are (in the worst case) not sufficient.

Summarized: Ruprecht has to do **ten** tests (in the worst case) to identify a bottle of apple wine.

3 Not-so-secret Santa

Author: Jacques Resing (TU Eindhoven)

3.1 Challenge

The five elves Armin, Bruno, Carlo, David, and Erich want to give presents to each other. Each elf receives exactly one present, and each elf has to give exactly one present. In order to determine who is going to give a present to whom, the five elves position themselves in a circle. Then they each point (randomly and simultaneously) at one of the other four elves. If no two elves point at the same elf, then everybody gives his present to the elf he is pointing at. Otherwise the procedure will be repeated.

Let p denote the probability that the procedure is executed only once. Which of the following statements holds true for p?

Artwork: Frauke Jansen

Possible answers:

1. $p \leq 0.002$
2. $0.002 < p \leq 0.004$
3. $0.004 < p \leq 0.008$
4. $0.008 < p \leq 0.016$
5. $0.016 < p \leq 0.032$
6. $0.032 < p \leq 0.064$
7. $0.064 < p \leq 0.128$
8. $0.128 < p \leq 0.256$
9. $0.256 < p \leq 0.512$
10. $0.512 < p$

3.2 Solution

The correct answer is: 6.

If there are no two elves that are pointing at the same elf, the elves form either a circle of five

$$X_1 \to X_2 \to X_3 \to X_4 \to X_5 \to X_1$$

or a combination of a circle of two

$$Y_1 \to Y_2 \to Y_1$$

and a circle of three

$$Y_3 \to Y_4 \to Y_5 \to Y_3.$$

Now, we calculate the number of possibilities for both cases:

1) The elves form a circle of five $X_1 \to X_2 \to X_3 \to X_4 \to X_5 \to X_1$:

Because of symmetry, one can choose Armin as X_1. Than, there are 24 possibilities left to arrange Bruno, Carlo, David, Erich as X_2, X_3, X_4, X_5.

2) The elves form a circle of two $Y_1 \to Y_2 \to Y_1$ and a circle of three $Y_3 \to Y_4 \to Y_5 \to Y_3$:

For the circle of two, one can choose two random elves, for which there are 10 possibilities. One can arrange the remaining three elves clockwise or counter-clockwise. In sum, there are $2 \cdot 10 = 20$ possibilities.

All in all there are $24 + 20 = 44$ possibilities for the case that no two of the elves are pointing at one. Since each one of the five elves points at one of the other four, the total number of possibilities is $4^5 = 1024$. Thus, the probability we are looking for is

$$p = 44/1024 \approx 0.043.$$

4 Sledge Test

Author: Hennie ter Morsche (TU Eindhoven)

4.1 Challenge

If you cross the Antarctic track from East to West, you will pass the six milestones A, B, C, D, E, F (in this order). The distance from milestone A to C is 116 miles; the distance from D to E is 126 miles, and the distance from E to F is 53 miles.

Today Santa's little helpers Yoda and Zeno have been testing their new sledges on the track. Both sledges were running the full distance at constant speed, and Yoda's sledge was slower than Zeno's sledge.

- Yoda traversed the track once from A to F.
- Zeno first drove from F to A; then he returned from A to F (without losing any time while turning the sledge at A).
- Yoda started in A at exactly the same time as Zeno started in F.
- When Yoda reached B, Zeno reached E on his way from F to A.
- Yoda and Zeno met each other at C.
- When Zeno was driving back from A to F, he overtook Yoda at milestone D.

What is the distance between the two milestones B and E?

Artwork: Rike Hofmann

Possible answers:

1. Roughly 351 miles.
2. Roughly 352 miles.
3. Roughly 353 miles.
4. Roughly 354 miles.
5. Roughly 355 miles.
6. Roughly 356 miles.
7. Roughly 357 miles.
8. Roughly 358 miles.
9. Roughly 359 miles.
10. Roughly 360 miles.

4.2 Solution

The correct answer is: 8.

Assume that Zeno rides exactly q-times as fast as Yoda. Because Yoda rides from A to B in the same time as Zeno rides from E to F, one has

$$|AB| \;=\; 53/q. \tag{1}$$

Because Yoda needs the same time for AC as Zeno needs for CF, one has $|CF| = 116q$. We conclude that

$$|CD| \;=\; |CF| - |DE| - |EF| \;=\; 116q - 179. \tag{2}$$

Because Yoda rides from C to D in the same time as Zeno rides from A to C and A to D, it holds $q|CD| = |AC| + |AD| = 2|AC| + |CD|$. With (2), we yield

$$\begin{aligned} 232 &= 2|AC| \\ &= (q-1)|CD| \\ &= (q-1)(116q-179) \\ &= 116q^2 - 295q + 179. \end{aligned} \tag{3}$$

One can write (3) as $116q^2 - 295q - 53 = 0$ or as

$$116q - \frac{53}{q} \;=\; 295.$$

Finally, we obtain

$$\begin{aligned} |BE| &= |AC| + |CF| - |AB| - |EF| \\ &= 116 + 116q - \frac{53}{q} - 53 \\ &= 116 + 295 - 53 \\ &= 358. \end{aligned}$$

If someone wants to know everything about the Antarctic track and the sledge test: One can easily calculate that

$$q = (295 + \sqrt{111617})/232 \approx 2.71.$$

Accordingly, the remaining distances between the milestones are

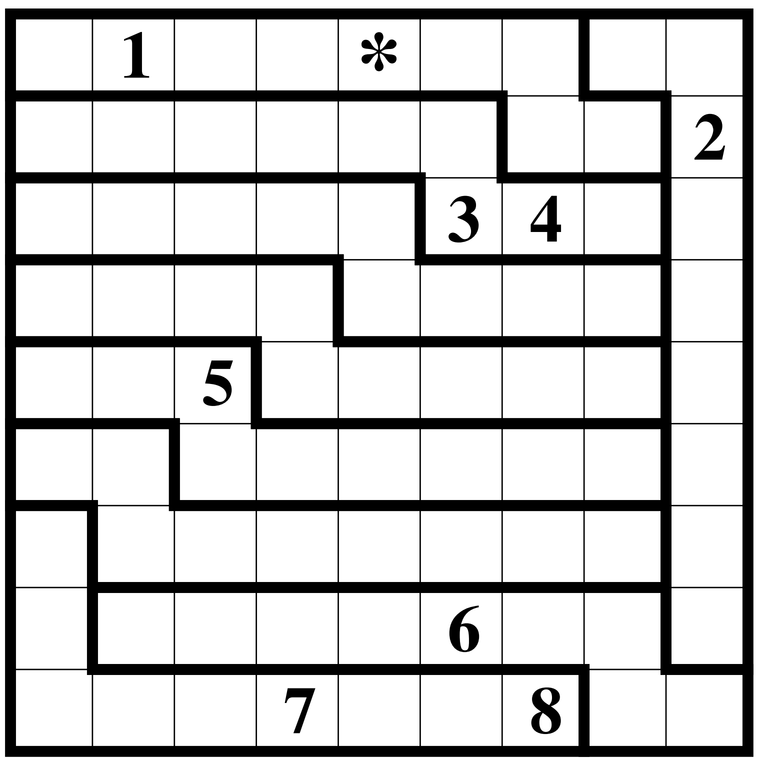
$$|AB| = 12296/(295 + \sqrt{111617}) \approx 19.55,$$
$$|BC| = (527 - \sqrt{111617})/2 \approx 96.45,$$
$$|CD| = (-63 + \sqrt{111617})/2 \approx 135.55.$$

5 Kudosu

Author: Cor Hurkens (TU Eindhoven)

5.1 Challenge

Kudosu is a variant of the well-known Sudoku puzzle. The 81 cells in the following 9×9 scheme are to be filled with the digits $1, 2 \ldots, 9$ so that each row and each column contains each of these nine digits exactly once. Furthermore, each of the nine regions (framed with bold lines) should contain each of these nine digits exactly once.



We would like to know: Which digit is going to show up in the cell marked by a star?

Artwork: Friederike Hofmann

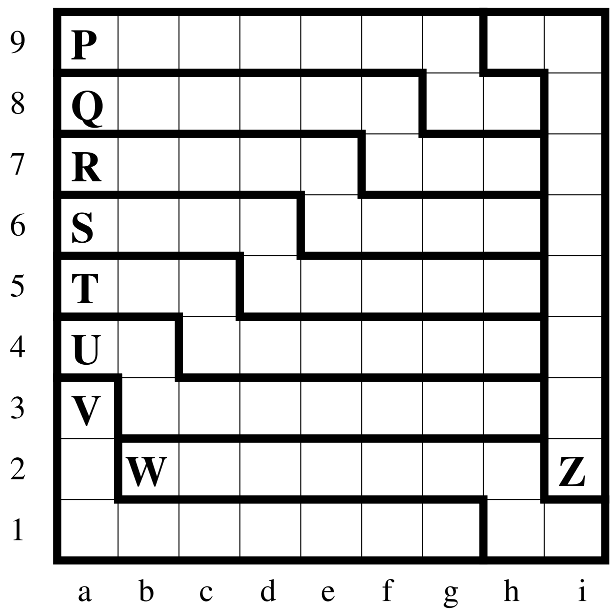
Possible answers:

1. The cell marked by a star contains the digit 1.
2. The cell marked by a star contains the digit 2.
3. The cell marked by a star contains the digit 3.
4. The cell marked by a star contains the digit 4.
5. The cell marked by a star contains the digit 5.
6. The cell marked by a star contains the digit 6.
7. The cell marked by a star contains the digit 7.
8. The cell marked by a star contains the digit 8.
9. The cell marked by a star contains the digit 9.
10. The contents of the cell marked by a star is not determined uniquely.

5.2 Solution

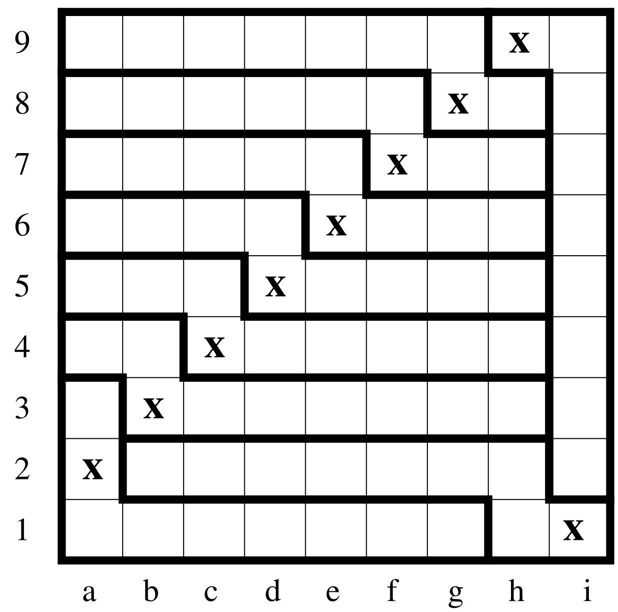
The correct answer is: 9.

We number the rows with 1 to 9 and the columns with a to i and name the nine regions $P, Q, R, S, T, U, V, W, Z$, as shown in the following picture:

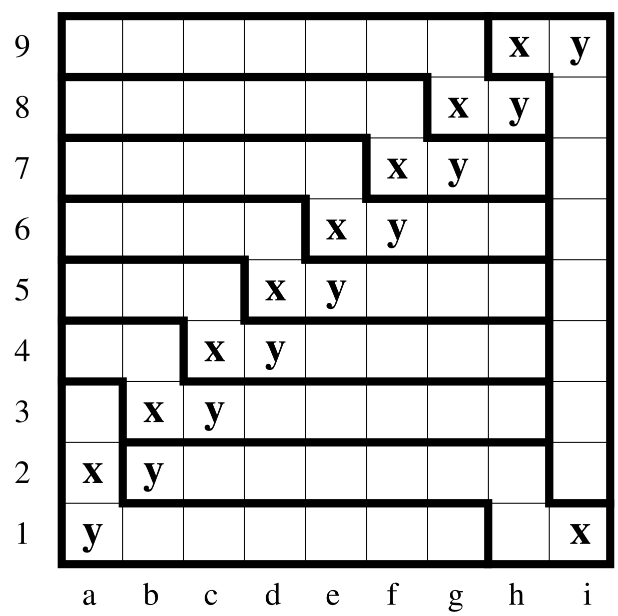


Now, we examine the entry x in the cell $i1$ (which is part of the region W). This entry has to appear in region Z too: Since x appears just once in the column i, the cell $h9$ has to contain x. The entry x has to appear also in region P: One easily notices that it has to be in the cell $g8$. Afterwards, one can work through the diagonal:

- In region Q the entry x is in the cell $f7$,
- in region R in the cell $e6$,
- in region S in the cell $d5$,
- in region T in the cell $c4$,
- in region U in the cell $b3$, and
- in region V in the cell $a2$:



Next, we examine the entry y in the cell $i9$. This entry has to appear also in region P, where its only possible location is $h8$. By similar reasoning, we deduce that the seven cells $g7$, $f6$, $e5$, $d4$, $c3$, $b2$, $a1$ also contain the entry y:



Analogous arguments along other diagonals show that every entry has to match the entry directly above right. Hence, one obtains the following unique solution of the Kudosu:

9	**2**	**1**	**7**	**6**	**9**	**8**	**5**	**3**	**4**
8	**1**	**7**	**6**	**9**	**8**	**5**	**3**	**4**	**2**
7	**7**	**6**	**9**	**8**	**5**	**3**	**4**	**2**	**1**
6	**6**	**9**	**8**	**5**	**3**	**4**	**2**	**1**	**7**
5	**9**	**8**	**5**	**3**	**4**	**2**	**1**	**7**	**6**
4	**8**	**5**	**3**	**4**	**2**	**1**	**7**	**6**	**9**
3	**5**	**3**	**4**	**2**	**1**	**7**	**6**	**9**	**8**
2	**3**	**4**	**2**	**1**	**7**	**6**	**9**	**8**	**5**
1	**4**	**2**	**1**	**7**	**6**	**9**	**8**	**5**	**3**
	a	b	c	d	e	f	g	h	i

6 The House of Saint Nicholas

Authors: Anna Maria Hartkopf (FU Berlin), Robert Wöstenfeld (Mathe im Leben)
Projects: www.polytopia.eu, SFB/Transregio 109 – Discretization in Geometry and Dynamics, Mathe im Advent
Translation: Ariane Beier (MATHEON)

6.1 Challenge

As every year on December 6th, this morning Saint Nicholas has filled the cleaned boots of the good children with sweets, fruit, and little toys. However, on the remaining days of the year, Saint Nicholas has not much to do. Then, he often sits at home and checks the news from Gnome Village with his Gnomebook app. As he does not move around much, he gained a lot of weight throughout the last years and his belly circumference is almost twice as big now. In consequence, the *House of Saint Nicholas* is simply to flat; he almost does not fit inside anymore.

He calls for gnome Friedensreich Tausendsassa, the most famous architect in Gnome Village, explains his situation, and asks for a solution. Indeed, Friedensreich has an idea: “Why does your home need to be flat? You could build a three-dimensional house. Then, you would have much more space inside.”

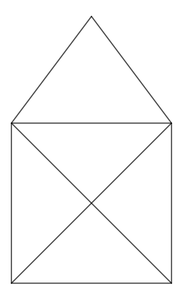


Figure 2: House of Saint Nicholas

“A three-dimensional house? That sounds quite strange to me...”, Saint

Nicholas murmurs into his thick white beard. He is sceptical: “My current house is a very peculiar one: You can draw it on a peace of paper with only one continuous stroke—without stopping in between and without painting any edge more than once (see Figure 2). Clearly, the new three-dimensional (!) house has to fulfil this condition too.” Friedenreich is positive: “Of course, this is definitely possible. I will draw some drafts and you can decide if one of them suits you.”

Actually, Friedensreich is not quite sure if he can fulfil Saint Nicholas’s wish. So he returns to his studio and works out some drafts. He puts together the four nets (see drafts A, B, C, D in Figure **??**) and obtains the according three-dimensional models. Afterwards, he tries to trace the edges of each model with only one continuous stroke—without stopping in between and without painting any edge more than once.

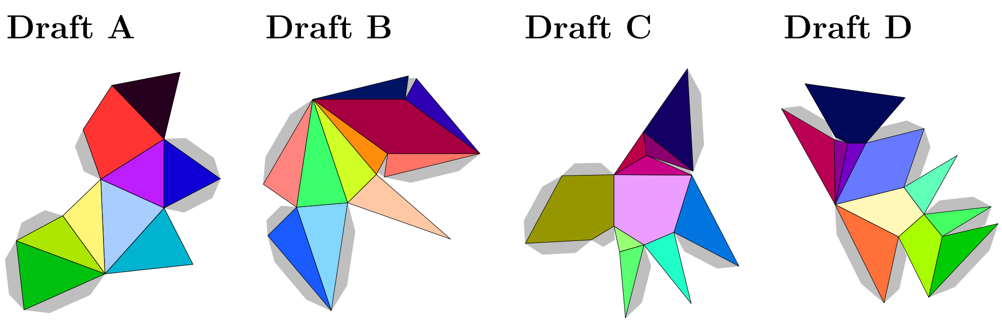


Figure 3: Drafts A to D. You will find the working material for each model on the following pages.

But for which of the four models, given by the nets A to D in Figure 3, is this possible?

Possible answers:

1. This is not possible for any of the four given models.
2. This is only possible for model A and B.
3. This is only possible for model A and D.
4. This is only possible for model B and C.
5. This is only possible for model B and D.
6. This is only possible for model C and D.
7. This is only possible for model A, B and C.
8. This is only possible for model A, C and D.
9. This is only possible for model B, C and D.
10. It is possible for all four models.

Project reference:

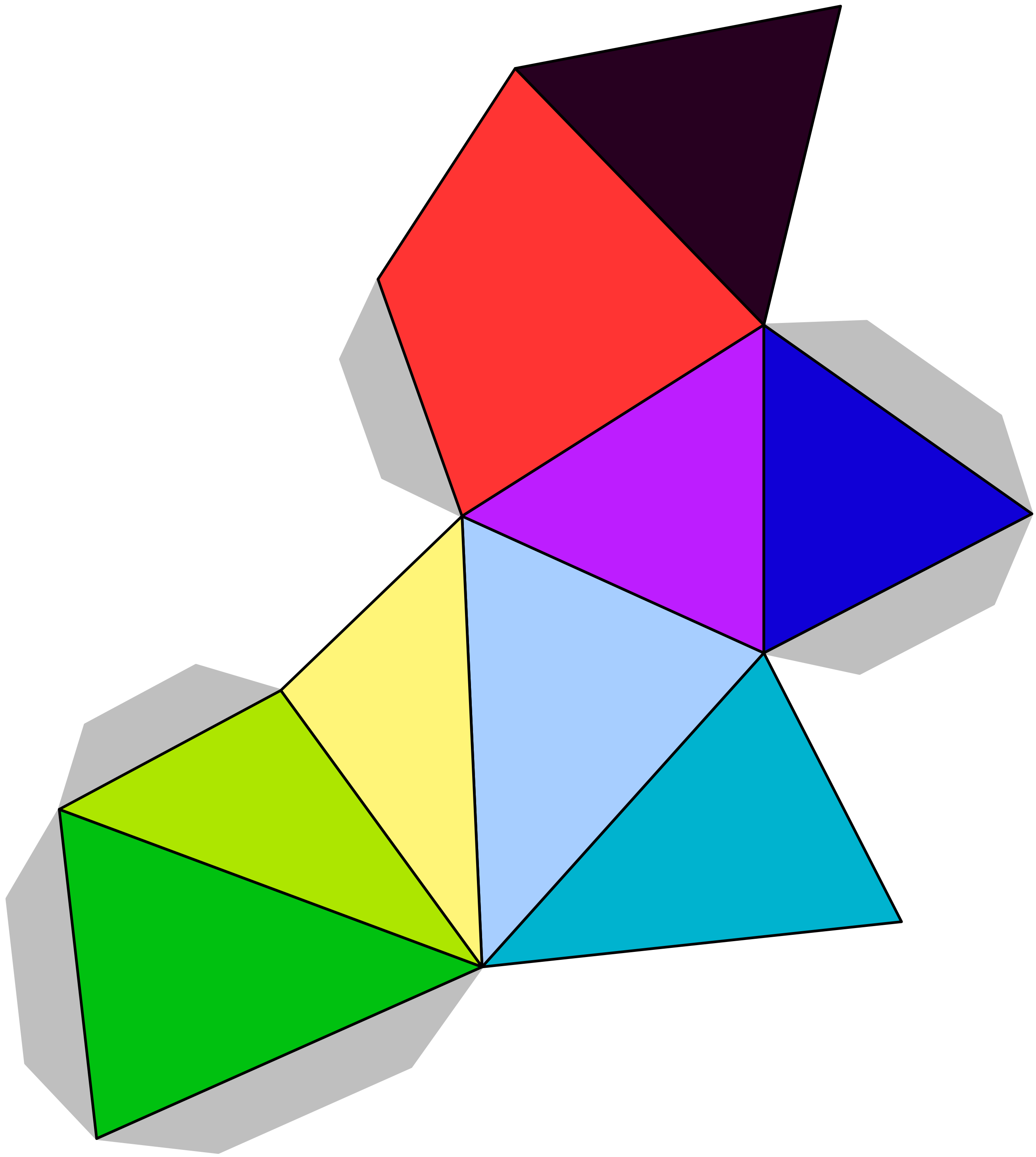
The geometry project *Adopt a Polyhedron* provides an exciting insight into the world of polyhedra. Everybody is invited to name, adopt, and craft a polyhedron. The promotion is free of charge for all participants.

Pharaoh's tombs, origami figures, and 3D objects in film animations and computer games differ only little from a geometric point of view. They occur—mathematically speaking—in the form of so-called polyhedra. Mathematicians describe polyhedra as a piece of space bounded by flat surfaces. The best known polyhedra are the Platonic solids, including, e. g., the cube. For thousands of years and until today, polyhedra and their relatives have been the subject of mathematical research.

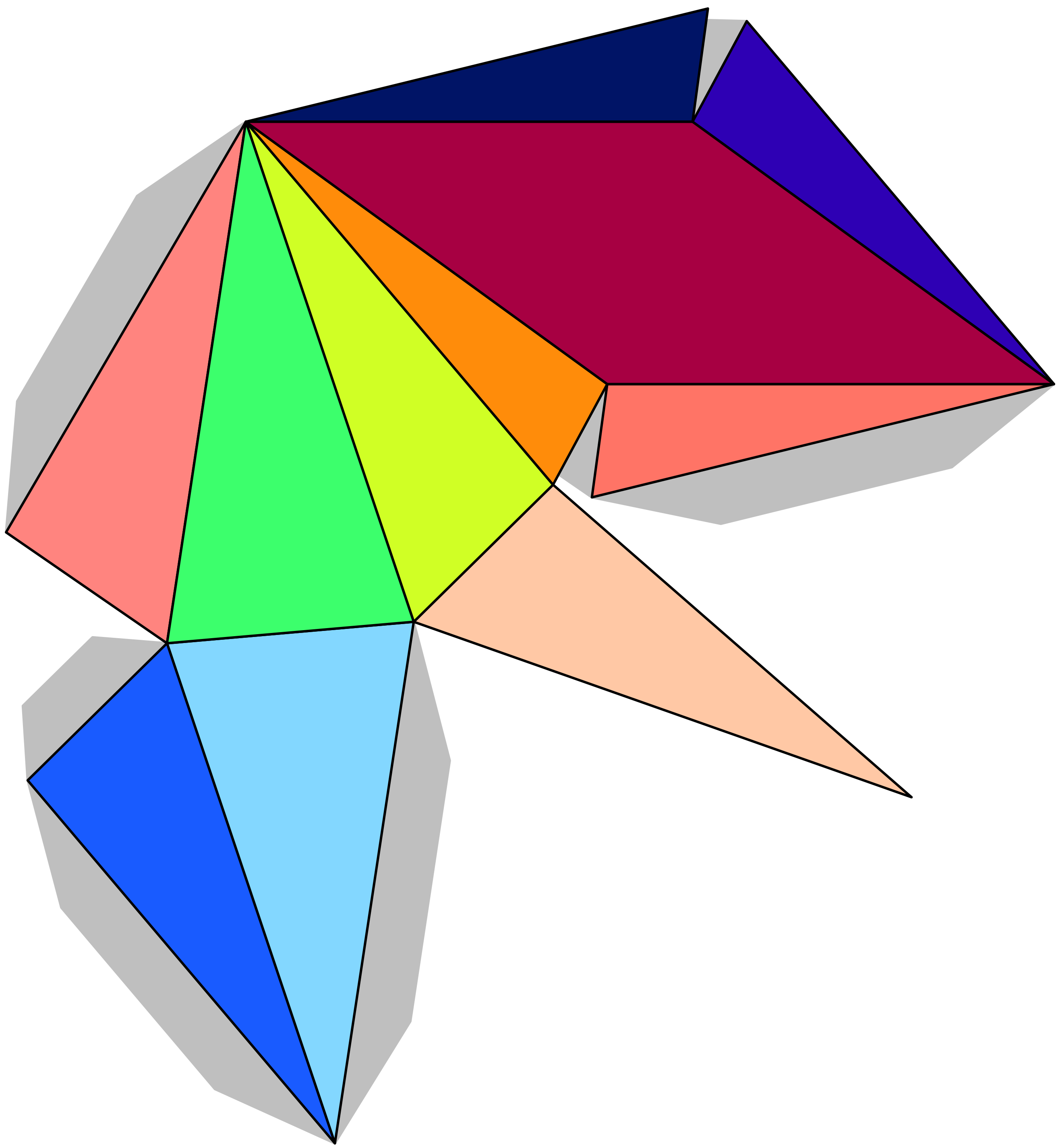
The project *Adopt a polyhedron* ranks at the interface between research and science communication. It aims to make the inexhaustible wealth of forms of polyhedra accessible to everybody. With `www.polytopia.eu`, Berlin mathematicians have created an adoption network. Participants can take on a symbolic sponsorship for a polyhedron of their choice. The adoption process takes place online with just a few mouse clicks and is free of charge for the users. Anyone who adopts a polyhedron at `www.polytopia.eu` can name it and bring it to life. There is a lot of information on our website. If you want, you can print out an individual crafting sheet at `www.polytopia.eu` and have your own polyhedron physically shaped as well. All math enthusiasts are invited to join the project and adopt a polyhedron!

The project is presented by the Collaborative Research Center (CRC) *Discretization in Geometry and Dynamics*, which consists of Berlin and Munich mathematicians as a so-called "Transregio". The CRC/Transregio 109 *Discretization in Geometry and Dynamics* is funded by the *Deutsche Forschungsgemeinschaft* (DFG) and is realized at Technische Universität Berlin and Technische Universität München. The Freie Universität Berlin is also affiliated. `www.polytopia.eu` is additionally supported by the German Mathematical Society.

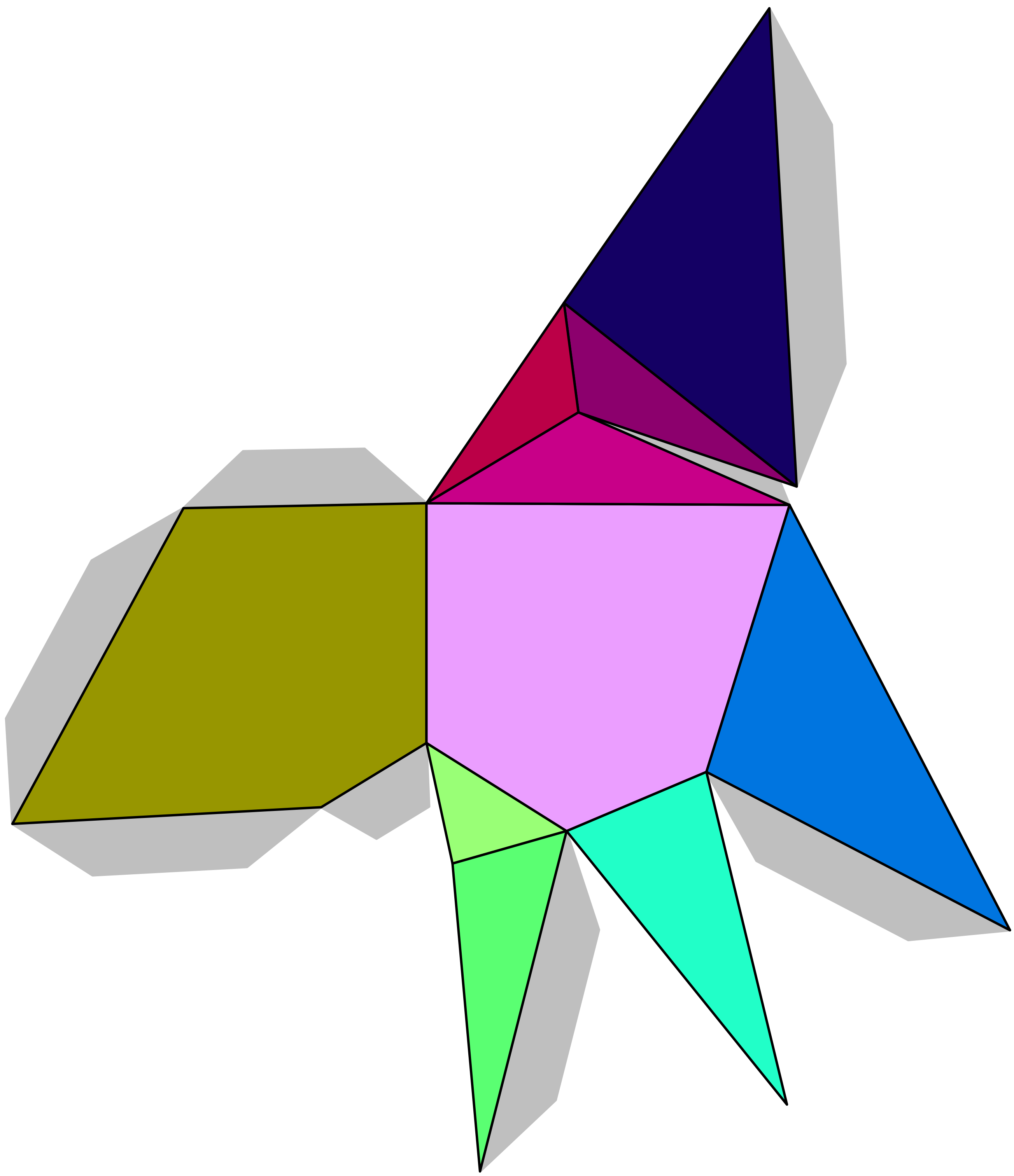
Working material for draft A



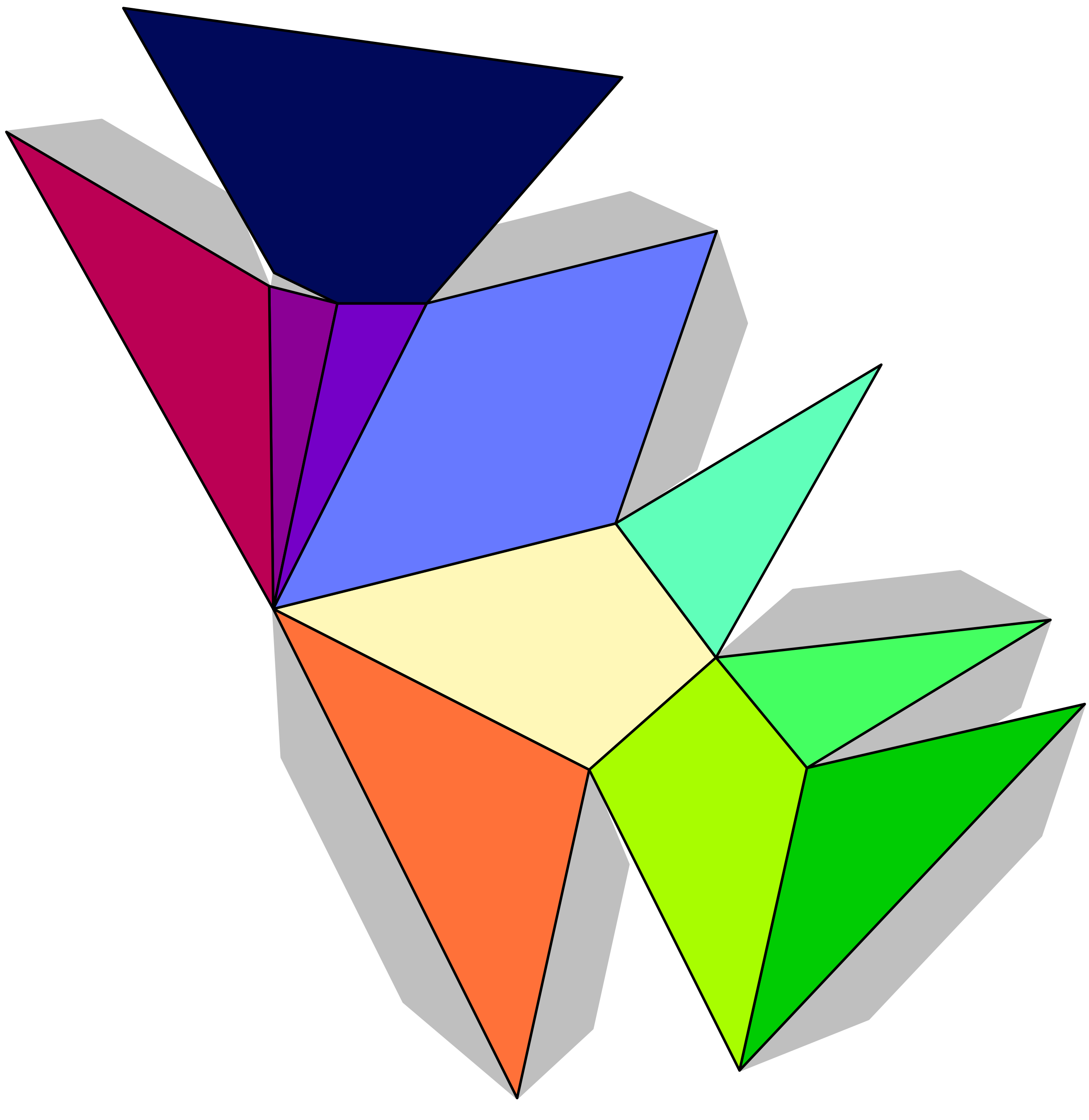
Working material for draft B



Working material for draft C



Working material for draft D



6.2 Solution

The correct answer is: 2.

Mathematically speaking, this challenge is a problem from *Graph Theory*. A *graph* is a structure consisting of *vertices* (also called *nodes*) and *edges* that connect two vertices. Two examples of a graph is depicted in Figure 4. The graph on the left is *finite*, because it consists of only finitely many vertices (A to F), and it is *connected*, because you will find a connection of any two vertices by consecutive edges, a so-called *path*. In the example above, one can connect the vertices A and F through the path e, h.

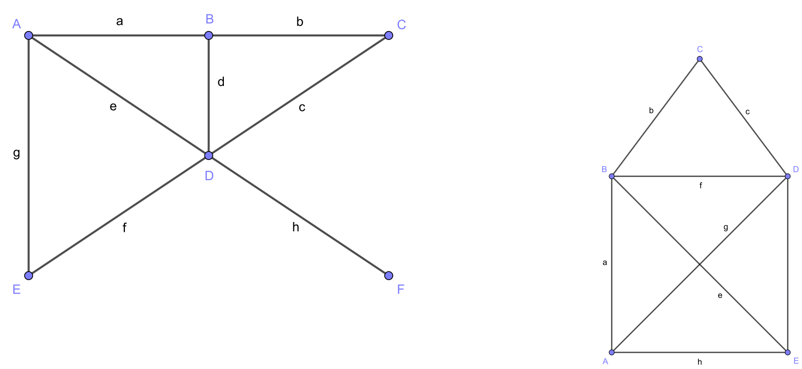


Figure 4: Left: A graph with six vertices (A to F) and eight edges(a to h). Right: The House of Saint Nicholas is a graph with five vertices (A to E) and eight edges (a to h).

The planar *House of Saint Nicholas* is also a graph (see Figure 4, right). Here, there is a path that contains the eight edges of the graph exactly once, e. g. the path a, b, c, d, e, f, g, h.

Such a path that contains every edge of a graph exactly once is called *Euler path*. Thus, we are looking for euler paths in our three-dimensional models A to D.

We will utilise the following theorem in order to solve our problem.

Theorem: On a finite connected graph there exists an Euler path if and only if none or exactly two of its vertices are of odd degree. Here, the *degree* of a vertex is the number of edges adjacent to this vertex.

For example, the vertex B in the House of saint Nicholas has degree 4, since exactly 4 edges (a, b, e, and f) are adjacent to it. Therefore, the degree of B is even. In the House of Saint Nicholas, solely the vertices A and E are of odd degree (namely, 3). Thus, the prerequisites of the above theorem are satisfied, and we already knew that there indeed exists an Euler path in the House of saint Nicholas.

We will now apply the above theorem to the models A to D:

Model A: If one builds model A from the given net, it looks similar to Figure 5. We count that there are three edges adjacent to each of the vertices A and B and four edges adjacent to each of the remaining vertices. Thus, vertices A and B are of odd degree, and the remaining ones are of even degree. We conclude from the above theorem that there exists an Euler path on model A. If you would like to see model A from different angles, have a look at the Polytopia webpage:

$$\texttt{https}: //\texttt{www.polytopia.eu/detailansicht?id} = \texttt{700032}$$

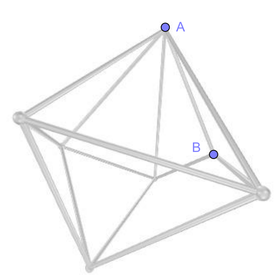


Figure 5: Model A

Model B: If one builds model B from the given net, it looks similar to Figure 6. We count that there are three edges adjacent to each of the vertices A and B and either four or six edges adjacent to each of the remaining vertices. Thus, vertices A and B are of odd degree, and the remaining ones are of even degree. We conclude from the above theorem that there exists an Euler path on model B. If you would like to see model B from different angles, have a look at the Polytopia webpage:

$$\texttt{https}://\texttt{www.polytopia.eu/detailansicht?id} = 800190$$

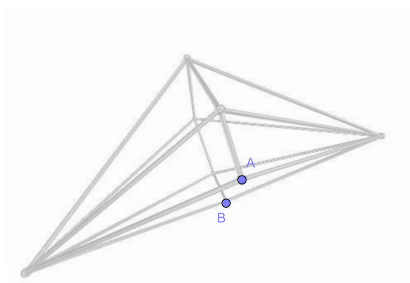


Figure 6: Model B

Model C: If one builds model C from the given net, it looks similar to Figure 7. We count that there are three edges adjacent to each of the four vertices A to D and five edges adjacent to the vertices E and F. Thus, there are six edges of odd degree in model C. We conclude from the above theorem that there is no Euler path on model C. If you would like to see model C from different angles, have a look at the Polytopia webpage:

$$\texttt{https}://\texttt{www.polytopia.eu/detailansicht?id} = 901265$$

Model D: If one builds model D from the given net, it looks similar to Figure 8. We count that there are three edges adjacent to each of the six vertices A to F and five edges adjacent to vertices G and H. Thus, there are eight edges of odd degree in model D. We conclude from the above theorem that there is no Euler path on model D. If you would like to see model D from different angles, have a look at the Polytopia webpage:

$$\texttt{https}://\texttt{www.polytopia.eu/detailansicht?id} = 1000119$$

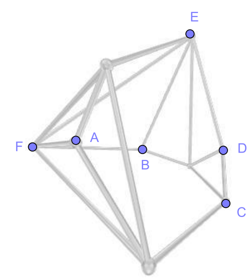


Figure 7: Model C

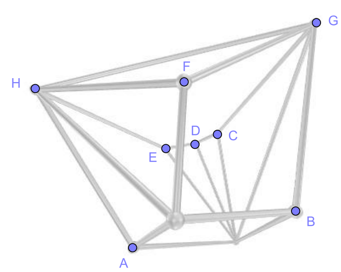


Figure 8: Model D

7 Rabbits

Authors: Cor Hurkens (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

7.1 Challenge

Ruprecht has divided his large square rabbit compound (with an edge length of 28 m) with to horizontal and three vertical fences, resulting in a total of twelve smaller rectangular compounds. The following illustration depicts the area (in square meters) of some of these smaller compounds:

	28		**F**
70		**35**	
14			**21**

Attention: The illustration **only** depicts the area of the compounds. Obviously, it is **not true to scale!**

Furthermore, Ruprecht measured the area of the compound in the upper right corner and therefore knows that its area F is less than 100 square meters.

We want to know: What is the area F of this compound?

Illustration: Julia Nurit Schönnagel

Possible answers:

1. One has $F \approx 7\,\mathrm{m}^2$.
2. One has $F \approx 13\,\mathrm{m}^2$.
3. One has $F \approx 28\,\mathrm{m}^2$.
4. One has $F \approx 34\,\mathrm{m}^2$.
5. One has $F \approx 49\,\mathrm{m}^2$.
6. One has $F \approx 51\,\mathrm{m}^2$.
7. One has $F \approx 63\,\mathrm{m}^2$.
8. One has $F \approx 76\,\mathrm{m}^2$.
9. One has $F \approx 84\,\mathrm{m}^2$.
10. One has $F \approx 98\,\mathrm{m}^2$.

7.2 Solution

The correct answer is: 1.

We denote the lengths and widths of the twelve small enclosures as depicted in the following figure by x_1, x_2, x_3, x_4 and y_1, y_2, y_3.

y_3		**28**		**F**
y_2	**70**		**35**	
y_1	**14**			**21**
	x_1	x_2	x_3	x_4

Because the whole enclosure is a square with the length $28\,m$, we obtain the two equations

$$x_1 + x_2 + x_3 + x_4 = 28 \qquad \text{and} \qquad y_1 + y_2 + y_3 = 28. \tag{4}$$

The five given areas lead to

$$x_1\, y_1 = 14, \quad x_1\, y_2 = 70, \quad x_2\, y_3 = 28, \quad x_3\, y_2 = 35, \quad x_4\, y_1 = 21. \tag{5}$$

The equations $x_1\, y_1 = 14$ and $x_1\, y_2 = 70$ result in $y_2 = 5y_1$.

By virtue of (5) and $y_2 = 5y_1$, we can now express the four numbers x_1, x_2, x_3, x_4 in terms of y_1 and y_3:

$$x_1 = \frac{14}{y_1}, \qquad x_2 = \frac{28}{y_3}, \qquad x_3 = \frac{35}{y_2} = \frac{7}{y_1}, \qquad x_4 = \frac{21}{y_1}. \tag{6}$$

From the equations in (4) and (6), we deduce

$$\frac{14}{y_1} + \frac{28}{y_3} + \frac{7}{y_1} + \frac{21}{y_1} = 28 \qquad \text{and} \qquad y_1 + 5y_1 + y_3 = 28.$$

We simplify them and obtain

$$\frac{3}{y_1} + \frac{2}{y_3} = 2 \qquad \text{and} \qquad 6y_1 + y_3 = 28.$$

This leads to $y_3 = 28 - 6y_1$ and $\frac{3}{y_1} + \frac{2}{28-6y_1} = 2$, which is equivalent to

$$y_1^2 - 6y_1 + 7 = 0. \tag{7}$$

With the quadratic equation (7), we get

$$y_1 = 3 \pm \sqrt{2}.$$

By using this formula, one can now determine the other variables:

y_1	y_2	y_3	x_1	x_2	x_3	x_4
$3-\sqrt{2}$	$15-5\sqrt{2}$	$10+6\sqrt{2}$	$6+2\sqrt{2}$	$10-6\sqrt{2}$	$3+\sqrt{2}$	$9+3\sqrt{2}$
$3+\sqrt{2}$	$15+5\sqrt{2}$	$10-6\sqrt{2}$	$6-2\sqrt{2}$	$10+6\sqrt{2}$	$3-\sqrt{2}$	$9-3\sqrt{2}$

The solution in the first line of the table leads to

$$F = x_4 y_3 = (10+6\sqrt{2})(9+3\sqrt{2}) = 126 + 84\sqrt{2} > 100,$$

which is a contradiction to Ruprechts statement. The solution in the second line of the table leads to

$$F = x_4 y_3 = (10-6\sqrt{2})(9-3\sqrt{2}) = 126 - 84\sqrt{2} \approx 7,20606.$$

8 Straight harmony

Author: Luise Fehlinger (HU Berlin)
Project: ZE-AP1 – Teachers at University
Translation: Ariane Beier (MATHEON)

8.1 Challenge

The leadlight window in Santa's workshop was destroyed. The window had the shape of a trapezium and was divided into two parts by a beam at its midsegment. The upper part was coloured red and the lower green (see Figure 9). Of course, the gnomes are eager to repair the broken window at once.

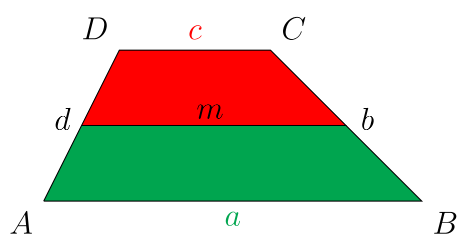


Figure 9: The window has the form of a trapezium with bases a and b, and legs b and d. The trapezium $ABCD$ is divided into two smaller trapezia by the parallel m.

However, Santa is quite stressed and thus requests a new partition that is more harmonic than the one before. He believes that, if one has to build a new window anyway, one can consider a new design which is much more pleasant:

The gnomes are to build a new beam m parallel to the bases such that the length of this beam m equals the harmonic mean of the bases a and c. The **harmonic mean** is defined to be the inverse of the arithmetic mean of the inverses. That is, the harmonic mean of a and c is given by

$$\frac{2}{a^{-1} + c^{-1}}.$$

Now, the gnomes are discussing how to realise Santa's demand.

Which of the following constructions does the trick?

Artwork: Sonja Rörig

Possible answers:

1. Marek says: Santa should not make such a fuss about the new window. The midline already has the required length.

2. Nadia suggests the following construction: Draw a circle of radius c around A. Denote the intersection of this circle with $\overline{AB}$ by E. Construct the perpendicular bisector of $\overline{AE}$. Denote an intersection of this bisector with the circle around B through A by F. Then, $\overline{AF}$ has the required length.

3. Ida is quite sure that the parallel beam has to be installed such that the areas of the red and green window are the same.

4. Jonas divides the height of the trapezium at the ratio of c (below) to a (above). There, he wants to install the parallel beam.

5. Hannah suggests to build the beam parallel to the bases through the barycentre of the trapezium.

6. Employing shearings, Rasmus constructs two rectangles from a square width edge length 1: one with edge lengths a and $\frac{1}{a}$, and one with edge lengths c and $\frac{1}{c}$. Afterwards, he consecutively marks the lengths $\frac{1}{a}$ and $\frac{1}{c}$ on a half-line, and bisects the resulting line segment. The thus constructed line segment has the requested length.

7. Lina wants to keep it short and simple: She wants to use the line parallel to the bases that runs through the intersection of the diagonals of the trapezium as the beam that separates the two parts of the window.

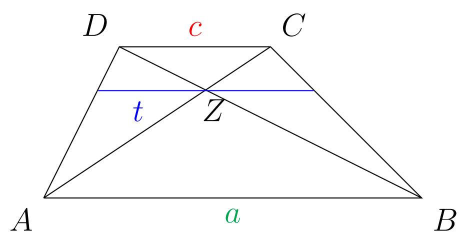
8. Jolanda draws the perpendicular through D onto the base a; then, the diagonal $\overline{AC}$. Through their intersection, she constructs the parallel to the bases.

9. Cornelius constructs the perpendicular bisectors of the legs b and d. Then, he draws the parallel to the bases through their intersection.

10. Milena is positive that Santa just wants to play tricks on the gnomes and that it is impossible to construct such a beam.

8.2 Solution

The correct answer is: 7.



We denote the intersection of the diagonals by Z and the parallel through Z to the bases by t. We utilise the intercept theorem for the line segments emerging from A through C and D and the parallel lines c and t. One has

$$\frac{c}{\frac{t}{2}} = \frac{|\overline{AC}|}{|\overline{AZ}|}.$$

For the similar triangles ΔZCD and ΔZAB emerging at Z, one has:

$$\begin{aligned}\frac{c}{a} &= \frac{|\overline{ZC}|}{|\overline{ZA}|}\\ &= \frac{|\overline{AC}| - |\overline{ZA}|}{|\overline{ZA}|}\\ &= \frac{|\overline{AC}|}{|\overline{ZA}|} - 1.\end{aligned}$$

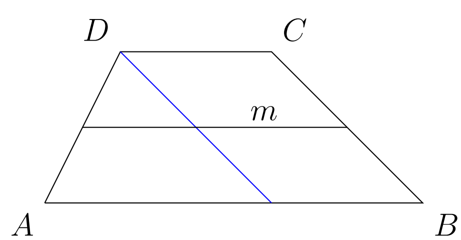
We solve this equation for $\frac{|\overline{AC}|}{|\overline{ZA}|}$, equate it with the first result, and deduce further:

$$\begin{aligned} & & \frac{c}{a}+1 &= \frac{c}{\frac{t}{2}} = \frac{2c}{t} \\ \Leftrightarrow & & t &= \frac{2c}{\frac{c}{a}+1} \\ \Leftrightarrow & & t &= \frac{2c}{\frac{c+a}{a}} \\ \Leftrightarrow & & t &= \frac{2}{\frac{c+a}{a\cdot c}} \\ \Leftrightarrow & & t &= \frac{2}{a^{-1}+c^{-1}} \end{aligned}$$

We conclude that Lina's suggestion yields the correct construction.

In what follows, we show that the other elves are not correct:

1. The length of the midline corresponds to the arithmetic mean of a and c. One way to show this fact is to construct the parallel to the leg $\overline{BC}$ through D and consider the intercept theorem for the similar triangle emerging at D. For a general trapezium, the arithmetic mean is bigger than the harmonic mean. (The proof remains an exercise!) Thus, Marek is not correct.

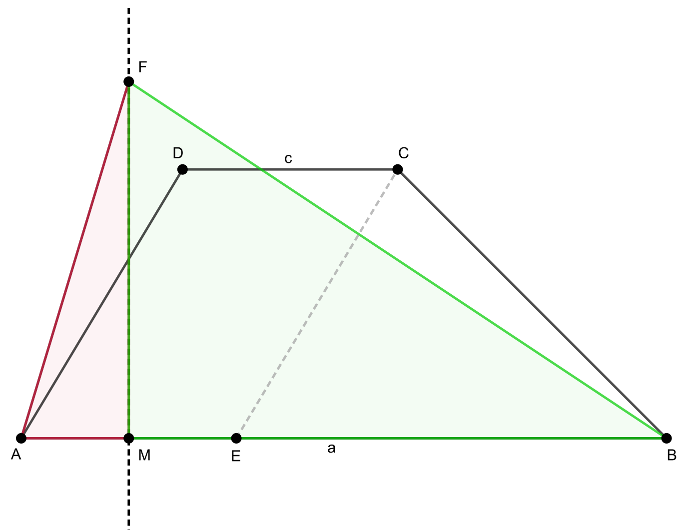


2. We denote the centre of $\overline{AE}$ by M, and one has
$$|\overline{BM}| = a - \frac{c}{2}.$$
The triangle ΔBFM has a right angle at M, and $|\overline{BF}| = a$. With the Pythagorean theorem one deduces
$$|\overline{MF}|^2 = a^2 - (a - \frac{c}{2})^2 = ac - \frac{c^2}{4}.$$

The triangle ΔAMF also has a right angle at M, and $|\overline{AM}| = \frac{c}{2}$. Again, we employ the Pythagorean theorem:

$$|\overline{AF}|^2 = \frac{c^2}{4} + \left(ac - \frac{c^2}{4}\right) = ac.$$

Thus, $|\overline{AF}| = \sqrt{ac}$, which is the geometric mean of a and c. However, the geometric mean is bigger than the harmonic mean in a general trapezium. (The proof is left as an exercise!) In consequence, Nadia is not correct.



3. If the areas of the upper and lower smaller trapezia were to coincide, the beam must lie closer to the longer base than to the shorter one. Thus, the beam would be longer than the midline and, in consequnce, too long. We conclude that Ida's construction is not the correct one.

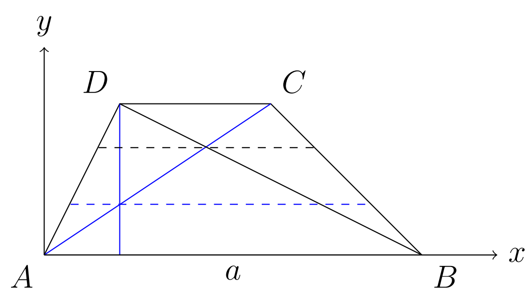
4. Since Jonas wants to align the shorter and longer section conversely to the shorter and longer base, the beam would again be longer than the arithmetic mean. Thus, this construction is not the correct one.

5. In a general trapezium, the barycentre is closer to the longer base than to the shorter one. Thus, the barycentre is located too low to satisfy Santa's demand.

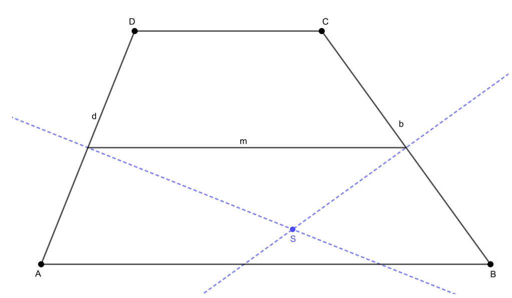
6. Rasmus's construction is a good start: he constructs the inverses and their arithmetic mean. Unfortunately, he forgets constructing the inverse of this number again. Thus, his solution is not correct. Besides, his construction is quite complicated.

7. As shown above, Lina's solution is correct.

8. If we draw a coordinate system such that the x-axis coincides with the base a with its origin at A, then we can regard the diagonal $\overline{AC}$ as a homogeneous linear function with positive slope. Since the x-coordinate of D is smaller than the x-coordinate of the intersection of the diagonals (which has, as we have shown, the right height), Jolanda's construction yields a beam that is located too low.



9. In a general trapezium, the intersection of the perpendicular bisectors is located below the midline.



10. Lina has already shown that it is indeed possible to accomplish such a construction.

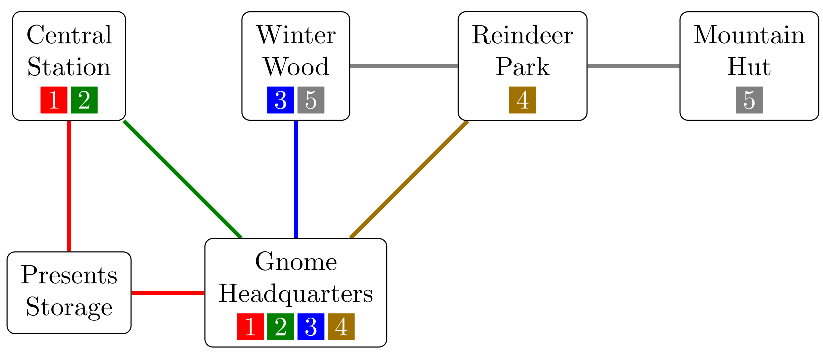
9 Timetable Trouble at the North Pole

Author: Niels Lindner
Project: ECMath MI7 – Routing Structures & Periodic Timetabling

9.1 Challenge

Santa Clause is not amused. Shortly before Christmas, the North Pole Transit Authority (NPTA) changed their timetables. Starting from his mountain hut, Santa needs now much more time to reach the central station, where the Polar Express departs.

The line network of NPTA is as follows:



The following table contains the departure times of the sleighs and their direction of travel. The timetable is repeated every 60 minutes.

Station	1		2		3		4		5	
Central Station	23 ↓	37 ↑	53 ↓	07 ↑						
Presents Storage	39 ↓	21 ↑								
Gnome Hq.	50 ↓	10 ↑	13 ↓	47 ↑	48 ↓	12 ↑	17 ↓	43 ↑		
Winter Wood					12 ↓	48 ↑			35 ↓	25 ↑
Reindeer Park							49 ↓	11 ↑	40 ↓	20 ↑
Mountain Hut									58 ↓	02 ↑

At intermediate stations, arrival and departure minutes are identical. No trip between two adjacent stations takes 60 minutes or more. A transfer is possible from one line to another line if the transfer time, i. e. the difference between departure and arrival time, is at least one minute. Transfers can only occur between two different lines. Passing through an intermediate station does not count as a transfer.

Which of the following statements is false?

Artwork: Sonja Rörig

Possible answers:

1. There are four different routes from Mountain Hut to Central Station with the property that no station is visited more than once.
2. Santa Claus needs at least 2 hours and 5 minutes for a trip from Mountain Hut to Central Station.
3. On the way back from Central Station to Mountain Hut, Santa spends in total at least 55 minutes waiting at stations.
4. Rudolph, the reindeer, can leave his sleigh at home: Using NPTA, he can make the trip from Reindeer Park to Central Station within less than an hour.

5. For every pair (A, B) of distinct stations holds: The shortest route from A to B takes exactly the same time as the shortest route from B to A.

6. Following Santa's request, Line 5 departs 15 minutes earlier in the direction of Winter Wood, so that at Reindeer Park, Santa can change comfortably to Line 4 towards Gnome Headquarters. If Santa departs at 09.47 from Mountain Hut, he is just in time for the Polar Express departing at 11.11 from Central Station.

7. However, the NPTA restores the timetable quickly: Due to departing 15 minutes earlier in the direction of Winter Wood, but not in the opposite direction, they had to put one more vehicle in operation on Line 5.

8. This upsets Santa again. He retreats to his hut and puzzles out a new timetable. The line stops and the travel times between station remain unchanged. Finally, he beams with joy: There is a timetable, where no single transfer at the stations Central Station, Gnome Headquarters and Winter Wood takes longer than 10 minutes.

9. Rudolph is thrilled by Santa's idea. He also constructs a timetable, where no transfer time at the stations Reindeer Park, Gnome Headquarters and Winter Wood takes longer than 10 minutes.

10. Finally, the NPTA agrees on a radical cut: Line 4 is discontinued without replacement. Then they install a timetable, where no transfer time at all is longer than 7 minutes.

Project reference:

The project *Routing Structures & Periodic Timetabling* deals with computing optimal timetables in public transit networks. This takes into account the impact a timetable has on the choice of passenger routes. Conversely, the different routes passengers can take are included into the timetable creation process. The main goal is to achieve shortest possible transfer times for all passengers.

9.2 Solution

The correct answer is: 9.

1. All routes from Mountain Hut to Central Station pass throguh Gnome Headquarters. From Mountain Hut to Gnome Headquarters, there are two distinct routes (Line 5 and either Line 3 or 4). From Gnome Headquarters to Central Station (using Line 1 or Line 2), there are two routes as well. In total, this adds up to four routes.

2. The travel times along the four possible routes are as follows:

		5 3 1		5 3 2	
Station		Minute	TTT	Minute	TTT
Mountain Hut	departure	02	0:00	02	0:00
Winter Wood	arrival	25	0:23	25	0:23
	departure	48	0:46	48	0:46
Gnome Headquarters	arrival	12	1:10	12	1:10
	departure	10	2:08	47	1:45
Central Station	arrival	37	2:35	07	2:05

		5 4 1		5 4 2	
Station		Minute	TTT	Minute	TTT
Mountain Hut	departure	02	0:00	02	0:00
Reindeer Park	arrival	20	0:18	20	0:18
	departure	11	1:09	11	1:09
Gnome Headquarters	arrival	43	1:41	43	1:41
	departure	10	2:08	47	1:45
Central Station	arrival	37	2:35	07	2:05

Minute: departure/arrival minute
TTT: total travel time (hours:minutes)

Hence, Santa Claus needs at least 2 hours and 5 minutes.

3. The following transfer times arise at the four possible routes:

Station	1 3 5	2 3 5	1 4 5	2 4 5
Gnome HQs	58	35	27	4
Winter Wood	23	23	–	–
Reindeer Park	–	–	51	51
Total	81	58	78	55

4. Using lines 4 and 2, Rudolph can make it in 56 minutes (see 2.)

5. The following observations will be of use: At each station, the departure minute a of a line in one direction and the arrival minute b of the same line in the opposite direction satisfy $a + b = 60$. Moreover, the travel times between two adjacent stations of a line are the same for both directions. Possible differences between the shortest travel times from A to B or from B to A, respectively, can only arise from transfers.

   At an arbitrary station, consider a transfer from line ℓ_1 to line ℓ_2 with transfer time $u_{1,2}$, and the reverse transfer from line ℓ_2 to line ℓ_1 with transfer time $u_{2,1}$. Because transfer times take at least one minute, we have $u_{1,2}, u_{2,1} \geq 1$. Since on a shortest route, it makes no sense to consider transfers longer than 60 minutes (otherwise, one could depart at least one hour earlier), we can also assume $u_{1,2}, u_{2,1} \leq 60$.

   If a_1 denotes the arrival minute of ℓ_1, the departure minute of ℓ_2 after changing is computed as either $a_1 + u_{1,2}$ or $a_1 + u_{1,2} - 60$, because departure minutes are between 0 and 59. In the opposite direction, if line ℓ_2 arrives at minute a_2, then ℓ_1 will depart either at $a_2 + u_{2,1}$ or $a_2 + u_{2,1} - 60$. With the above observation, $a_1 + a_2 + u_{1,2} \in \{60, 120\}$ and $a_1 + a_2 + u_{2,1} \in \{60, 120\}$. In any case, $u_{1,2} - u_{2,1}$ is an integer multiple of 60. As $u_{1,2} - u_{2,1}$ is at most $60 - 1 = 59$ and at least $1 - 60 = -59$, necessarily $u_{1,2} = u_{2,1}$ holds.

   This means that both driving times and transfer times are the same in both directions. In particular, the shortest route from A to B takes as long as the shortest route from B to A.

6. If Line 5 departs 15 minutes earlier in the direction of Winter Wood, a departure at 09.47 at Mountain Hut leads to an arrival at 10.05 at Reeindeer Park. Then Santa changes to Line 4, which departs at 10.11 and arrives at 10.43 at Gnome Headquarters. From there, he uses Line 2 at 10.47 and arrives at Central Station at 11.07. In particular, he can reach the Polar Express at 11.11.

7. If a vehicle departs from Mountain Hut at minute 02, it arrives at 25 at Winter Wood, departs at 35 again and reaches Mountain Hut at 58. From there, it can depart again at minute 02 after precisely one hour. Therefore, one vehicle suffices to operate Line 5 in the original timetable. If Line 5 departs 15 minutes earlier in the direction of Winter Wood, then a vehicle departing at 47 from Mountain Hut still reaches Mountain Hut again at minute 58 after 1 hour and 11 minutes. The next departure is then 49 minutes later at 47, a total of two hours after the last departure of the same vehicle from Mountain Hut. In particular, two vehicles are necessary.

8. Santa's timetable could for example look like this:

Station	1	2	3	4	5
Central Station	31 ↓ 29 ↑	35 ↓ 25 ↑			
Presents Storage	47 ↓ 13 ↑				
Gnome HQs	58 ↓ 02 ↑	55 ↓ 05 ↑	02 ↓ 58 ↑	01 ↓ 59 ↑	
Winter Wood			26 ↓ 34 ↑		33 ↓ 27 ↑
Reindeer Park				33 ↓ 27 ↑	38 ↓ 22 ↑
Mountain Hut					56 ↓ 04 ↑

Between any two distinct lines, the maximum transfer time at Central Station is 6 minutes, whereas it is 7 minutes at Gnome Headquarters and Winter Wood.

9. **False:** Such a timetable cannot exist. To see this, consider a trip with lines 3, 4 and 5 following the cycle Winter Wood – Gnome Headquarters – Reindeer Park – Winter Wood. The travel time without transfers is $24 + 32 + 5 = 61$ minutes. On top of that there are the two transfers at Gnome Headquarters and at Winter Wood. Adding a third transfer from Line 5 to Line 3 at Winter Wood, we reach again a departure of Line 3 at Winter Wood. This departure is at least 61 minutes later than the original departure. Because the timetable is repeated every 60 minutes only, it must be 120 minutes later. In particular, the three transfers at Gnome Headquarters, Reindeer Park, and Winter Wood and take 59 minutes in total. Therefore, at least one of these transfers takes $59/3 > 10$ minutes or more.

10. Take the timetable from 8. without Line 4.

10 Xmasium

Authors: Frits Spieksma (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

10.1 Challenge

In one of Santa Claus's research laboratories the scientists have discovered a new chemical element and named it Xmasium (in analogy with the famous elements Rubidium, Caesium and Francium). On Ruprecht's table there is a cooking pot that contains infinitely many Xmasium atoms: There are exactly N Xmasium atoms of atomic weight 0, exactly N Xmasium atoms of atomic weight 1, exactly N Xmasium atoms of atomic weight 2, exactly N Xmasium atoms of atomic weight 3, and so on. For every integer $w \geq 0$, this cooking pot contains exactly N Xmasium atoms of atomic weight w.

Now, Ruprecht starts to remove atoms from the pot. In the first step, he removes four atoms with overall atomic weight 1 from the pot. In the second step, he removes four atoms with overall atomic weight 2 from the pot. In the third step, he removes four atoms with overall atomic weight 3 from the pot. In the fourth step, he removes four atoms with overall atomic weight 4 from the pot. And so on, and so on: In the k-th step (with $k \geq 1$), Ruprecht removes four atoms with overall atomic weight k from the pot.

We would like to know: What is the smallest integer N so that Ruprecht (with an appropriately chosen strategy) is able to perform infinitely many such steps?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. The smallest integer is $N = 11$.
2. The smallest integer is $N = 12$.
3. The smallest integer is $N = 13$.
4. The smallest integer is $N = 14$.
5. The smallest integer is $N = 15$.
6. The smallest integer is $N = 16$.
7. The smallest integer is $N = 17$.
8. The smallest integer is $N = 18$.
9. The smallest integer is $N = 19$.
10. For every integer N, Ruprecht can only perform finitely many steps.

10.2 Solution

The correct answer is: 6.

First, we want to find a **lower bound**: therefore, we take an arbitrary number N that allows Ruprecht to perform infinitely many steps. After the first $6N$ steps, Ruprecht has taken exactly 24 atoms with an overall weight of exactly

$$1 + 2 + \cdots + 6N = 3N(6N + 1)$$

out of the pot.

The $24N$ atoms with the smallest atom weight have

- N-times weight 0,
- N-times weight 1,
- N-times weight 2, and so on, and so on, and so on, and finally
- N-times weight 23.

Thus, the smallest possible overall weight of $24N$ atoms in the pot is:

$$(0 + 1 + 2 + \cdots + 23)N = 23$$

We conclude that

$$3N(6N + 1) \geq 23 \cdot 12N.$$

This inequality can be simplified to $6N \geq 91$, which implies the lower bound $N \geq 16$.

Now, we have a look at the **upper bound**: We describe a possible strategy in which Ruprecht makes the following four steps for every number $g \geq 1$:

- The step $4g-3$ takes out the atoms with the weight $g-1, g-1, g-1, g$.
- The step $4g - 2$ takes out the atoms with the weight $g - 1, g - 1, g, g$.
- The step $4g - 1$ takes out the atoms with the weight $g - 1, g, g, g$.
- The step $4g$ takes out the atoms with the weight g, g, g, g.

This strategy uses all in all just six atoms of weight 0. The other atoms with weights $g \geq 1$ will be treated as follows:

- One atom will be taken out of the pot during step $4g - 3$,
- two more during step $4g - 2$,
- three during step $4g - 1$,
- four during step $4g$,
- three during step $4g + 1$,
- two during step $4g + 2$, and finally
- one during step $4g + 3$.

All in all, there will be

$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$$

atoms taken out with a weight $g \geq 1$. The following table shows the first twenty steps of this strategy, where the k th column shows the weight of the atoms taken out in the k th step:

0	0	0	l	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	⋯
0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	⋯
0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	⋯
1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	⋯

To sum up, Ruprecht is able to perform infinitely many steps for $N = 16$.

11 Kerfuffle at Sugarloaf Mountain

Authors: David Rueda, Ariane Beier (MATHEON)

11.1 Challenge

A tense atmosphere wafts around Sugarloaf Mountain, a snowy mountain in the middle of the Erz Mountains near the small town Rüodeschanerow. The cable car, which runs from Candy Valley to Point Jelly Bean, is out of order. Hence, Santa Claus has to take his sleigh to deliver all the presents to the young and old inhabitants of Point Jelly Bean.

The reindeers are not happy to hear that: The shortest path from Candy Valley to Point Jelly Bean along the cable car is far too steep; so they have to gallop all around the mountain. However, road engineer Greta Gnome is able to calm the grumpy animals: "The shortest path from Candy Valley to Point Jelly Bean that goes all around the Sugarloaf Mountain is not awfully long. Besides, you can sled downhill on a quite significant part of this path."

Now, we want to know: How long is the shortest path x from Candy Valley to Point Jelly Bean that goes all around Sugarloaf Mountain? And how long is the part y of this path that leads downhill?

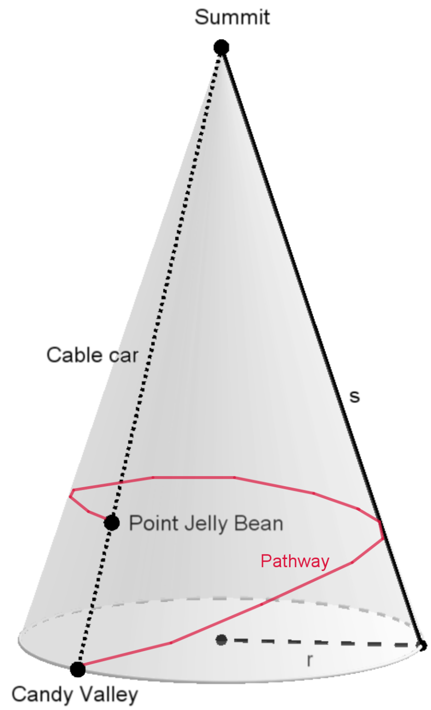


Figure 10: Sugerloaf Mountain is exactly formed like a right circular cone. Its directrix has radius $r = 20\,\text{km}$; its generatrices s are $60\,\text{km}$ long. Candy Valley is situated at the foot of Sugarloaf Mountain. Point Jelly Bean lies on the generatrix joining Candy Valley and the summit of the Mountain; its distance from Candy Valley on this generatrix is $10\,\text{km}$.

Artwork: Sonja Rörig

Possible answers:

1. One has $x \approx 92\,\text{km}$ and $y \approx 39\,\text{km}$.
2. One has $x \approx 92\,\text{km}$ and $y \approx 40\,\text{km}$.
3. One has $x \approx 93\,\text{km}$ and $y \approx 40\,\text{km}$.
4. One has $x \approx 93\,\text{km}$ and $y \approx 41\,\text{km}$.
5. One has $x \approx 94\,\text{km}$ and $y \approx 41\,\text{km}$.
6. One has $x \approx 94\,\text{km}$ and $y \approx 42\,\text{km}$.
7. One has $x \approx 95\,\text{km}$ and $y \approx 42\,\text{km}$.
8. One has $x \approx 95\,\text{km}$ and $y \approx 43\,\text{km}$.
9. One has $x \approx 96\,\text{km}$ and $y \approx 43\,\text{km}$.
10. One has $x \approx 96\,\text{km}$ and $y \approx 44\,\text{km}$.

11.2 Solution

The correct answer is: 7.

The cone's lateral surface can be viewed as a circular sector of radius $s = 60\,\text{km}$ (see Figure 11). We want to calculate the shortest path x from Candy Valley (C) to Point Jelly Bean (P) with the aid of the *law of cosines*:

$$x^2 = (50\,\text{km})^2 + (60\,\text{km})^2 - 2 \cdot 50\,\text{km} \cdot 60\,\text{km} \cdot \cos\gamma$$

$$x = \sqrt{50^2 + 60^2 - 2 \cdot 50 \cdot 60 \cdot \cos\gamma}\ \text{km}.$$

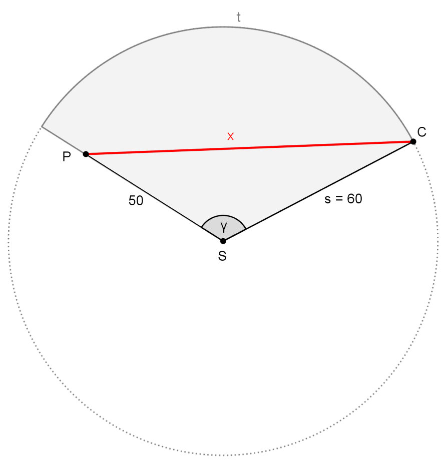


Figure 11: The cone's lateral surface depicted as a circular sector

In order to determine the angle γ, we calculate the ratio of the circular sector as part of the whole circle: The arc length t of the circular sector is $t = 2\pi r$. The perimeter of the whole circle is

$$u = 2\pi \cdot s = 2\pi \cdot 3r = 3t.$$

That is, $\gamma = \frac{1}{3} \cdot 360° = 120°$ and $\cos\gamma = \cos(120°) = -\frac{1}{2}$. We conclude:

$$\begin{aligned} x &= \sqrt{50^2 + 60^2 + 50 \cdot 60}\ \text{ km} = \sqrt{2500 + 3600 + 3000}\ \text{ km} \\ &= \sqrt{9100}\ \text{ km} = 10\sqrt{91}\ \text{ km} \\ &\approx 95,39\,\text{km} \end{aligned}$$

Now, we want to calculate the length of the path y, the part of x that goes downhill: The highest point H on the path x is precisely the point where x is tangential to a level curve of the cone. The cone's level curves are perpendicular to its directrices s, i. e. the radii of the big circle. Hence, H is determined as the point where x and a radii s are perpendicular (see Figure 12).

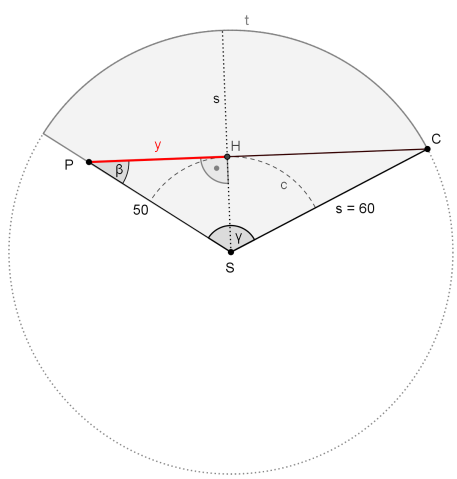


Figure 12: The highest point H on the path x and the part y that goes downhill

In the right triangle ΔGHW, y is the desired length of the path that goes downhill. One has

$$y = \cos\beta \cdot 50\,\text{km}.$$

We determine $\sin\beta$ with the help of the *law of sines*:

$$\sin\beta = \frac{\sin\gamma\cdot s}{x} = \frac{\sin(120^\circ)\cdot 60}{10\sqrt{91}} = \frac{\frac{\sqrt{3}}{2}\cdot 60}{10\sqrt{91}}$$
$$= \frac{3\sqrt{3}}{\sqrt{91}}.$$

Futhermore,

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{3\cdot 9}{91}} = \sqrt{\frac{91-27}{91}} = \sqrt{\frac{64}{91}}$$
$$= \frac{8}{\sqrt{91}}.$$

Hence,

$$y = \frac{8}{\sqrt{91}}\cdot 50\,\text{km}$$
$$= \frac{400}{\sqrt{91}}\,\text{km}$$
$$\approx 41,93\,\text{km}.$$

12 Tournament

Author: Onno Boxma (TU Eindhoven)

12.1 Challenge

In the knock-out tournament today the blue team with 24 elves is competing against the yellow team with 24 elves. Every elf enters the tournament with a single ticket. In every round of the tournament either team nominates one of its combatants. Then, all the tickets of these two combatants are put into a cylinder hat. The winner of the round is determined by randomly drawing a ticket from the hat. The winner receives all the tickets of the loser and stays in the tournament. The loser is eliminated from the tournament. Once the last elf of one team is eliminated, the other team has won the tournament. After 41 rounds there are only seven elves left in the tournament:

- Blue team: Atto (16 tickets), Bilbo (6 tickets), Chico (5 tickets), Dondo (1 ticket)
- Yellow team: Espo (10 tickets), Frodo (6 tickets), Gumbo (4 tickets)

The yellow team nominates Frodo as combatant for the next round.

Who should become the blue combatant, in order to maximize the probability of winning the tournament for the blue team?

Artwork: Sonja Rörig

Possible answers:

1. Only Atto maximizes the probability of winning for the blue team.
2. Only Bilbo maximizes the probability of winning for the blue team.
3. Only Chico maximizes the probability of winning for the blue team.
4. Only Dondo maximizes the probability of winning for the blue team.
5. Only Atto and Bilbo maximize the probability of winning for the blue team.
6. Only Atto and Chico maximize the probability of winning for the blue team.
7. Only Atto and Dondo maximize the probability of winning for the blue team.
8. Only Bilbo and Chico maximize the probability of winning for the blue team.
9. Only Bilbo and Dondo maximize the probability of winning for the blue team.
10. It does not matter at all who becomes the blue combatant.

12.2 Solution

The correct answer is: 10.

The blue team wins with the probability 7/12—completely independent of the combatant they nominate.

We even want to prove the following statement: If the blue team has b contestants with altogether B tickets and the yellow team has g contestants with altogether G tickets, then the blue team wins the game with the probability $B/(B+G)$—independent of the nomination of the combatants.

First, we note that the stated probability is symmetric in b (or B) and g (or G). Now, we want to show the statement by mathematical induction:

Inductive base: If $b = 0$ or $g = 0$, the statement is obviously true.

Inductive hypothesis: We suppose that the statement is true for $(b-1)+g$ and $b+(g-1)$.

Inductive assumption: We want to show the statement holds for $b+g$.

Inductive step: The blue team nominates a player with x tickets, and the yellow team nominates a player with y tickets. The blue player wins this round with probability $x/(x+y)$. If the blue player wins, the probability of winning the game for the blue team is $(B+y)/(B+G)$, according to the inductive hypothesis. The blue player loses this round with probability $y/(x+y)$. Then, the blue team wins the game with probability $(B-x)/(x+y)$. Thus, the probability of winning the game for the blue team is:

$$\frac{x}{x+y} \cdot \frac{B+y}{B+G} + \frac{y}{x+y} \cdot \frac{B-x}{B+G} = \frac{B}{B+G}.$$

Hence, we have proven the statement and have shown that the probability of winning the game depends solely on B and G and not on b, g or the nominations of the contestants.

Remark: The challenge can be solved easily with a computer program, which calculates the probabilities of winning for all duels possibly occurring in the tournament. For example, for the first round there are only eight situations possible:

(1) Atto wins against Frodo.

(2) Atto loses against Frodo.

(3) Bilbo wins against Frodo.

(4) Bilbo loses against Frodo.

(5) Chico wins against Frodo.

(6) Chico loses against Frodo.

(7) Dondo wins against Frodo.

(8) Dondo loses against Frodo.

Since there are only six duels to consider, the branches and the total number of possible situations remain manageable.

Another approach is to simulate the tournament some million times with a computer. Then, one will realise that the probabilities do not depend on the teams' particular nominations of contestants.

13 Treasure Island

Author: Hennie ter Morsche (TU Eindhoven)

13.1 Challenge

Today the elves go treasure hunting on the circle-shaped Christmas island. There are nine old palm trees standing at the periphery of the island: Four of these palm trees are at the northernmost, easternmost, southernmost and westernmost point of the island. Another palm tree is exactly at north-northeast, one at northeast, one at south-southwest, one at west-southwest, and the last one at west-northwest.

The treasure map says:

> The three palm trees, where the three red-bearded pirates have their hammocks, form a triangle. The orthocentre of this triangle contains the anthill A. The three palm trees, where the black-bearded pirates have their hammocks, form also a triangle. The orthocentre B of this triangle contains a blue flower. The remaining three palm trees form yet another triangle, whose orthocentre C contains our compass. At the barycentre of triangle ABC, our treasure has been buried.

"Oh, no!", laments Wailo the wailing-elf. "The nine palm trees are still around. But the red-bearded and black-bearded pirates have left the island long ago, the anthill A has disappeared, the flower in B has withered away, and the pirates must have taken the compass in C with them. There are hundreds of possibilities of dividing the nine palm trees into three triangles. Hence, we will have to dig at hundreds of points, and we will end up with thick calluses on our hands."

"I think that our situation is much better than it looks!", says Geobald the geometry-elf. "Actually, there are not many different candidate points for the position of the treasure. Suppose for instance that the three red-bearded pirates had their hammocks at the three palm trees in north, in north-northeast and in northeast. Then the orthocentre of the resulting triangle would not be located on the island, and the anthill A would have to be located somewhere in the ocean. Hence, this particular case may be ignored and eliminated right away. And if we ponder a little bit more, we certainly will be able to eliminate further cases."

How many different points on the island are candidates for the location of the treasure?

Artwork: Sonja Rörig

Possible answers:

1. Exactly one point
2. Exactly 3 points
3. Exactly 9 points
4. Exactly 12 points
5. Exactly 24 points
6. Exactly 27 points
7. Exactly 35 points

8. Exactly 105 points
9. Exactly 420 points
10. Exactly 1680 points

13.2 Solution

The correct answer is: 1.

First, we introduce a coordinate system in which the centre of the circular Christmas island coincides with the coordinate origin $(0, 0)$ and the nine palm trees are nine points $(x_k, y_k),\ 1 \leq k \leq 9$, on the unit circle, i. e.

$$x_k^2 + y_k^2 = 1.$$

We consider three arbitrary palm trees $U = (x_u, y_u)$, $V = (x_v, y_v)$, and $W = (x_w, y_w)$. The orthocentre of the triangle ΔUVW is located on the line through U that is orthogonal to the edge VW; the points (x, y) on this line satisfy the equation

$$(x_v - x_w)(x - x_u) + (y_v - y_w)(y - y_u) = 0. \tag{8}$$

You may deduce this by first computing the linear equation for VW and then calculating its orthogonal through U or by computing the scalar product of these two lines.
In addition, the orthocentre of ΔUVW is located on the line that runs through V and that is orthogonal to UW too:

$$(x_u - x_w)(x - x_v) + (y_u - y_w)(y - y_v) = 0. \tag{9}$$

With the help of a computer algebra system, one easily calculates (the calculation with pen and paper is a little tedious) that these two lines (8) and (9) intersect at the point

$$(x_u + x_v + x_w,\ y_u + y_v + y_w),$$

which is the orthocentre of ΔUVW.

In consequence, the anthill, the blue flower, and the compass are located at the points

$$\begin{aligned} A &= (x_i + x_j + x_k,\ y_i + y_j + y_k), \\ B &= (x_l + x_m + x_n,\ y_l + y_m + y_n), \\ C &= (x_o + x_p + x_q,\ y_o + y_p + y_q) \end{aligned}$$

with

$$\{i,j,k\} \cup \{l,m,n\} \cup \{o,p,q\} = \{i,j,k,l,m,n,o,p,q\} = \{1,2,3,4,5,6,7,8,9\}.$$

The x-coordinate of the barycentre of the triangle ΔABC is defined as the arithmetic mean of the x-coordinates of the points A, B, C. The y-coordinate of the barycentre is the arithmetic mean of the y-coordinates of the points A, B, C. Hence, the treasure is buried at the point with the following coordinates

$$\begin{aligned} x &= \frac{1}{3}\,(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9), \\ y &= \frac{1}{3}\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9). \end{aligned}$$

Thus, the location of the treasure is independent of the distribution of the nine palm trees into three triangles. In conclusion, there is only **one point** on the Christmas island that qualifies as the treasure site.

Remark: The challenge is also easily solved with a computer program, which runs over all combinations of possible triangles defined by the nine palm trees. Furthermore, the use of a construction software like GeoGebra leads to the right idea.

14 A Slightly Different Christmas Star

Authors: Khai Van Tran, Daniel Schmidt genannt Waldschmidt, Sven Jäger, Rico Raber
Project: Combinatorial Optimization & Graph Algorithms (COGA), TU Berlin

14.1 Challenge

"This is unfair!" exclaims reindeer Miri, spokeswoman of the *United Reindeer Transport Union (URTU)*, "We toiled in the cookie factory throughout the winter! We transported flour, cinnamon, and sugar in record time! But on Christmas Day, when we finally get to the interesting part, only a few of us may draw Santa's sleigh."

After negotiations with URTU it is decided to divide the earth into 15 zones with each zone being assigned to a different team of reindeer in order to give everyone the opportunity to draw Santa's sleigh at least once. Therefore, on Christmas Day, Santa will start his tour at the North Pole, deliver the presents in one of the zones; then he will return to the North Pole in order to swap the reindeer. Afterwards, the presents in the next zone will be delivered and so on. After the last present is delivered, the empty sleigh will return to the North Pole completing the star shaped tour.

"Since the sleigh will return to the North Pole after visiting a zone anyway, we should only load those presents that will be delivered in the respective next zone onto the sleigh," suggests gnome Rico. "Then the reindeer will have to draw a lighter load and will be faster." Santa flinches while remembering last year's Christmas when the gnomes started some rounds of *clay gift shooting* the day just before Christmas and Rudolph had to get some new presents wrapped the last minute. So leaving the gnomes alone with the gifts is out of the question—and not just due to Santa's professional pride forbidding him to start his route with a mostly empty sleigh.

In order to plan the best route, Rudolph looks at a world map (see Figure 13), on which the 15 zones are recorded together with their respective distances to the North Pole and the weight of the respective presents (see also Table 1). Rudolph knows that the transport time is not only proportional to the distance traversed but also proportional to the weight of the sleigh including all the gifts still loaded onto it: The empty sleigh weighs 30 tons (t) and drawing the empty sleigh 10 air miles (am) takes 300 moon seconds (mon). With presents weighing $70\,t$ loaded on the sleigh (so including the sleigh yields a total weight of $100\,t$) the reindeer would need $2000\,mon$ to draw the sleigh $20\,am$. Since the reindeer waiting at the North Pole are nervously anticipating their turn, changing the reindeer takes no time at all. Once the sleigh arrives in one of the zones, Santa with his millennia of experience can distribute the gifts without loosing any time too.

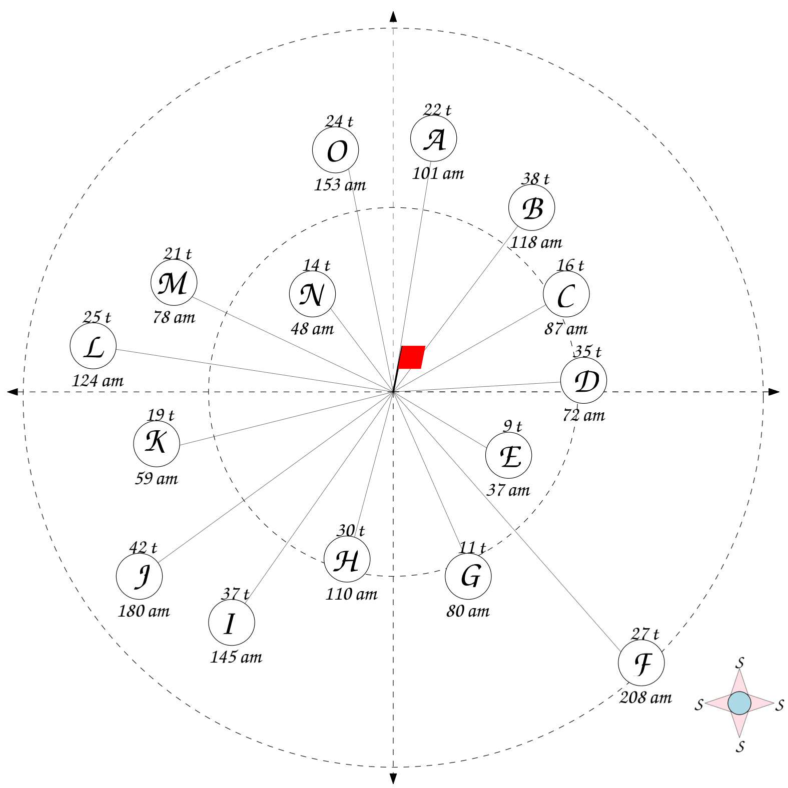


Figure 13: World map with the 15 zones. The flag marks the North Pole.

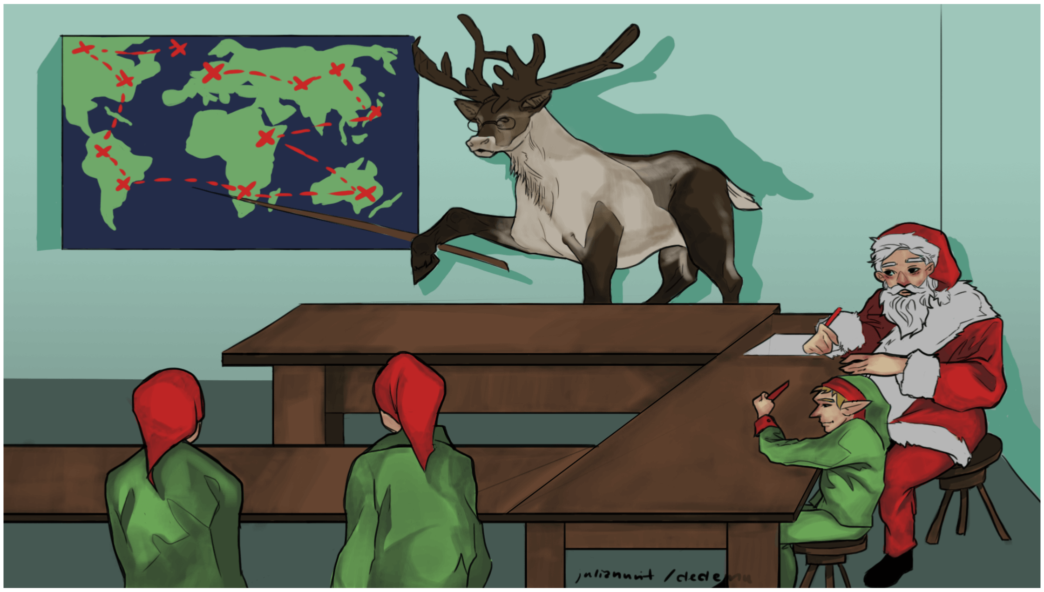
Zone	Weight in t	Distance in am
A	22	101
B	38	118
C	16	87
D	35	72
E	9	37
F	27	208
G	11	80
H	30	110
I	37	145
J	42	180
K	19	59
L	25	124
M	21	78
N	14	48
O	24	153

Table 1: Weight of the presents to be delivered and distance of the zones from the North Pole.

All that remains to be done is to determine the order in which the zones are visited...

The travel time we want to minimize is the time from Santa starting at the North Pole with the fully loaded sleigh until Santa returns to the north Pole with an empty sleigh. We say that a route is **optimal** if it minimizes that travel time. We say that an optimal route is **unique** if there is only one order of zones that minimizes the travel time.

Which one of the following statements is correct?



Artwork: Julia Nurit Schönnagel

Possible answers:

1. An optimal route must visit J before E before O and the optimal route is unique.
2. An optimal route must visit F before M before E and the optimal route is unique.
3. An optimal route must visit N before H before L and the optimal route is unique.
4. An optimal route must visit K before G before I and the optimal route is unique.
5. An optimal route must visit L before J before M and the optimal route is unique.
6. An optimal route must visit J before E before O and the optimal route is not unique.
7. An optimal route must visit F before M before E and the optimal route is not unique.

8. An optimal route must visit N before H before L and the optimal route is not unique.

9. An optimal route must visit K before G before I and the optimal route is not unique.

10. An optimal route must visit L before J before M and the optimal route is not unique.

14.2 Solution

The correct answer is: 8.

Santa would very much like to ditch heavy presents early. On the other hand, he would prefer to only cover short distances at first while the sleigh is still heavily loaded.

We argue that Santa should calculate the ratio between the distance to the North Pole and weight of the presents for each zone (in the following just called **ratio**) and then visit the zones according to increasing ratio. In order to see that this might be a reasonable strategy, we present a "proof by picture" first:

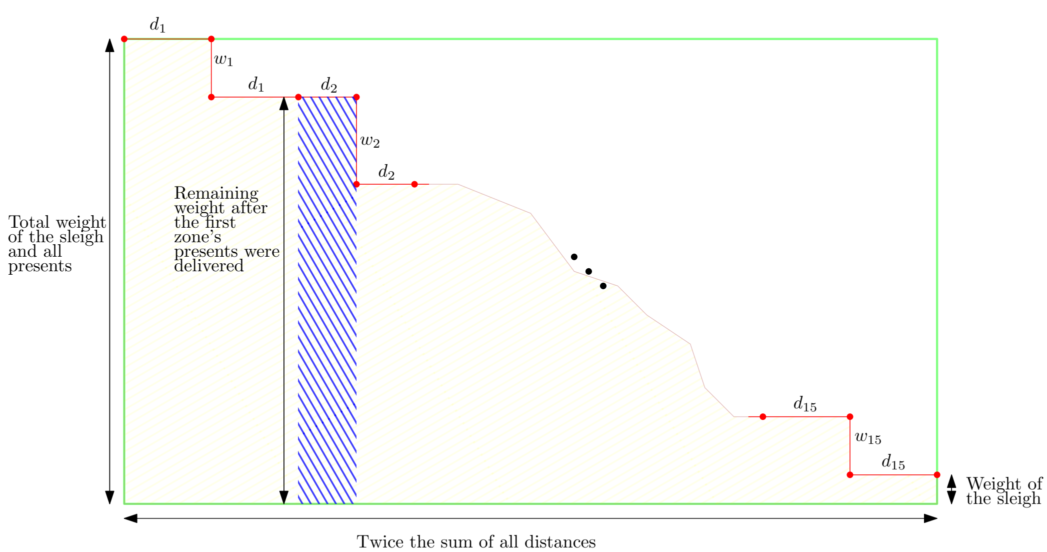


Figure 14: d_1, d_2 are the respective distances for the first two zones; w_1, w_2 the corresponding weights.

Provided we have fixed a certain order of zones, we can draw a diagram as above. Travelling a certain distance takes time according to the product of the remaining weight of the sleigh multiplied with the distance. In the picture, the time needed to travel to the second zone is exactly the area of the blue rectangle. Therefore, the total time corresponds to the yellow hatched area under the red curve.

Additionally, we note that the sides of the green rectangle, which determines the beginning and end of the red curve, correspond exactly to the total weight of the sleigh at the very beginning and twice the sum of all distances, respectively. Thus, the green rectangle is independent of the order in which the zones are visited. In order to minimize the yellow hatched area, one should try to "bend" the red curve towards the left and the bottom. This is accomplished by making sure that the red curve is dropping steeply at first, which corresponds to a low ratio, and is becoming flat, which corresponds to a high ratio, towards the end.

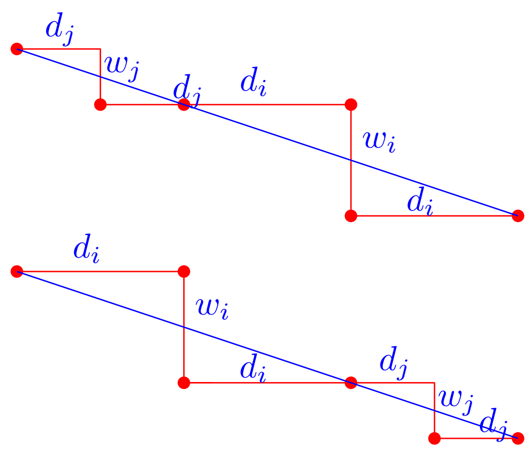


Figure 15: The two zones i and j have the same ratio.

If there are zones with equal ratio, then they must be visited successively. Looking at Figure 15, the areas under the red stairs and under the blue line are equal. (The steepness of the blue line is determined by the ratio of the zones.) The figure demonstrates that it does not matter in which order zones with equal ratios are visited, and thus, if they exist, the optimal route is not unique.

Of course, we can also calculate this exactly: Assume we have fixed an order, then let us look at two consecutive zones i and j (i. e. we visit i directly before j). Let w_i,w_j be their respective weights and d_i,d_j their respective distances. If one were to switch the order in which i and j are visited, this of course will not change the time needed for the zones which are visited before

i and j. The remaining weight on the sleigh after i and j also remains the same and thus, all the travel times after i and j stay the same.

So in order to see how the travel time changes due to the swap, one would only have to look at the travel times to and from i and j. Let R be the total weight of the sleigh directly after visiting i and j. Then we compare the travel times without the swap (LHS of the inequality) with the travel time with the swap (RHS of the inequality):

$$\begin{aligned} & ((R+w_i+w_j)+(R+w_j))\cdot d_i + ((R+w_j)+R)\cdot d_j \leq \\ & \qquad ((R+w_i+w_j)+(R+w_i))\cdot d_j + ((R+w_i)+R)\cdot d_i \\ \Leftrightarrow \quad & 2R(d_i+d_j)+w_id_i+w_jd_j+2w_jd_i \leq 2R(d_i+d_j)+w_id_i+w_jd_j+2w_id_j \\ \Leftrightarrow \quad & 2w_jd_i \leq 2w_id_j \\ \Leftrightarrow \quad & \frac{d_i}{w_i} \leq \frac{d_j}{w_j} \end{aligned}$$

Therefore, a swap reduces the travel time if and only if the ratio of i is greater than the ratio of j. In a route in which we do not visit the zones by increasing ratio, we can always find such a swappable pair. We can iterate these kind of swaps (thereby improving the total travel time) until we visit the zones by increasing ratio. Thus, this is always done in an optimal route.

With a basically identical calculation (just replacing the $\leq$ in the middle with a $=$) we can also see that swapping two consecutive zones with equal ratio does not change the total travel time. So the optimal route is unique if and only if there are two zones with equal ratios.

In the following table, we have sorted the zones by their ratio (we rounded the ratios):

Zone	Weight in t	Distance in am	Ratio in am/t
D	35	72	2,057
B	38	118	3,105
K	19	59	3,105
N	14	48	3,429
H	30	110	3,667
M	21	78	3,714
I	37	145	3,919
E	9	37	4,111
J	42	180	4,286
A	22	101	4,591
L	25	124	4,960
C	16	87	5,438
O	24	153	6,375
G	11	80	7,273
F	27	208	7,704

First, we notice that zones B and K have the same ratio (their distances and weights only differ by a factor of 2, respectively). Thus, the optimal route is not unique. The only correct order mentioned in the answer possibilities is the following: N must be visited before H and H before L.

15 Sum and Product

Authors: Aart Blokhuis (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

15.1 Challenge

Santa Claus tells the super-smart elves Prodo and Summo: "I am thinking of two secret integers x and y with $2 \leq x \leq y$ and $x + y \leq 47$. I have written the sum $S = x + y$ on one card and the product $P = xy$ on another card. Prodo has received the card with the product, and Summo has received the card with the sum. You may now look at your cards!"

Prodo and Summo are staring at their cards and start to ponder about the secret numbers. Then Summo tells Prodo: "You are not able to determine the sum S."

Prodo says: "Aha! I have learnt something new. But I still do not know x and y."

Summo says: "Aha! I have learnt something new. Now I do know x and y."

Prodo says: "Aha! Finally I do know the secret numbers x and y."

Which of the following ten statements holds true?

Artwork: Friederike Hofmann

Possible answers:

1. The unit digit of S is 1.
2. The unit digit of S is 2.
3. The unit digit of S is 3.
4. The unit digit of S is 4.
5. The unit digit of S is 5.
6. The unit digit of S is 6.
7. The unit digit of S is 7.
8. The unit digit of S is 8.
9. The unit digit of S is 9.
10. The unit digit of S is 0.

15.2 Solution

The correct answer is: 1.

(1) If the product P has a prime factor $a \geq 23$, Prodo is able to conclude $x = P/a$, $y = a$, and $S = (P/a) + a$, because: One of the numbers x and y has to be a multiple of this prime factor a. Since the multiple $y = 2a$ (and any higher multiple $3a$, $4a$, $5a$, etc.) would violate the bound $x + y \leq 47$, it holds $y = a$.

The first statement of Summo (*"You are not able to determine the sum S."*) now implies that S cannot be written as a sum of a prime factor ≥ 23 and a number ≥ 2. This eliminates all numbers ≥ 25 as possible candidates for S:

- Every number $S \geq 25$ can be written as a sum of the prime factor 23 and the number $S - 23 \geq 2$.

At this point of our analysis, there are only the numbers ≤ 24 left as candidates for S.

(2) If the product P is the product of two prime factors c and d with $c \leq d$, Prodo can conclude $x = c$, $y = d$ and $S = c + d$. The first statement of Summo (*"You are not able to determine the sum S."*) now implies that S cannot be written as a sum of two prime factors. This eliminates further candidates for S:

- All even numbers between 4 and 24 can be written as a sum of two prime factors:

$$\begin{array}{llll} 4 = 2 + 2; & 6 = 3 + 3; & 8 = 3 + 5; & 10 = 5 + 5; \\ 12 = 5 + 7; & 14 = 3 + 11; & 16 = 3 + 13; & 18 = 5 + 13; \\ 20 = 3 + 17; & 22 = 5 + 17; & 24 = 5 + 19. & \end{array}$$

- Furthermore, the following odd numbers can be written as a sum of two prime factors:

$$\begin{array}{llll} 5 = 2 + 3; & 7 = 2 + 5; & 9 = 2 + 7; & 13 = 2 + 11; \\ 15 = 2 + 13; & 19 = 2 + 17 & 21 = 2 + 19. & \end{array}$$

We conclude that only the numbers 11, 17, 23 are left as candidates for S.

(3) Now, we count the products xy for each remaining candidate S that fulfil $2 \leq x \leq y$ and $x + y = S$:

- $S = 11$: 18, 24, 28, 30.
- $S = 17$: 30, 42, 52, 60, 66, 70, 72.
- $S = 23$: 42, 60, 76, 90, 102, 112, 120, 126, 130, 132.

One can prove easily that every listed product has a second factorisation whose two factors add up to a sum ≤ 47. Because of the first statement of Summo we (and Prodo) are able to deduce $S \in \{11, 17, 23\}$.

(4) After the first statement of Summo, Prodo is still not able to determine the numbers x and y (*"But I still do not know x and y."*). This means that the product P is compatible with at least two of the remaining three candidates for S. We omit all products of the enumeration that are assigned to only one candidate and obtain:

- $S = 11$: 30.
- $S = 17$: 30, 42, 60.
- $S = 23$: 42, 60.

(5) After Prodo's last statement, Summo is finally able to determine both numbers. If $S = 17$ or $S = 23$, there would be at least two other possibilities left; Summo would not know, which one of the compatible products is P. Hence, $S = 11$. Furthermore, we get $P = 30$ and $x = 5$ and $y = 6$. Of course, Summo and Prodo draw the same conclusions.

16 Temperature

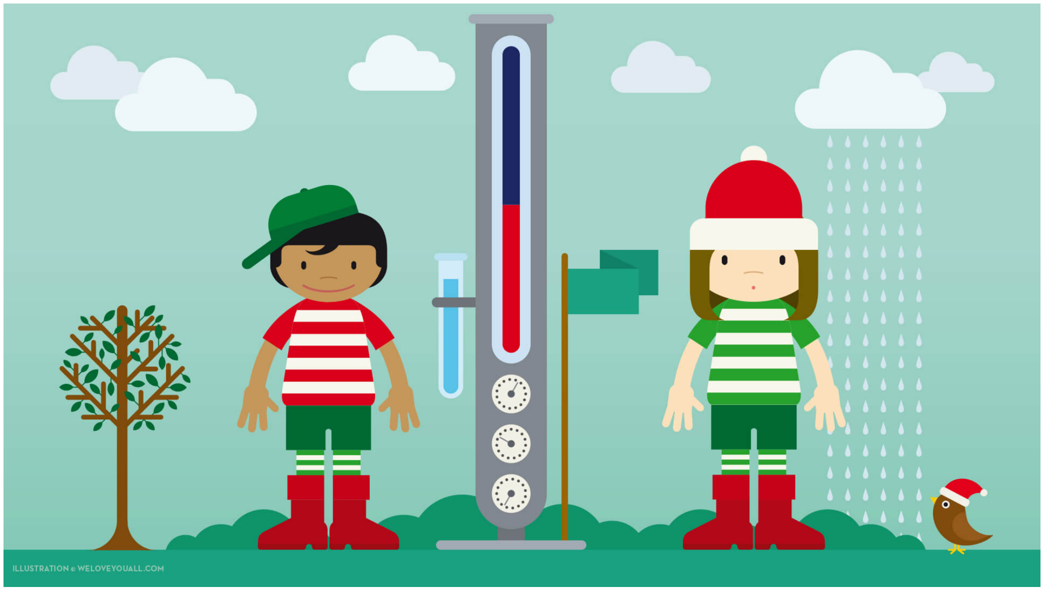
Author: Judith Keijsper (TU Eindhoven)

16.1 Challenge

"Gosh, that's really strange," shouts Wendelin the weather-elf. "Here I am sitting in my arm-chair, leisurely studying the temperature values of the last N days as measured by our weather station *Omicron-702*, enjoying the evening, and suddenly I am hit by the flabbergasting truth: The average temperature in each group of seven consecutive days has always been positive. And the striking surprise is that the average temperature in each group of ten consecutive days has always been negative. I am curious to know whether this pattern will also continue tomorrow."

Matto, the omniscient mathematics-elf has listened to Wendelin's story and says, "No, no, no. There is absolutely no way that this pattern will also continue tomorrow. After $N+1$ days, there must be a group of seven consecutive days with non-positive average temperature, or a group of ten consecutive days with non-negative average temperature."

Can you tell us the value of this miraculous number N?



Artwork: Friederike Hofmann

Possible answers:

1. The number is $N = 9$.
2. The number is $N = 10$.
3. The number is $N = 11$.
4. The number is $N = 12$.
5. The number is $N = 13$.
6. The number is $N = 14$.
7. The number is $N = 15$.
8. The number is $N = 16$.
9. The number is $N = 17$.
10. The number is $N = 18$.

16.2 Solution

The correct answer is: 7.

For a proof of contradiction, we first assume that there is a sequence of $N = 16$ temperatures $T_1, \ldots, T_{16}$ such that the average value of any seven consecutive days is always positive and the average value of any ten consecutive days is always negative. Then, we have a look at the following table with seven rows and ten columns:

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}
T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}
T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}
T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}

Because each row contains the measurements of 10 consecutive days, the sum of each row is negative. Thus, the sum of all $7 \cdot 10$ values always has to be negative. Since each column contains the measurements of 7 consecutive days, the sum of each column is positive. Thus, the sum of all $10 \cdot 7$ values always has to be positive. Contradiction! We conclude that it is $N \leq 15$.

Next, we show that $N = 15$ days is indeed possible. To this end, we have a look at the following sequence of $N = 15$ measurements:

$$5,\ 5,\ -12,\ 5,\ 5,\ -12,\ 5,\ 5,\ 5,\ -12,\ 5,\ 5,\ -12,\ 5,\ 5$$

Each group of seven consecutive values in this sequence contains 5 times the value 5 and 2 times the value -12; the average value is $1/7$ and thus positive. Each group of ten consecutive values in this progression contains 7 times the value 5 and 3 times the value -12; the average value is $-1/10$ and thus negative.

Our discussion implies that the wanted number is $N = 15$.

17 Twinkle, Twinkle, Little Star

Author: Falk Ebert (Herder-Gymnasium, HU Berlin)
Translation: Ariane Beier (MATHEON)

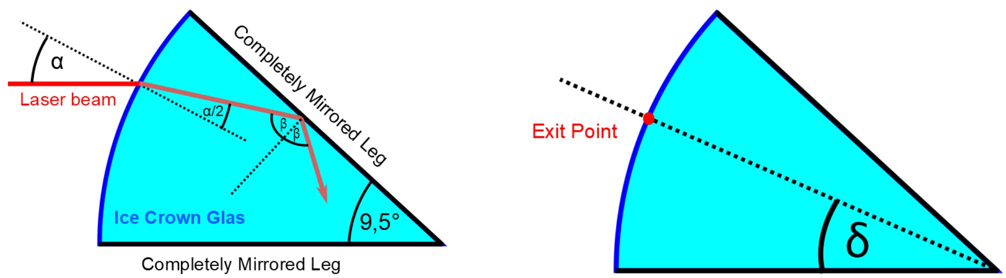
17.1 Challenge

Also at the North Pole, Christmas is the time to decorate house and garden festively. To adorn his Christmas tree, Santa has a very special star. The so-called *Twinkle Star* is made of glass and has 24 points. In fact, it *had* 24 points—until a ginger-haired little boy broke one point while flying low on a tremendously fast sledge last year (everyone who recognises this reference, is now allowed to gloat over it).

Glenn, the glass-blower elf, immediately crafted a new point for Santa's Twinkle Star. But unfortunately, somehow it will not twinkle as bright as the original points. Stan, the standardisation elf, points out how to identify a standardised point of a Twinkle Star:

> A standardised Twinkle Star's point is a section of an intangibly thin glass plate in the form of a plane circular sector. The circular sector's radius is 10 cm and its central angle is 9.5°. The legs of the circular sector are completely mirrored; that is, incoming light is reflected according to the law of reflection "angle of incidence = angle of reflection" (see Figure 16a). The material, the Twinkle Star's points are made of, is *Ice Crown Glass* with a north polar refractive index of 2; that is, at the transition from the air to the glass, the angle between the laser beam and the perpendicular is halved (see Figure 16a). Besides the described reflection and refraction, there occurs no other reflection, refraction, diffraction and absorption of light in the considered materials.
>
> In order to identify a standardised point of a Twinkle Star, one has to ray that point with a laser beam that is parallel to one leg (hereafter called *lower leg*), at a distance of at most 1.5 cm. Accordingly, the beam will be refracted at the transition to the Ice Crown Glass, then reflected multiple times at the mirrored legs, and finally will exit the circular sector at its arc, i. e. at the

curved side of the star's point. Now, one has to measure the angle δ of the exit point of the reflected laser beam respective to the lower leg (see Figure 16b). If the measured angle is **not** equal to the *comparison angle*, then the point is **not** a standardised Twinkle Star's point.



(a) Sketch (**not true to scale**) (b) Position of the exit point

Figure 16: A Twinkle Star made of Ice Crown Glass.

Since Stan's memory is not the best anymore, he would have to look up the value of this comparison angle.

We ask you to help Stan and tell him the value of the comparison angle δ between the lower leg and the exit point of standardised Twinkle Star's point.

Artwork: Frauke Jansen

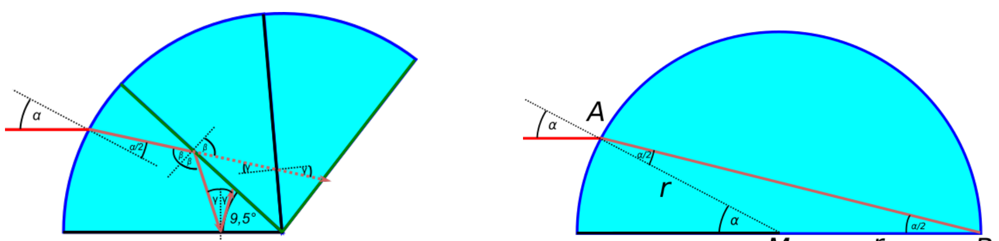
Possible answers:

1. 1°
2. 2°
3. 3°
4. 4°
5. 5°
6. 6°
7. 7°
8. 8°
9. 9°
10. It is not possible to determine the angle without knowing the exact distance of the laser beam.

17.2 Solution

The correct answer is: 9.

First, we want to eliminate the nasty multiple reflections occurring in the Twinkle Star. To this end, observe that the reflected beam may as well pass straight through the mirrored side: After the beam entered the circular sector of Ice Crown Glass and hit the first wall, we may mirror the whole circular sector at that wall (Figure 17a, green) and straight extend the beam beyond it, then mirror the extended beam at the second wall and so on and so forth. Since there are many such reflections occurring, we have to mirror the circular sector multiple times. The scheme is illustrated in Figure 17a.



(a) Multiple reflections in the mirrored world (not true to scale)

(b) Exit point in the mirrored world

Figure 17: Beweisskizze

In order to determine the exit point, we examine the circle our circular sector made of Ice Crown Glass is part of. In the "mirrored world" the exit point coincides with the point where the straight beam hits the circle for the second time (see point B in Figure 17b). Having a close look at Figure 17a, we deduce that the exit point is located on the extended lower leg: The entering beam is parallel to the lower leg. Thus, the radius MA and the lower leg form the angle α. The triangle ΔAMB is an isosceles, since its legs are two radii of the same circle. So according to the exterior angle theorem, the angle at B is $\alpha/2$. Moreover, the angle $\measuredangle MAB$ is the refractive angle of the laser beam, which is also $\alpha/2$. Accordingly, the extended beam moves along AB in the mirrored world.

Now, one has to reflect where the exit point is located in our original, non-mirrored, Twinkle Star point. In Figure 17a, the reflection of the upper leg is depicted in green, whereas the the reflection at the lower leg is depicted in black. It is easy to see that the first black copy of the lower leg is rotated by $2 \cdot 9.5^\circ = 19^\circ$ w. r. t. the original. The ninth is, accordingly, rotated by $9 \cdot 19^\circ = 171^\circ$ w. r. t. the original. An angle of 9° is further missing to complete the half circle. Those $\mathbf{9}^\circ$ determine the exit point of the laser beam.

18 Santa's sick stomach—a logical analysis of cookies

Authors: Katinka Becker (FU Berlin), Alexander Bockmayr (FU Berlin)
Project: CH5 – Model classification under uncertainties for cellular signaling networks
Translation: Ariane Beier (MATHEON)

18.1 Challenge

Like a thick white carpet, the snow rests upon Winter Wonder Land. The House of Father Christmas appears quiet and placid, almost drowsy. However, the first impression is misleading: If you open one of the hidden little doors, you will find yourself in the biggest hustle and bustle you could ever imagine. Since there are only a few days left until Christmas, the prearrangements run at full blast. And in between all the jolly and busy sawing, painting, sewing, and wrapping, you will find two of Santa's little helpers with very worried expressions on their faces.

"We have to figure something out, Ella!" gnome Gustav sighs. "We must prevent what happened last year. A sick Santa for Christmas—that is completely out of the question!"
Elf Ella agrees, "You are right. We need a plan."

Last Christmas, as the years before, Santa found a plate of cookies at each of the children's houses that he had given presents to. Since Santa has a very sweet tooth, he ate all of these delicious cookies: the vanilla crescents, the speculoos, the cinnamon and spritz biscuits, and of course the ginger nuts too. Until last Christmas, this self-indulgent consumption had never posed a problem. However, last year Santa had terrible stomach aches after emptying some of the cookie plates. In consequence, he needed to rest for 30 minutes until he was able to resume his journey. Then, he made his way to the next house and to the next plate of cookies... Besides Santa feeling uncomfortable every now and then that night, there is absolutely no time to rest at Christmas Eve. Thus, Ella and Gustav need to consider how to prevent this disaster from happening again.

"We cannot prohibit Santa from nibbling cookies. They are his absolute favourite, and all this giving and receiving is such an important part of Christmas," Gustav says.

"No, we certainly cannot... But I think, we do not have to: After all, Santa ate several plates of cookies without getting sick afterwards. It seems that there are certain cookies or certain selections of cookies that cause Santa pain," Ella replies. "Last year, I recorded which selections of cookies Santa ate and if he became sick afterwards or if not. Have a look!"

	Vanilla crescents x_1	Specu-loos x_2	Cinnamon biscuits x_3	Spritz biscuits x_4	Ginger nuts x_5	Coconut macaroon x_6	Anise cookies x_7	
Plate no. 1	1	0	0	0	0	0	0	Stomach ache
Plate no. 2	0	1	0	0	0	1	1	
Plate no. 3	1	0	1	0	1	1	0	
Plate no. 4	0	1	1	0	1	1	0	
Plate no. 5	1	0	0	1	1	0	1	
Plate no. 6	0	0	1	0	1	0	0	No stomach ache
Plate no. 7	0	0	0	1	0	0	1	
Plate no. 8	0	1	0	1	1	1	1	

Figure 18: Eight observations from last Christmas. Every row displays a plate/selection of cookies that Santa ate. Every column represents a kind of cookie. In the first five cases, Santa had a stomach ache; in the three last cases, he did not.

Ella's recordings (see Figure 18) contain eight observations of eight cookie plates Santa ate on Christmas Eve. In each case, Ella wrote down which of the seven kinds of cookies (other kinds of cookies have not been on the plates) Santa ate (marked with 1) or he did not (marked with 0). The order in which the cookies were eaten does not matter. Additionally, Ella marked in which cases Santa became sick or not.

Gustav remarks doubtfully, "Santa will not be able to remember all this information. You know that his memory is not the best anymore?!"
"He will not have to memorize all the combinations," Ella replies. "We will investigate our notes and will look for shorter patterns that explain Santa's stomach aches."

Since Ella is a mathematician during the rest of the year, she grabs pen and paper and starts to work:

We are looking for logical patterns hidden in our recordings. We have seven different kinds of cookies $x_1, \dots, x_7$. Every pattern contains a set of cookie recommendations.

A **cookie recommendation** L is a variable x with values in the set $\{0, 1\}$ or its negation $\overline{x} = 1 - x$. For every kind of cookie, the recommendation is either *"Eat the cookie!"* or *"Do not eat the cookie!"*.

A **cookie rule** t consists of a set of cookie recommendations that are combined via multiplication—logically speaking this corresponds to the operation *and*. We write

$$t = L_1 \cdot L_2 \cdot \ldots \cdot L_k = L_1 L_2 \ldots L_k,$$

where $\{L_1, \dots, L_k\}$ is the set of cookie recommendations in t. Thus, a cookie rule is nothing else than a combination of cookie recommendations. As usual when working with sets, the order of the cookie recommendation in a cookie rule does not matter; that is, the cookie rules $L_1 \cdot L_2$ and $L_2 \cdot L_1$ both correspond to the set $\{L_1, L_2\}$. Furthermore, each cookie recommendation is to appear at most once in a cookie rule: either positive as x_i, negative as $\overline{x}_i$ or not at all.

The **degree** of a cookie rule t is the number of different cookie recommendations (i.e. the number of different kinds of cookies) contained in t.

"Gosh, Ella! How is an ordinary gnome to comprehend this complicated stuff?" Gustav complains.
"Wait a sec! I will explain it to you with help of an example," Ella tries to calm her colleague. "For $n = 2$, the cookie rules of degree 1 are x_1, $\overline{x_1}$, x_2, $\overline{x_2}$, and

$$x_1 x_2,\ x_1 \overline{x_2},\ \overline{x_1} x_2,\ \overline{x_1}\,\overline{x_2}$$

are the cookie rules of degree 2, where the order of the cookie recommendations does not matter. For example, the cookie rule $x_1 \overline{x}_2$ says, *'Eat cookies of kind no. 1, but do not eat cookies of kind no. 2!'*"
"Ok, I get it now," Gustav says relieved.

Contentedly, Ella resumes scribbling her formulae: By evaluating the negation and multiplication, one obtains a function

$$f_t : \{0,1\}^n \to \{0,1\}$$

for every cookie rule t. We write $t(v) = f_t(v)$ for the value of the function f_t applied to a selection of cookies $v \in \{0,1\}^n$. Furthermore, we say that a cookie rule **covers** a selection of cookies $v \in \{0,1\}^n$ if $t(v) = 1$.

"Stop!!!" Gustav interrupts again. "Do you have another example to demonstrate this?"
"Sure!" Ella replies. "For $n = 4$, the cookie rule $t = x_1\overline{x}_2x_4$ defines the following function

$$f_t : \{0,1\}^4 \to \{0,1\}, (v_1, v_2, v_3, v_4) \mapsto f_t(v_1, v_2, v_3, v_4) = v_1 \cdot (1 - v_2) \cdot v_4.$$

If we apply f_t to the cookie selection $v = (1,0,0,1)$, we obtain

$$t(v) = f_t(v) = f_t(1,0,0,1) = 1 \cdot (1-0) \cdot 1 = 1 \cdot 1 \cdot 1 = 1.''$$

Ella continues to ponder: A cookie rule t is called **pattern** of our recordings if t covers at least one of the cookie selections that caused stomach ache and none of the selections that did not.

"In our recordings in Figure 18, one pattern is marked in orange: the pattern $t = x_2\overline{x}_4$ covers two of the plates that caused pain and no plate that did not cause pain," Ella explains.

"Amazing!" Gustav is left almost speechless. "This is great! But remember that Santa's memory is not very good! I don't think that he is able to memorize patterns that consist of more than three cookie recommendations."
"Yes, I do remember," Ella says eagerly. "We will look for short patterns, for prime patterns! A pattern t is called **prime pattern** if no true subset of cookie recommendations in t is a pattern. Furthermore, two prime patterns t and t' are called **distinct** if the sets of cookie recommendations in t and t' are distinct; recall that the order of the cookie recommendations is not important and cookie recommendations (or kinds of cookies) do not appear more than once in such prime patterns.

Now, Ella and Gustav want to know: How many distinct prime patterns of degree 3 are contained in the recordings displayed in Figure 18?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. The number of prime patterns of degree 3 is 1.
2. The number of prime patterns of degree 3 is 2.
3. The number of prime patterns of degree 3 is 3.
4. The number of prime patterns of degree 3 is 4.
5. The number of prime patterns of degree 3 is 5.
6. The number of prime patterns of degree 3 is 6.
7. The number of prime patterns of degree 3 is 7.
8. The number of prime patterns of degree 3 is 8.
9. The number of prime patterns of degree 3 is 9.

10. The number of prime patterns of degree 3 is 0 (that is, there are no prime patterns of degree 3).

Remark: We suggest to first consider the cookie rules of degree 2 together with those plates covered by them that caused stomach ache and with those plates covered by them that did not cause stomach ache.

Project reference:

Mathematical modelling in biological and medical applications is almost always faced with the problem of incomplete and noisy data. Rather than adding unsupported assumptions to obtain a unique model, a different approach generates a pool of models in agreement with all available observations. Analysis and classification of such models allow linking the constraints imposed by the data to essential model characteristics and showcase different implementations of key mechanisms.

Logical Analysis is a method for data classification. Here we use Logical Analysis to search for patterns in data sets consisting of protein measurements in cellular signalling networks. Patterns of activated or inactivated proteins can among other things help to differentiate between healthy and mutated cells.

18.2 Solution

The correct answer is: 6.

In the following, we will write **literal** for a cookie recommendation and **term** for a cookie rule.

Recall that prime patterns are patterns that do not contain patterns of lower degree. The only pattern of degree 1 is x_1, which is also a prime pattern. The prime patterns of degree 2 are those patterns that do not contain the pattern x_1. The terms of degree 2 are listed in Table 2 together with the covered positive or negative observations. An entry of the form

$$A \;\; L_1L_2 \;\; B$$

with $A \subseteq \{1,2,3,4,5\}$ and $B \subseteq \{6,7,8\}$ denotes that the term L_1L_2 covers the positive observation in A and the negative observations in B. The prime patterns are exactly those terms L_1L_2 for which $A \neq \emptyset$ and $B = \emptyset$.

Definition 1 (Pattern candidate) *A term L_1L_2 is called a* ***pattern candidate*** *of degree 2, if L_1L_2 covers at least one positive and at least one negative observation.*

Satz 1 *If the term $L_1L_2L_3$ is a prime pattern of degree 3, then L_1L_2, L_2L_3, and L_1L_3 are pattern candidates of degree 2.*

Proof: As a pattern $L_1L_2L_3$ covers at least one positive observation, which is then also covered by L_1L_2, L_2L_3, and L_1L_3.
If a term L_iL_j $(1 \leq i < j \leq 3)$ does not cover a negative observation, then L_iL_j would be a pattern and thus $L_1L_2L_3$ would not be a prime pattern. □

Satz 2 *Let L_1L_2, L_2L_3 be two pattern candidates with a common literal L_2. Then, the term $L_1L_2L_3$ is a prime pattern of degree 3 if and only if the following three conditions hold:*

1. *There is at least one positive observation that is covered by both L_1L_2 and L_2L_3.*

2. *No negative observation is covered by both L_1L_2 and L_2L_3.*

3. *There is a negative observation that is covered by L_1L_3.*

Proof: A (positive or negative) observation is covered by $L_1L_2L_3$ if and only if it is covered by both L_1L_2 and L_2L_3. Thus, $L_1L_2L_3$ is a pattern if and only if the two conditions 1 and 2 hold.
If $L_1L_2L_3$ is a prime pattern, then L_1L_3 is a pattern candidate according to Theorem 1. Thus, it covers at least one negative observation, i. e. condition 3 holds.
On the other hand, assume that the three conditions 1–3 hold. We have seen that $L_1L_2L_3$ is a pattern then. Since L_1L_2, L_2L_3 (as pattern candidates) and L_1L_3 (due to condition 3) cover each at least one negative observation, L_1L_2, L_2L_3, and L_1L_3 are not patterns themselves. Hence, $L_1L_2L_3$ is a prime pattern. □

To determine all prime patterns of degree 3, it is sufficient to scan Table 2 for entries of the form

$$A \;\; L_1L_2 \;\; B \quad \text{and} \quad C \;\; L_2L_3 \;\; D$$

with a common literal L_2 such that:

1. $A, B \neq \emptyset$ and $C, D \neq \emptyset$, i. e. L_1L_2 and L_2L_3 are pattern candidates.
2. $A \cap C \neq \emptyset$, i. e. $L_1L_2L_3$ covers at least one positive observation.
3. $B \cap D = \emptyset$, i. e. $L_1L_2L_3$ does not cover a negative observation.
4. For the remaining term $E \;\; L_1L_3 \;\; F$, one has $F \neq \emptyset$, i. e. L_1L_3 covers at least one negative observation (and thus is a pattern candidate as well).

Beispiel 1 *We start with the entry* $\{1,5\} \;\; \overline{x_2}\,\overline{x_3} \;\; \{7\}$ *and look for entries* $C \;\; \overline{x_3}L_3 \;\; D$ *with* $C, D \neq \emptyset, \{1,5\} \cap C \neq \emptyset$ *and* $\{7\} \cap D = \emptyset$*. The only admissible entry is* $\{5\} \;\; \overline{x_3}\,x_5 \;\; \{8\}$*. Furthermore, since* $\{3,5\} \;\; \overline{x_2}\,x_5 \;\; \{6\}$ *holds, the term* $\overline{x_2}x_5$ *is not a pattern. Thus,* $\overline{x_2}\,\overline{x_3}\,x_5$ *is a prime pattern that covers the positive observation* $\{5\}$*.*

Analogously, we examine all possible combinations of pattern candidates (while the number of possibilities decreases rapidly). To sum up, the prime patterns of degree 3 are:

$$\overline{x_2}\,\overline{x_3}\,x_5, \;\; \overline{x_2}\,x_4\,x_5, \;\; \overline{x_2}\,x_5\,x_7, \;\; \overline{x_3}\,x_5\,\overline{x_6}, \;\; x_4\,x_5\,\overline{x_6}, \;\; x_5\,\overline{x_6}\,x_7$$

Hence, the number of prime patterns of degree 3 is **6**.

$\{1,5\}$	$\overline{x_2}\,\overline{x_3}$	$\{7\}$	$\{1,2\}$	$\overline{x_3}\,\overline{x_4}$	$\emptyset$	$\{1,2\}$	$\overline{x_4}\,\overline{x_5}$	$\emptyset$	$\{1\}$	$\overline{x_5}\,\overline{x_6}$	$\{7\}$	$\{1\}$	$\overline{x_6}\,\overline{x_7}$	$\{6\}$
$\{3\}$	$\overline{x_2}\,x_3$	$\{6\}$	$\{5\}$	$\overline{x_3}\,x_4$	$\{7,8\}$	$\{3,4\}$	$\overline{x_4}\,x_5$	$\{6\}$	$\{2\}$	$\overline{x_5}\,x_6$	$\emptyset$	$\{5\}$	$\overline{x_6}\,x_7$	$\{7\}$
$\{2\}$	$x_2\,\overline{x_3}$	$\{8\}$	$\{3,4\}$	$x_3\,\overline{x_4}$	$\{6\}$	$\emptyset$	$x_4\,\overline{x_5}$	$\{7\}$	$\{5\}$	$x_5\,\overline{x_6}$	$\{6\}$	$\{3,4\}$	$x_6\,\overline{x_7}$	$\emptyset$
$\{4\}$	$x_2\,x_3$	$\emptyset$	$\emptyset$	$x_3\,x_4$	$\emptyset$	$\{5\}$	$x_4\,x_5$	$\{8\}$	$\{3,4\}$	$x_5\,x_6$	$\{8\}$	$\{2\}$	$x_6\,x_7$	$\{8\}$
$\{1,3\}$	$\overline{x_2}\,\overline{x_4}$	$\{6\}$	$\{1,2\}$	$\overline{x_3}\,\overline{x_5}$	$\{7\}$	$\{1\}$	$\overline{x_4}\,\overline{x_6}$	$\{6\}$	$\{1\}$	$\overline{x_5}\,\overline{x_7}$	$\emptyset$			
$\{5\}$	$\overline{x_2}\,x_4$	$\{7\}$	$\{5\}$	$\overline{x_3}\,x_5$	$\{8\}$	$\{2,3,4\}$	$\overline{x_4}\,x_6$	$\emptyset$	$\{2\}$	$\overline{x_5}\,x_7$	$\{7\}$			
$\{2,4\}$	$x_2\,\overline{x_4}$	$\emptyset$	$\emptyset$	$x_3\,\overline{x_5}$	$\emptyset$	$\{5\}$	$x_4\,\overline{x_6}$	$\{7\}$	$\{3,4\}$	$x_5\,\overline{x_7}$	$\{6\}$			
$\emptyset$	$x_2\,x_4$	$\{8\}$	$\{3,4\}$	$x_3\,x_5$	$\{6\}$	$\emptyset$	$x_4\,x_6$	$\{8\}$	$\{5\}$	$x_5\,x_7$	$\{8\}$			
$\{1\}$	$\overline{x_2}\,\overline{x_5}$	$\{7\}$	$\{1,5\}$	$\overline{x_3}\,\overline{x_6}$	$\{7\}$	$\{1,3,4\}$	$\overline{x_4}\,\overline{x_7}$	$\{6\}$						
$\{3,5\}$	$\overline{x_2}\,x_5$	$\{6\}$	$\{2\}$	$\overline{x_3}\,x_6$	$\{8\}$	$\{2\}$	$\overline{x_4}\,x_7$	$\emptyset$						
$\{2\}$	$x_2\,\overline{x_5}$	$\emptyset$	$\emptyset$	$x_3\,\overline{x_6}$	$\{6\}$	$\emptyset$	$x_4\,\overline{x_7}$	$\emptyset$						
$\{4\}$	$x_2\,x_5$	$\{8\}$	$\{3,4\}$	$x_3\,x_6$	$\emptyset$	$\{5\}$	$x_4\,x_7$	$\{7,8\}$						
$\{1,5\}$	$\overline{x_2}\,\overline{x_6}$	$\{6,7\}$	$\{1\}$	$\overline{x_3}\,\overline{x_7}$	$\emptyset$									
$\{3\}$	$\overline{x_2}\,x_6$	$\emptyset$	$\{2,5\}$	$\overline{x_3}\,x_7$	$\{7,8\}$									
$\emptyset$	$x_2\,\overline{x_6}$	$\emptyset$	$\{3,4\}$	$x_3\,\overline{x_7}$	$\{6\}$									
$\{2,4\}$	$x_2\,x_6$	$\{8\}$	$\emptyset$	$x_3\,x_7$	$\emptyset$									
$\{1,3\}$	$\overline{x_2}\,\overline{x_7}$	$\{6\}$												
$\{5\}$	$\overline{x_2}\,x_7$	$\{7\}$												
$\{4\}$	$x_2\,\overline{x_7}$	$\emptyset$												
$\{2\}$	$x_2\,x_7$	$\{8\}$												

Table 2: Terms of degree 2 with the covered positive and negative observations

19 Slipping Rudi

Author: Sarah Roth (ZIB)
Project: ECMath MI7 – Routing Structures and Periodic Timetabling
Translation: Ariane Beier (MATHEON)

19.1 Challenge

In order to secure on time delivery all over the world, right after their production in the Christmas Elve's Workshop, the Christmas presents are brought to the warehouse Greenland Gifts. From there, the presents are distributed to the present warehouses on the different continents.

As every year, the shipment of presents is organised by the international company *Running Rudi*. Currently, *Running Rudi* employs 168 reindeer, among them Rudolf (called Rudi) and his friends Ren-Qing, Rashid, Raha, Riley, and Rolando. This year, the friends were lucky to be assigned together to the yellow line, which runs back and forth from the station Greenland Gifts to the station Lama Logistics.

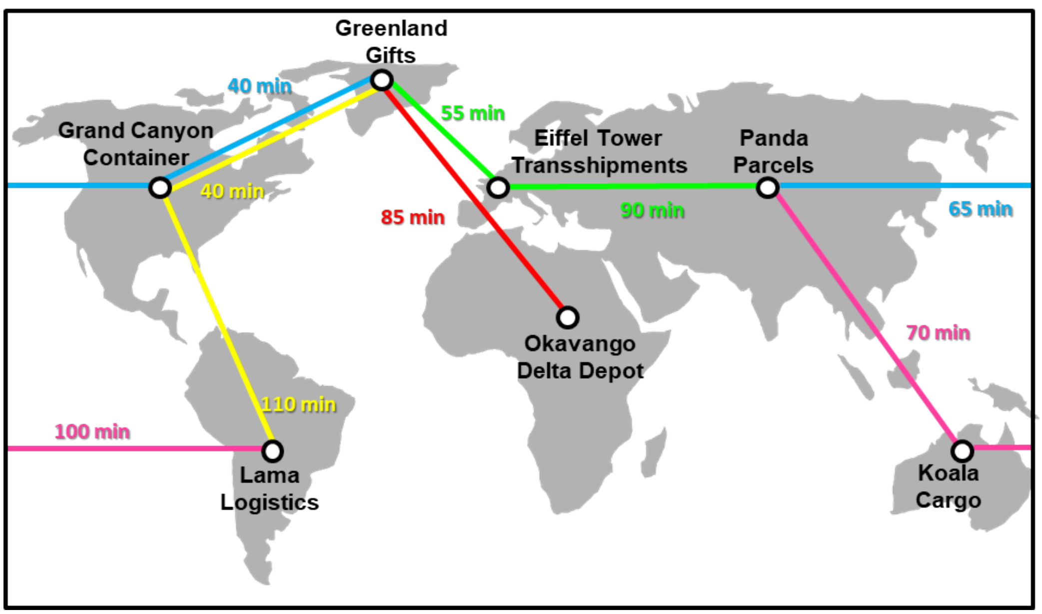


Figure 19: Network with times stated in minutes.

The following timetable is realised by *Running Rudi*. The timetable contains the departure times (= arrival times) and the direction of travel. The timetable is repeated every 60 minutes.

Station	1		2		3		4		5	
Greenland Gifts	30 ↓	40 ↑	35 ↓	50 ↑	55 ↓	15 ↑	05 ↓	20 ↑		
Grand Canyon Container					35 ↓	35 ↑	45 ↓	40 ↑		
Okavango Delta Depot	55 ↓	15 ↑								
Eiffel Tower Transshipments			30 ↓	55 ↑						
Lama Logistics							35 ↓	50 ↑	20 ↓	45 ↑
Koala Cargo									00 ↓	05 ↑
Panda Parcels			00 ↓	25 ↑	40 ↓	30 ↑			10 ↓	55 ↑

A reindeer cart consists of six reindeer. It is assigned to a fixed line and pulls the same sleigh all day long. Thus, all of the 168 reindeer are occupied simultaneously.

Unfortunately, Rudi slipped during the touchdown at Greenland Gifts this morning. Besides having a crimson nose now, he also sprained his ankle. It is out of the question for Rudi to work on the next two days. Since Rudi's friends are not able to pull the sleigh by themselves, the timetable gets mixed up completely.

Rachel, the manager at *Running Rudi*, tries to implement a new schedule for the remaining reindeer, such that the existing timetable (see above) can be adhered. In doing so, she allows the reindeer to switch in between lines at the termini. However, the change of six reindeer from a sleigh of one line to the sleigh of another line takes 15 minutes.

Rudi's friends—who of course still want to work together in a cart—have some additional requests for the new schedule:

Ren-Qing, "I have never been in Europe. I want to go there."
Riley, "And I want to deliver presents to my friends in Australia."
Rolando, "In any case, I do not want to be on the road for more than 12 hours from our starting point at Greenlanidc to the endpoint, which is at

Greenland Gifts as well."

What is the maximal number of reindeer carts that can be saved with a new schedule that still realises the timetable given above? Is there a route in this schedule that fulfils all the requests made by Rudi's friends?

Illustration: Frauke Jansen

Possible answers:

1. There does not exist a schedule with less than 168 reindeer.
2. At most one cart can be saved with a new schedule. However, this schedule realises only two requests.
3. At most one cart can be saved with a new schedule. This schedule realises all of the requests.
4. At most two carts can be saved with a new schedule. However, this schedule realises only one request.
5. At most two carts can be saved with a new schedule. This schedule realises all of the requests.

6. At most three carts can be saved with a new schedule. However, this schedule realises only one request.
7. At most three carts can be saved with a new schedule. However, this schedule realises only two requests.
8. At most four carts can be saved with a new schedule. However, this schedule realises only one request.
9. At most four carts can be saved with a new schedule. However, this schedule realises only two requests.
10. At most four carts can be saved with a new schedule. This schedule realises all of the requests.

Project reference:

The project *Routing structures and Periodic* attends to develop optimal timetables for local public transport networks. The transport companies prefer timetables that use vehicles as efficient as possible. At the same time, requests of the employees need to be incorporated as well.

19.2 Solution

The correct answer is: 7.

The number of reindeer required on one tour may be determined by counting the hours needed to run this tour. For every hour, one needs to employ a sleigh with six reindeer.

Example: Red line.
In the original timetable, the reindeer circle only on one line. For the red line, 85 min are spent for the rides in each direction (see Figure 1). In addition, there are layovers of 20 min at *Okavango Delta Depot* and 50 min at *Greenland Gifts* (see the timetable). In total, one has:

$$85 + 85 + 20 + 50 = 240$$

Thus, one cart needs 240 min (i. e. four hours) for a whole tour on the red line. Since the timetable is repeated every 60 min, four carts with all in all $4 \cdot 6 = 24$ reindeer are required on that line.

In order to minimize the number of required carts, we need to construct a timetable that minimizes the hours spent on the tours. The number of hours for a particular tour may be determined by counting the number of times the minute 00 is exceeded on this tour. Our goal is to minimize the latter.

The pure travel time on each tour is the same as in the original timetable. Thus, carts may only be saved by shortening the layover time at the termini.

There are three stations at which more than one line terminates. At these stations, one may change the itinerary planning of the reindeer. The stations are *Lama Logistics*, *Panda Parcels*, and *Greenland Gifts*. In the following, we will examine all possible combinations for redrafting the reindeer at these stations:

A **1** in the table of arrival and departure times indicates that the minute 00 will be exceeded; whereas, a **0** indicates that this is not the case. The 15 min required to reharness a reindeer cart are already included in the timetable. For an admissible itinerary, every arrival time needs to be assigned to exactly one departure time, since the reindeer need to know when and on which line they have to continue their tour. Thus, we have to choose exactly one entry from each row and each column to construct an itinerary. Because we want to minimize the number of times 00 is exceeded, we choose those connections with as many zeros as possible. These connections are marked in orange. The original connections may be found on the diagonal.

Lama Logistics

Arr. \ Dep.	50	20
35	0	1
45	1	1

In this case, the original connection was already optimal. The reindeer continue to ride on the line that took them to Lama Logistics. It is not possible to shorten the travel time by switching to another line, since the change of a reindeer cart from one line to another takes 15 min, which would exceed the 5 min layover from the magenta to the yellow line. As we have not changed the itinerary, we did not save a reindeer cart at this station.

Panda Parcels

Arr. \ Dep.	25	55	30
00	0	0	0
10	0	0	0
40	1	0	1

Arr. \ Dep.	25	55	30
00	0	0	0
10	0	0	0
40	1	0	1

At this station, there exist two possibilities to construct an optimal route. Both itinerary changes lead to one saved reindeer cart compared to the original connection (on the diagonal). Either the connection green line stays the same and the reindeer carts switch only between the magenta and blue line, or the reindeer carts change from the green to the blue line, from the magenta to the green line, and from the blue to the magenta line.

Greenland Gifts

Arr. \ Dep.	35	30	05	55
50	1	1	1	1
40	1	1	1	0
20	0	1	1	0
15	0	0	1	0

Here, there is again only one optimal itinerary: The green and the yellow line will be connected as well as the red and blue line. In doing so, one is able to save two reindeer carts compared to the original schedule (on the diagonal).

Altogether, we are able to save three reindeer carts compared to the original itinerary. But what about the reindeer's demands?

According to the first possibility chosen at *Panda Parcels*, we obtain a union of the yellow and green lines as well as a union of the red, blue, and magenta lines. Thus, a reindeer cart is able to visit either Europe or Australia, but not both. A tour on the yello-green line takes 11 h. Hence, it is possible to fulfil the wishes of Ren-Qing and Rolando, but not Riley's.

By choosing the second possibility at *Panda Parcels*, the itinerary becomes a circle running through all lines and all continents. Thus, it is possible to fulfil Ren-Qing's and Riley's wish. However, the whole tour takes more than 25 h, in particular more than 24 h. Hence, Rolando will not be satisfied.

No matter which of the two possibilities Rachel will choose, in an optimal itinerary that saves three reindeer carts, only two of three wishes can be fulfilled.

Calculation of travel times:

Line	1	2	3	4	5
Pure Travel Time	$2 \cdot 85$ $= 170$	$2 \cdot 55 + 2 \cdot 90$ $= 290$	$2 \cdot 65 + 2 \cdot 40$ $= 210$	$2 \cdot 40 + 2 \cdot 110$ $= 300$	$2 \cdot 70 + 2 \cdot 100$ $= 340$

Layover Time	Panda Parcels Possibility 1		Panda Parcels Possibility 2
	1 3 5	2 4	1 2 3 4 5
Greenland Gifts	$15 + 15 = 30$	$15 + 15 = 30$	$2 \cdot 30 = 60$
Grand Canyon Container	0	0	0
Okavango Delta Depot	20	0	20
Eiffel Tower Transshipments	0	0	0
Lama Logistics	35	15	$15 + 35 = 50$
Koala Cargo	0	0	0
Panda Parcels	$20 + 15 = 35$	25	$15 + 30 + 15 = 60$
Sum	120	70	190

According to possibility 1, we obtain a travel time (pure travel time + layover) of

$$290\,\text{min} + 300\,\text{min} + 70\,\text{min} = 660\,\text{min},$$

i. e. 11 h, for the union of the green and yellow lines. For the union of the red, blue, and magenta lines, we get a travel time of

$$170\,\text{min} + 210\,\text{min} + 340v\,\text{min} + 100\,\text{min} = 840\,\text{min},$$

i. e. 14 h.

According to possibility 2, we obtain a overall travel time of

$$170\,\text{min} + 290\,\text{min} + 210\,\text{min} + 300\,\text{min} + 340\,\text{min} + 190\,\text{min} = 1500\,\text{min},$$

i. e. 25 h, for the union of all five lines as described above.

20 Hat Challenge 2018

Authors: Aart Blokhuis (TU Eindhoven), Gerhard Woeginger (TU Eindhoven)

20.1 Challenge

Father Christmas addresses the three super clever elves Atto, Bilbo, and Chico, “My dear elves! Challenging brain teasers with coloured hats on elves’ heads have a long tradition in the mathematical advent calendar. For this reason, I would like to invite you to a cosy afternoon with coffee and cake tomorrow.”
“Great, we love to come!” the elves answer enthusiastically.

Father Christmas is also delighted: “Very well. I will prepare five hats this evening: A white one, a yellow one, a red one, a blue one, and a black one. Tomorrow, you position yourselves one after the other. Then, I will put one of the five hats on each of your heads—lightning-fast and from behind your backs: Atto will stand at the back of the short line, and he can see only the hats on the heads of the two gnomes in front of him. Bilbo is in the middle; he can only see the hat on Chicos head. Chico stands in the front; he is not able to see any of the hats.”

Father Christmas continuous, “At first, Atto has to guess his hat’s colour—he has to utter his guess loud and clear so that Bilbo and Chico are able to hear it. Secondly, Bilbo has to guess the colour of the hat on his head, while Atto and Chico are listening. At last, Bilbo has to guess his hat’s colour. Afterwards, you can remove the hats from your heads and have a close look at their colour. If all three of you managed to guess the colour right, then you will receive a delicious piece of *Sacher torte* and a good cup of coffee. If, however, only one of you fails, then all of you will only get some bitter brew made of coffeeweed and a dry piece of cake baked a month ago.”

Atto interrupts Father Christmas by asking, “How do you choose our hats?”
“I will choose them completely at random; that is, such that each of the 60 possible colour combinations on your heads is exactly equiprobable,” Father Christmas answers.

Bilbo asks, “May we have a look at the hats you did not put on our heads?”
“No!” replies Father Christmas. “The unused hats will be put in a closet.”

Chico asks, “Are we allowed to communicate any other information besides the colour we guess the hat on our head has?”
“No, of course not!” Father Christmas says. “That would be a violation of the super clever elves’ code of honour.”

The three elves start to ponder. They discuss and contemplate. They contemplate, and they discuss. Then, they discuss even more and contemplate even longer. Finally, they work out a truly ingenious strategy that maximises the number N of colour combinations that lead to coffee and Sacher torte.

Our question is: How big is N?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. $N = 12$.
2. $N = 13$.

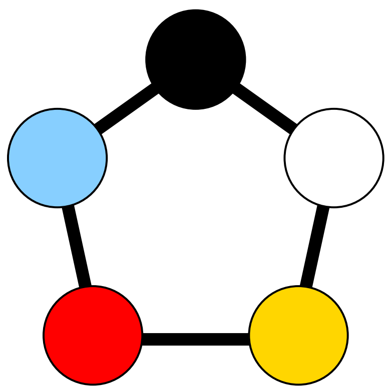
3. $N = 14$.
4. $N = 15$.
5. $N = 16$.
6. $N = 17$.
7. $N = 18$.
8. $N = 19$.
9. $N = 20$.
10. $N = 21$.

20.2 Solution

The correct answer is: 9.

First, one easily sees that $N \leq 20$: Atto is just able to see the hats of Bilbo and Chice; thus, he has three candidates left that may equiprobably be his hat colour. Thus, Atto will only say the right colour in a third of the 60 possible cases.

Now, we describe a strategy that guarantees the elves success, coffee, and cake in at least 20 of the possible cases. The elves arrange the five colours in a pentagon as follows:



In the pentagon, for every two colours there is a third colour that has the same distance from both of them (and which we call the **mid colour** of these two colours):

From the colours WHITE and RED just YELLOW has the same distance (one side length); thus, YELLOW is the mid colour of WHITE and RED. From WHITE and BLACK just RED has the same distance (two side lengths), and thus RED is the mid colour of WHITE and BLACK. And so on.

The strategy of the elves is based on the assumption that Atto's hat colour A is the mid colour of Bilbo's and Chico's hat colours B and C:

- Atto sees B and C and identifies A as the mid colour of B and C.
- Bilbo hears A, sees C, and is able to calculate the colour B.

- Chico hears A and B and identifies the colour C.

For example, consider the following situation: Atto sees the colour B=BLUE and C=YELLOW on the heads of Bilbo and Chico:

- Since RED is the mid colour of BLUE and YELLOW, Atto will guess A=RED in this situation.
- Bilbo hears Atto guessing A=RED and sees C=YELLOW. Since Bilbo knows that RED is the mid colour of YELLOW and his own hat colour, he accordingly identifies B=BLUE as his hat colour.
- Chico hears Atto guessing A=RED and Bilbo guessing B=BLUE. Since Chico knows that RED is the mid colour of BLUE and his own hat colour C, he correctly identifies C=YELLOW as his hat colour.

How successful is this strategy?

If A actually is the mid colour of B and C, then Attos guesses right; if A is one of the other two possible colours, then Atto is guessing wrong. Since Father Christmas chooses the combination of the colours completely randomly, Atto will guess correctly in exactly one third of the possible cases.

And now comes the central observation: Bilbo and Chico will **always** be correct with this strategy! Atto's guessed color A is always the mid colour of B and C. If one knows C and the mid colour, one can determine B. If one knows B and the mid colour, one can determine C.

In conclusion:

- Bilbo always guesses correctly.
- Chico always guesses correctly.
- Atto guesses in a third of the possible cases correctly.

Hence, it is $N = 20$.

21 A cuboid of presents

Authors: Ulrich Reitebuch (FU Berlin), Martin Skrodzki (FU Berlin)
Project: GV-AP16 – Computational and structural aspects of point set surfaces
Translation: Ariane Beier (MATHEON)

21.1 Challenge

The elves want to store a particular pile of 512 presents as space-saving as possible. All of these presents

$$G_i^{(0)} = (x_i^{(0)}, y_i^{(0)}, z_i^{(0)}), \quad i = 1, \ldots, 512,$$

are rectangular cuboids with lengths x_i, widths y_i, and heights z_i, respectively. In addition, each present has an unambiguous orientation, given by the two labels "top" and "front" attached to it.

The elves note that each present belongs to a pair of presents

$$(G_{j_1}^{(0)}, G_{j_2}^{(0)}), \quad j_1, j_2 \in \{1, \ldots, 512\}, \quad j_1 \neq j_2,$$

that are *yz-compatible* and may be pushed together. That is, the widths and the heights of presents belonging to one pair are equal:

$$y_{j_1}^{(0)} = y_{j_2}^{(0)} \quad \text{and} \quad z_{j_1}^{(0)} = z_{j_2}^{(0)}.$$

Hence, two presents $G_{j_1}^{(0)}$ and $G_{j_2}^{(0)}$ of one pair yield a new rectangular cuboid with

- length $x_j^{(1)} = x_{j_1}^{(0)} + x_{j_2}^{(0)}$,
- width $y_j^{(1)} = y_{j_1}^{(0)} = y_{j_2}^{(0)}$ and
- height $z_j^{(1)} = z_{j_1}^{(0)} = z_{j_2}^{(0)}$,

see Figure 20. Furthermore, each of the 512 presents appears exactly once in one of such pairs. Thus, the elves are able to always push two presents together to yield 256 new rectangular cuboids $G_j^{(1)}$, $j = 1, \ldots, 256$, with

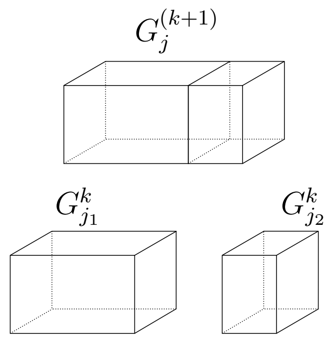


Figure 20: Two rectangular cuboids of the same widths and heights result in a new rectangular cuboid.

lengths $x_j^{(1)}$, widths $y_j^{(1)}$, and heights $z_j^{(1)}$, respectively.

The elves give it another try and observe that these 256 new rectangular cuboids may also be pushed together to yield even bigger rectangular cuboids. However, this time, always two of the 256 rectangular cuboids $G_j^{(1)}$, $j = 1, \ldots, 256$, are *xz-compatible*; that is, the lengths and heights within a pair of presents are the same. In the third step, the elves build 64 rectangular cuboids by always pushing together two *xy-compatible* rectangular cuboids that have the same length and width. In this way, they push together pairs of rectangular cuboids for additional six times, where the pair of rectangular cuboids are

- yz-compatible in the 1st, 4th, and 7th step,
- xz-compatible in the 2nd, 5th, and 8th step, and
- xy-compatible in the 3rd, 6th, and 9th step.

Finally, the elves obtain a big rectangular cuboid $G^{(9)}$ that consists of all 512 presents $G_i^{(0)}$. This rectangular cuboid may be stored efficiently, and the elves are very satisfied and take a break.

In order to easily cut wrapping paper, the elves use a LIGHT-powered Slicing And BrEaking Ressource, or in brief: LIGHTSABER. One elf is about to cut some wrapping paper with that LIGHTSABER, as he trips and causes a plane, non-axes-parallel cut through the stacked rectangular cuboid $G^{(9)}$, resulting in $G^{(9)}$ being divided into two non-empty parts.

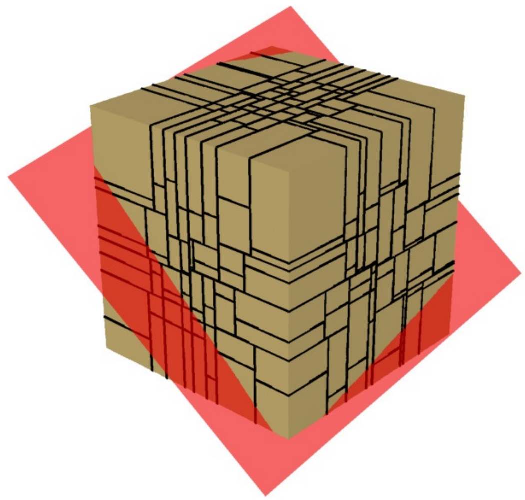


Figure 21: Example for a plane, non-axes-parallel cut through the rectangular cuboid $G^{(9)}$.

The arising kerfuffle is striking: Destroyed presents lie everywhere in the elves' workshop. However, the elves can only guess how many of the 512 presents were actually hit by the LIGHTSABER, because an over-eager administration elf has already shredded the documents with the exact measurements $(x_i^{(0)}, y_i^{(0)}, z_i^{(0)}),\ i = 1, \ldots, 512$. The elves only recall that the 512 presents allowed to be pushed together as described above.

What is the maximal number of the 512 presents $G_i^{(0)}$ that were divided into two non-empty parts by that cut?

Illustration: Sonja Rörig

Possible answers:

1. 64
2. 96
3. 120
4. 133
5. 142
6. 170
7. 176
8. 384
9. 448
10. 512

Project reference:

The partition of space, as described by the presents $G_i^{(0)}$, plays an important role when sampling points. If one first places data points in space and then partitions this space according to the described instructions of "pushing together" subsets, one is able to efficiently answer the following questions: Which of the data points lie in a particular domain and which data points are closest to the sampling point? The bigger the set of data points is, the more crucial are such efficient structures as a basis of any processes involving these data points. In the project *Computational and structural aspects of point set surfaces*, especially those data points are considered that were obtained from real geometries using 3D scan technologies and that digitally depict these geometries.

21.2 Solution

The correct answer is: 10.

We will show that at worst all 512 presents are divided by the LIGHTSABER by constructing presents $G_i^{(0)}$ with respective measurements. To this end, consider an arbitrary rectangular cuboid $G^{(9)}$ and an arbitrary non-axes-parallel plane H, which divides $G^{(9)}$ into two non-empty parts.

During the elves' last step, the remaining pair of cuboids was xy-compatible. Thus, we need to divide $G^{(9)}$ into two cuboids $G_1^{(8)}$ and $G_2^{(8)}$ of the same length and width, respectively. For this purpose, observe the height at which the cuboid $G^{(9)}$ is cut by H:

$$z_{\min} := \min\{z \in \mathbb{R} \mid \exists (x,y) \in \mathbb{R}^2 : (x,y,z) \in H \cap G^{(8)}\},$$
$$z_{\max} := \max\{z \in \mathbb{R} \mid \exists (x,y) \in \mathbb{R}^2 : (x,y,z) \in H \cap G^{(8)}\}.$$

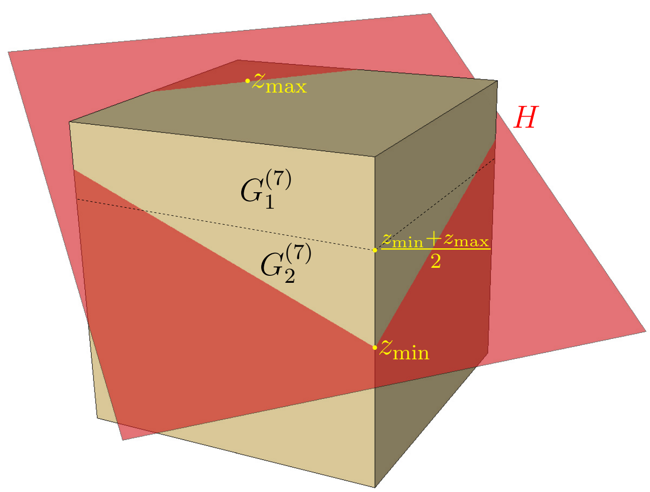


Figure 22: The division of $G^{(9)}$ into cuboids $G_1^{(8)}$ and $G_2^{(8)}$.

Note that $z_{\min}$ and $z_{\max}$ are the minimal and maximal height, respectively, in which $G^{(9)}$ is cut by H. Since H is non-axes-parallel, one has $z_{\min} > z_{\max}$.

If we divide $G^{(9)}$ exactly at the arithmetic mean of these heights,

$$\frac{z_{\max} + z_{\min}}{2}$$

we obtain two cuboids $G_1^{(8)}$ and $G_2^{(8)}$ with matching lengths and widths, and each of them is cut by H into two non-empty parts; see Figure 22.

We repeat this procedure recursively for each of the cuboids $G_1^{(8)}$ and $G_2^{(8)}$. Now, we have to construct xz-compatible pairs. For that reason, we have to determine the minimal and maximal widths—$y_{\min k}$ and $y_{\max k}$, $k = 1, 2$— and divide the cuboid at the y-coordinate $\frac{y_{\min k} + y_{\max k}}{2}$, $k = 1, 2$. We obtain four cuboids $G_j^{(7)}$, $j = 1, \ldots, 4$. Each of them has to be recursively divided according to its compatibility. Finally, this procedure yields 512 cuboids $G_i^{(0)}$, and each of them is cut by H into two non-empty parts.

In conclusion, we constructed an example in which all 512 cuboids were destroyed by the LIGHTSABER.

22 Cookie Explosion

Author: Christian Hercher (Universität Flensburg)
Translation: Ariane Beier (MATHEON)

22.1 Challenge

The union of reindeer calls for industrial action. After hours! At Christmas! Of course this commitment needs to be properly appreciated. Thus, Rudolph, the reindeer representative, is sent to Santa, the employer's representative:

Rudolph: We, the reindeer, request suitable wages for the valuable work we perform throughout the year. We want cookies as a compensation for the after hours at Christmas.

Santa: Cookies? Well, that's ok! We do have a lot of them... How many do you claim?

Rudolph (grinning whimsically): Santa, you like to play chess, don't you? We ask you to place one cookie on the first square of a normal chess board, then two on the second, then four on the third, and on the next square always twice as many as on the square before...

Santa: Wait! I will not be fooled with this old trick. No, my best Rudolph! Not on my watch!

Rudolph (still grinning whimsically): Well... We are willing to comply and play this game with these new rules:

1. We only use the baseline of a chessboard, that is, only eight squares. At the beginning, you have to place one cookie on the first square.
2. Every cookie that is located on one of the first seven squares may be exchanged for two cookies being placed on the next square.
3. Furthermore, we are allowed to interchange two piles of cookies that are adjacent and located on two of the squares no. 2 to 8. For this transaction, we have to pay one cookie that lies on the square directly before these two squares. One of the "interchange squares" is allowed to be empty at the interchange of the two

cookie piles. However, the "payment square", which lies directly before the two interchange squares, has to be non-empty before the payment.

4. In order to not confuse too much with all that cookie interchanging business, I suggest that at any time at most three of the eight squares are allowed to contain cookies. Furthermore, these three squares need to be direct neighbours. Of course, these three occupied squares are not fixed but rather are allowed to change throughout the game.

Santa: Fine! This sounds like a fair proposal. Let's close this deal by handshake. Or, wait a sec...

Now, we want to know: What is the maximal number of cookies that Rudolph is able to secure for his fellow reindeer? More precisely, what is the last digit (i. e. the unit digit) of this number?

Artwork: Julia Nurit Schönnagel

Possible answers:

1. 1.
2. 2.
3. 3.
4. 4.
5. 5.
6. 6.
7. 7.
8. 8.
9. 9.
10. 0.

Follow up question no. 1:

What is the second last digit (i. e. the tens place) of this number?

Follow up question no. 2:

How many cookies can Rudolph secure for the reindeer if he loosens the condition that only three adjacent squares are occupied by cookie piles to allow four, five or six squares?

22.2 Solution

The correct answer is: 2.

For Rudolph, it is only possible to increase the number of cookies on the board by applying a draw of

Type 1: Exchange one cookie from square k for two cookies on square $k+1$.

However, thereby as well as by a draw of

Type 2: Interchange the cookies on squares $k+1$ and $k+2$ by "paying" a cookie from square k.

the number of cookies on square k decreases. In particular, the game is finite, since no draw changes the number of cookies in the squares $< k$.

By the virtue of constraint no. 4, it suffices to divide the game in phases that only regard three consecutive squares. At the beginning of each phase, there is a positive number of cookies on the square with the smallest number; at the end of the phase, there are no cookies left on this square. Then, this particular square in not of interest for the game anymore, and the next phase begins with the considered squares shifted by one (or the end of the game is almost reached if only the two second last squares are occupied).

To increase the readability of the upcoming discussion, we denote by (a_1, a_2, a_3) a constellation, where there are a_1 cookies on the k th square, a_2 cookies on square no. $(k+1)$, and a_3 cookies on square no. $(k+2)$. Furthermore,

$$(a_1, a_2, a_3) \to (b_1, b_2, b_3),$$

means that a valid series of draws leads from the first to the second constellation.

Thus, each phase starts with $(a_1, a_2, 0)$, where $a_1 > 0$, and ends with $(0, b_2, b_3)$, where at least one of the numbers b_1, b_2 is strictly positive. In the following, we will show ("inductively" over the phases) that Rudolph begins a phase with a constellation $(a_1, 0, 0)$ and ends it with $(0, b_2, 0)$.

Obviously, the first phase starts with $(1,0,0)$, that is as desired. If a phase ends as described, a new phase starts (by shifting the considered squares by one) also as desired. It remains to show that this strategy is indeed the best one for Rudolph:

As you might guess, it is convenient (from Rudolph's point of view) to store as many cookies on the squares with the lowest possible numbers (because these cookies may then be "doubled" more often by a draw of type 1). In this respect, a draw of type 2 is beneficial only if a large number of cookies move to a square with a lower number. In consequence, it is most beneficial if the square $k+1$ is empty, i. e. in a constellation like $(a_1, 0, a_3)$.

At the beginning of each phase, we have to deal with such a situation, but interchanging the two empty squares gains nothing and costs a cookie from square k. Thus, the first draw needs to be of type 1 applied to square k. If in consequence this square becomes empty, a new phase begins.

If not, there is at least one cookie left on square k. This cookie may be exchanged for two new cookies on square $k+1$, or it may be paid to interchange the cookie piles on square $k+1$ and $k+2$. To this end, we first have to empty square $k+1$ by draws of type 1:

$$(a_1 - 1, 2, 0) \to (a_1 - 1, 0, 2 \cdot 2) \to (a_1 - 2, 2 \cdot 2, 0)$$

or, in general,

$$(a, b, 0) \to (a - 1, 2 \cdot b, 0),$$

where $a \geq 1$ and $b \geq 2$.

Thus, if there are at least two cookies on square $k+1$ (and indeed there are after a draw of type 1), then the best approach is to shovel and double all these cookies onto square $k+2$ by draws of type 1 (applied to square $k+1$), before moving them back onto square $k+1$ by paying only one cookie from the pile on square k.

Gradually, we double the cookies on square $k+1$ for every cookie that was on square k. That is, we start a phase with $(a, 0, 0)$ and end it with $0, 2^a, 0$. Then, we start the following phase with $(2^a, 0, 0)$.

In the game's final phase, there are only cookies left on square no. 7 and 8, and draws of type 2 are not applicable anymore. Hence, the game ends with the draws $(a, 0) \to (0, 2a)$, which is the final number of cookies Santa has to pay the reindeer.
We conclude:

1st phase (squares 1 to 3): $(1, 0, 0) \to (0, 2, 0)$,
2nd phase (squares 2 to 4): $(2, 0, 0) \to (0, 2^2, 0)$,
3rd phase (squares 3 to 5): $(2^2, 0, 0) \to (0, 2^{2^2}, 0)$,
etc.

For $1 \leq n \leq 5$ let P_n defined by

$$P_1 = 2 \text{ und } P_{n+1} = 2^{P_n},$$

i. e. P_n is a tetration with base 2 and height n. The nth phase terminates with the situation $(0, P_n, 0)$. That is, the 6th phase (squares 6 to 8) concludes with $(0, P_6, 0)$. The final phase starts with $(P_6, 0)$ and ends with $2 \cdot P_6$ cookies on the 8th square.

As an illustration of the numbers involved, one has

$$P_1 = 2, \quad P_2 = 2^2 = 4, \quad P_3 = 2^4 = 2^{2^2} = 16, \quad P_4 = 2^{16} = 2^{2^{2^2}} = 65536$$

and $P_5 = 2^{65536} = 2^{2^{2^{2^2}}}$ is a number with 19729 digits. The numbers P_6 and $2 \cdot P_6$ are not even tangible. In particular, any computer algebra system would be overly exhausted by an expression as a decimal number.

However, we are only interested in the unit digit of $2 \cdot P_6$. Hence, we only need to determine the unit digit of P_6 and multiply it by 2. Since P_6 is a (rather large) power of two, we have a look at the first powers of two:

The unit digit of 2^1 is 2.
The unit digit of 2^2 is 4.
The unit digit of 2^3 is 8.
The unit digit of 2^4 is 6.

The unit digit of 2^5 is 2.

The unit digit of 2^6 is 4.
The unit digit of 2^7 is 8.
The unit digit of 2^8 is 6.

etc.

We observe that the unit digits appear in cycles of four. (You may as well easily prove this, since the unit digit of a product coincides with the unit digit of the product of the unit digits of the factors.)

Hence, we have shown that, for every $n \geq 0$ and $1 \leq k \leq 4$, the numbers 2^k and 2^{4n+k} have the same unit digit.

Since $P_6 = 2^{P_5}$ and P_5 is, as a power of two with exponent $P_4 > 2$), divisible by 4, P_6 and 2^4 share the same unit digit, which is 6.

In conclusion, the number of cookies $2 \cdot P_6$ has unit digit **2**.

Concerning follow up question no. 1:

In principal, one may answer this question as before, but one may as well pursue a better-structered plan:

Since we are interested in the second last digit of, it suffices to consider $2 \cdot P_6$ $(mod\ 100)$.

> **Remark:** We denote by
>
> $$a \equiv r \ (mod\ b)$$
>
> the remainder r of a after division by b. For example,
>
> $$7 \equiv 1 \ (mod\ 3),$$
>
> since 1 is the remainder of 7 after division by 3.
>
> More general, one requires that a and r have the same remainder after division by b, i. e. that $a - r$ is divisible by b. These congruences may almost be treated as common equations. The only subtlety arises when performing division, which we will not employ in the following considerations.

More precisely, it suffices to consider P_6 $(mod\ 50)$ and to multiply the solution by 2. Furthermore, we know that P_6 is an even number, and, thus, we are actually interested in P_6 $(mod\ 25)$.

We know that $2^{10} = 1024 \equiv -1$ $(mod\, 25)$ and, consequently, $2^{20} \equiv 1$ $(mod\, 25)$. And since $P_6 = 2^{P_5}$ is a power of two, we may reduce the exponent modulo 20. Hence, we want to determine P_5 $(mod\ 20)$. Again P_5 is divisible by 4. Hence, it suffices to consider P_5 $(mod 5)$. Since P_5 is a power of two, we proceed analogously as previously:
We know that $2^4 = 16 \equiv 1$ $(mod\ 5)$, i.e. the remainders of powers of two $(mod\ 5)$ reoccur in cycles of four. Thus, we have to determine the remainder of the exponent of $P_5 = 2^{P_4}$, i.e. of P_4, after division by 4. But since P_4 is again divisible by 4, P_5 has the same remainder $(mod\ 5)$ as 2^0 (and 2^4), which is 1.

We now know the remainder of P_5 $(mod\ 5)$, but still need to determine P_5 $(mod\ 5)$. More precisely, we are looking for remainder $0 \leq r < 20$ that is divisible by 4 and $\equiv 1$ $(mod\ 5)$. (The existence and uniqueness of this remainder is guaranteed by the *Chinese remainder theorem.*) The remainder is 16. Hence, the remainder of P_5 after division by 20 is 16.

In consequence, the remainder of $P_6(mod\ 25)$ is the same as the remainder of $2^{16} = 65536(mod\ 25)$, which is 11. Applying the Chinese remainder theorem again, we obtain

$$P_6 \equiv 36\ (mod\ 50) \quad \text{and} \quad 2 \cdot P_6 \equiv 72\ (mod\ 100).$$

The last two digits of the maximal number of cookies Rudolph can secure for his fellow reindeer is 72; thus, the wanted digit is **7**.

Concerning follow up question no. 2:

If four consecutive squares are allowed to be occupied with cookies, one gets phase transitions $(a, 0, 0, 0) \to (0, P_a, 0, 0)$. In each phase, the height of the tetration does not increase by one, but is rather given by the value of tetration from the phase before... By increasing the number of squares the height of the tetration iterates further.

Here, we encounter a disguised version of an *Ackermann function*, which is of great interest in Theoretic Computer Science. It is the earliest-discovered example of a totally computable function that is not primitive recursive (it simply grows much to fast).

However, we do not need to be intimidated by the growth of the considered numbers: The maximal number of cookies just remains a power of two, and the unit digit and the second last digit may thus be determined as shown above.

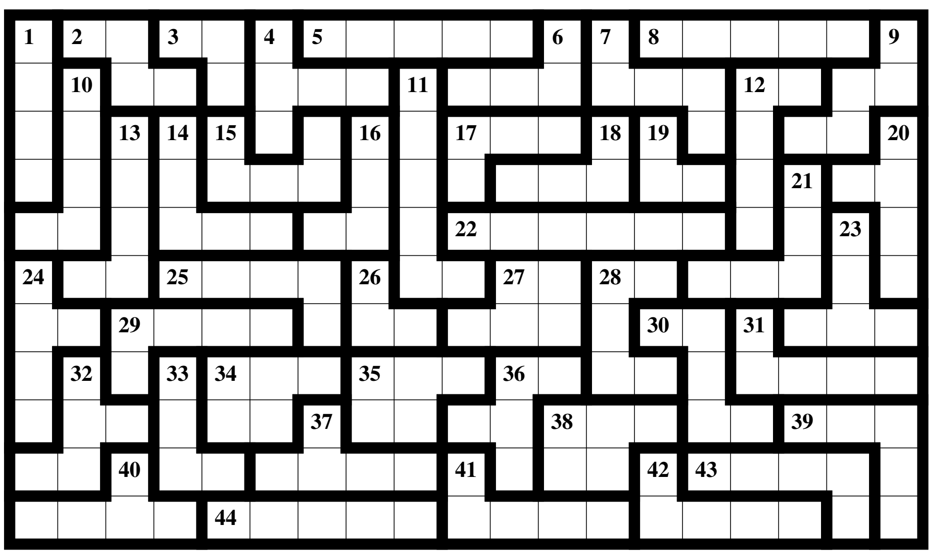
Remark: The challenge is a modification (and slight simplification) of Problem 5 from the IMO 2010. There, the contestants had to show that it is possible to earn $2010^{2010^{2010}}$ cookies with only six squares that were each equipped with one cookie at the beginning (without any constraints concerning which squares are allowed to be occupied or empty).

23 Mondrian

Authors: Hajo Broersma (Universiteit Twente), Cor Hurkens (TU Eindhoven)

23.1 Challenge

Mondrian the painter-elf has designed a new Christmas card, which he has subdivided into 44 regions. Mondrian calls two regions **adjacent** if they share a horizontal or a vertical edge. (For instance: Region 2 is adjacent to regions 1, 10, 13, 14 and 3. But region 2 is not adjacent to region 15, as these two regions only share a single point.)



Mondrian paints each of the regions with one of the four colors red, blue, yellow, and black so that no two adjacent regions have the same color. Mondrian starts by painting one of the regions black and thereby depletes his black color pencil. Hence the remaining 43 regions are painted red, blue, and yellow. It turns out that in the end the total red area, the total blue area, and the total yellow area all are equally large.

We want to know: Which of the regions has been painted black?

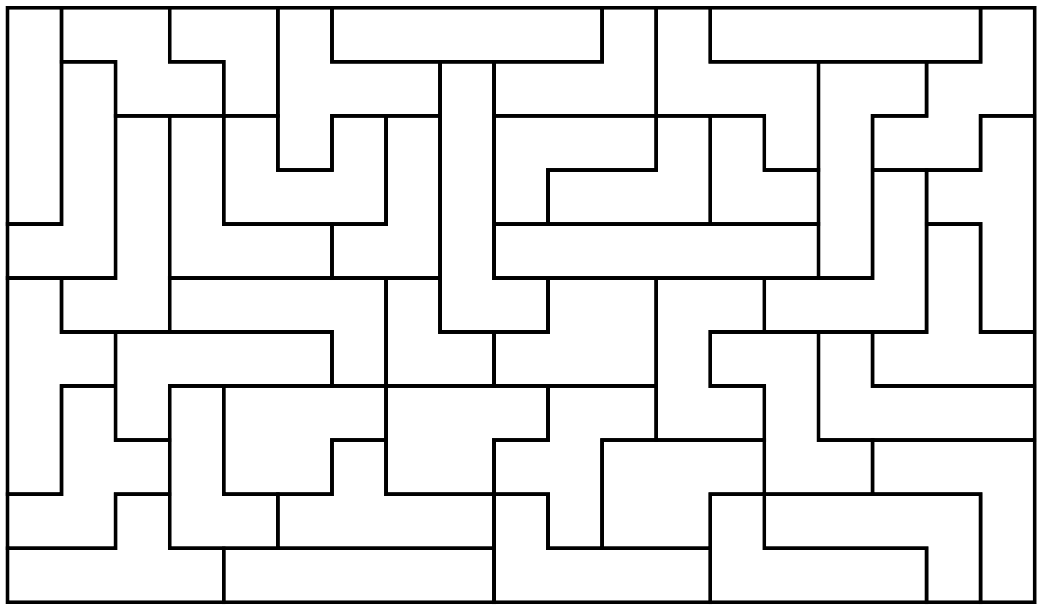
Artwork: Friederike Hofmann

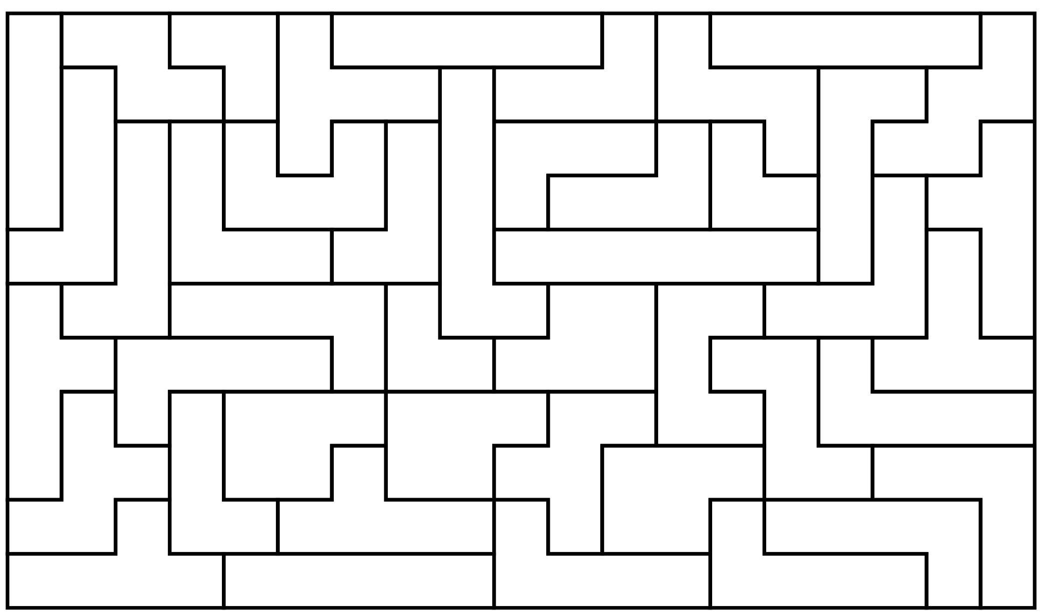
Possible answers:

1. One of the regions 1, 11, 21, 31, 41 has been painted black.
2. One of the regions 2, 12, 22, 32, 42 has been painted black.
3. One of the regions 3, 13, 23, 33, 43 has been painted black.
4. One of the regions 4, 14, 24, 34, 44 has been painted black.
5. One of the regions 5, 15, 25, 35 has been painted black.
6. One of the regions 6, 16, 26, 36 has been painted black.
7. One of the regions 7, 17, 27, 37 has been painted black.
8. One of the regions 8, 18, 28, 38 has been painted black.
9. One of the regions 9, 19, 29, 39 has been painted black.
10. One of the regions 10, 20, 30, 40 has been painted black.

Working material

This page may be printed and then be used as working material.

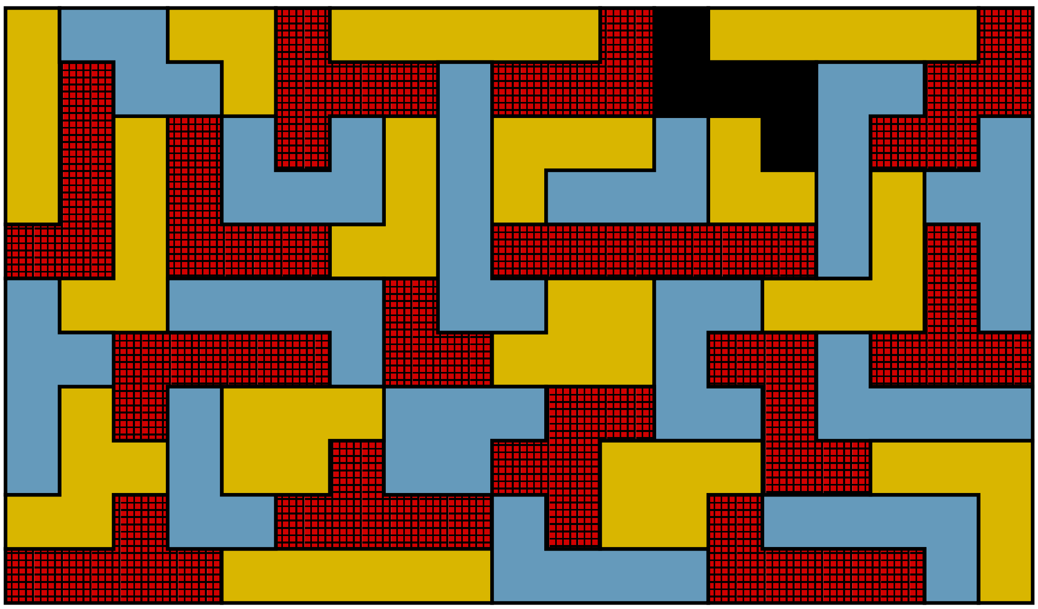




23.2 Solution

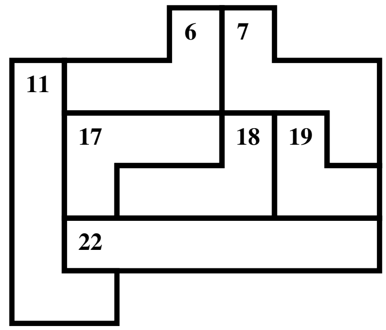
The correct answer is: 7.

The following figure shows a possible colouring of the Christmas card, where the red (checked), the blue (dark coloured), and the yellow (bright coloured) area each covers 68 squares, and where the region 7 is painted black.



Now, we want to show that the only region that can be coloured black is region 7. To this end, we first point out some useful observations: The area of the whole Christmas card is $11 \times 19 = 209$ unit squares. Since the red, the blue, and the yellow areas are equal in size, the whole red-blue-yellow area is $\equiv 0 \bmod 3$. Thus, the area F of the black region is of the form $F \equiv 2 \bmod 3$. The area of the smallest regions (for example, regions 3 and 26) is 3 unit squares and the area of the biggest regions (for example, regions 11 and 22) is 6 unit squares. As a consequence, the black region has to have an area of 5 unit squares.

Next, we assume for a proof by contradiction that none of the seven regions 6, 7, 11, 17, 18, 19, 22 is painted black. That is, Mondrian just used the colours red, yellow, and blue for these regions.



- Since the regions 18 and 19 are adjacent, they have two different colours (without loss of generalisation, assume: the colours red and blue).
- Since the region 7 is adjacent to the regions 18 and 19, it has the colour yellow.
- Since the region 22 is also adjacent to the regions 18 and 19, it is also yellow.
- Since the regions 11 and 17 are both adjacent to each other and to the region 22, one of them has to be blue and the other red.

The region 6 is adjacent to the region 7 (yellow) and to the two regions 11 and 17 (red and blue). Now, we have encountered a contradiction, because there is no valid colour left for region 6.

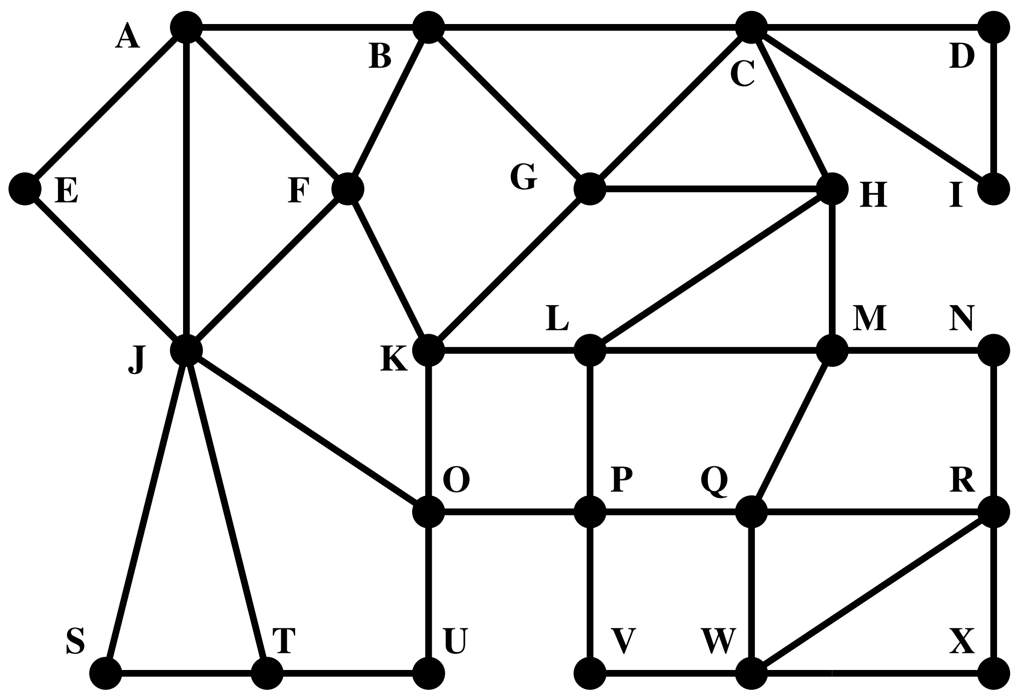
Now, we know that one of the seven regions 6, 7, 11, 17, 18, 19 or 22 is painted black. The areas of these regions take the values 4, 5, 6, 4, 4, 3 or 6 unit squares, respectively. Since the black region has an area of 5 squares, Mondrian painted the **region 7** black.

24 Nightwatch

Author: Frits Spieksma (TU Eindhoven)

24.1 Challenge

Wachfried the nightwatch-elf reports, "Last night I have been patrolling the 41 streets in our village. My patrol started at the northern border of the village (in one of the four points A, B, C, D). I have traversed two of the streets five times, one street four times, three streets three times, one street twice, and each of the thirty-four remaining streets exactly once. My patrol eventually ended directly in front of my house."



How many of the twentyfour points $A, B, C, \ldots, X$ in the picture are potential candidates for Wachfried's house?

Artwork: Frauke Jansen

Possible answers:

1. Only a single point.
2. Exactly five points.
3. Exactly six points.
4. Exactly seven points.
5. Exactly eight points.
6. Exactly nine points.
7. Exactly ten points.
8. Exactly eleven points.
9. Exactly twelve points.
10. Exactly thirteen points.

24.2 Solution

The correct answer is: 6.

If Wachfried strolls along a street an odd number of times, we call this street *odd*; otherwise, it is called *even*. A point adjacent to an odd number of odd streets is called *odd*.

Crucial observation: If there is an odd point, then Wachfried has to either begin or end his nightwatch at this point. Thus, if an odd point is not one of the four points A, B, C, D, then Wachfried has to terminate his nightwatch at this point.

According to Wachfried, there are 39 odd and 2 even streets. We will distinguish the following cases:

1. If none of the three streets TJ, TS and TU are even, then T is an odd point.
2. If TJ is the only even street adjacent to J, then J is an odd point.

   If TS is the only even street adjacent to S, then S is an odd point.

   If TU is the only even street adjacent to U, then U is an odd point.
3. If both streets are adjacent to S are even, then J is an odd point.

   If both streets are adjacent to U are even, then O is an odd point.
4. If TJ is an even street and if also the second even street is adjacent to J, then A or E or F or O or S is an odd point.

In the first case, Wachfried lives, according to the above observation, at point T. It is easy to construct a route starting in A and ending at T (where AB and BC are the only even streets).

In the second case, Wachfried lives at one of the points J, S, U. One easily constructs a route that begins in B and ends at J or S or U (where BC is the second even street).

In the third case, Wachfried lives at J or O. One easily constructs a route starting in C that ends in J or O.

The fourth case is the most interesting one: If one of the points E, F, O, S is odd, then, according to the crucial observation, Wachfried has to live at this odd point. In this case, it is fairly easy to construct a route that starts in C and terminates at the respective point.
In the remaining subcase, the streets TJ and JA are the two even streets, making A and C odd points. Thus, Wachfried may commence his route in C and terminate it at C; or he starts in A and ends in C.

In conclusion, the **nine points** $A, C, E, F, J, O, S, T, U$ qualify as Wachfried's house.

Remark: The following theorem is due to the Swiss mathematician Leonhard Euler (1707–1783):

> If a system of streets is connected (that is, if any street may be reached from any other street through the street network) and contains exactly two odd points, then there is a route that starts at one of the odd points and ends at the other one such that each street of the system is passed exactly ones.

If we regard streets that are traversed multiple times as multiple parallel streets that are traversed only once, the above theorem gives us all the desired routes for the solution. If you would like to know more about this problem, have a look at:

`https://en.wikipedia.org/wiki/Eulerian_path`